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THE $L(2,1)$-LABELING PROBLEM ON GRAPHS*

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Abstract. An $L(2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$. The $L(2,1)$-labeling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)$-labeling with $\max\{f(v) : v \in V(G)\} = k$. In this paper, we give exact formulas of $\lambda(G \cup H)$ and $\lambda(G + H)$. We also prove that $\lambda(G) \leq \Delta^2 + \Delta$ for any graph G of maximum degree Δ. For odd-sun-free (OSF)-chordal graphs, the upper bound can be reduced to $\lambda(G) \leq 2\Delta + 1$. For sun-free (SF)-chordal graphs, the upper bound can be reduced to $\lambda(G) \leq \Delta + 2\chi(G) - 2$. Finally, we present a polynomial time algorithm to determine $\lambda(T)$ for a tree T.

Key words. $L(2,1)$-labeling, T-coloring, union, join, chordal graph, perfect graph, tree, bipartite matching, algorithm

AMS subject classifications. 05C15, 05C78

1. Introduction.

The *channel assignment problem* is to assign a channel (nonnegative integer) to each radio transmitter so that interfering transmitters are assigned channels whose separation is not in a set of disallowed separations. Hale [11] formulated this problem into the notion of the *T-coloring* of a graph, and the T-coloring problem has been extensively studied over the past decade (see [4, 5, 7, 13, 14, 16, 17, 19]).

Roberts [15] proposed a variation of the channel assignment problem in which "close" transmitters must receive different channels and "very close" transmitters must receive channels that are at least two channels apart. To formulate the problem in graphs, the transmitters are represented by the vertices of a graph; two vertices are "very close" if they are adjacent in the graph and "close" if they are of distance two in the graph. More precisely, an *$L(2,1)$-labeling* of a graph G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) = 2$. A *k-$L(2,1)$-labeling* is an $L(2,1)$-labeling such that no label is greater than k. The *$L(2,1)$-labeling number* of G, denoted by $\lambda(G)$, is the smallest number k such that G has a k-$L(2,1)$-labeling.

Griggs and Yeh [10] and Yeh [21] determined the exact values of $\lambda(P_n)$, $\lambda(C_n)$, and $\lambda(W_n)$, where P_n is a *path* of n vertices, C_n is a *cycle* of n vertices, and W_n is an *n-wheel* obtained from C_n by adding a new vertex adjacent to all vertices in C_n. For the n-cube Q_n, Jonas [12] showed that $n + 3 \leq \lambda(Q_n)$. Griggs and Yeh [10] showed that $\lambda(Q_n) \leq 2n + 1$ for $n \geq 5$. They also determined $\lambda(Q_n)$ for $n \leq 5$ and conjectured that the lower bound $n + 3$ is the actual value of $\lambda(Q_n)$ for $n \geq 3$. Using a coding theory method, Whittlesey, Georges, and Mauro [20] proved that

$$\lambda(Q_n) \leq 2^k + 2^{k-q+1} - 2, \quad \text{where } n \leq 2^k - q \text{ and } 1 \leq q \leq k + 1.$$

In particular, $\lambda(Q_{2^k-k-1}) \leq 2^k - 1$. As a consequence, $\lambda(Q_n) \leq 2n$ for $n \geq 3$.

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For a tree T with maximum degree $\Delta \geq 1$, Griggs and Yeh [10] showed that $\lambda(T)$ is either $\Delta+1$ or $\Delta+2$. They proved that the $L(2,1)$-labeling problem is NP-complete for general graphs and conjectured that the problem is also NP-complete for trees.

For a general graph G of maximum degree Δ, Griggs and Yeh [10] proved that $\lambda(G) \leq \Delta^2+2\Delta$. The upper bound was improved to be $\lambda(G) \leq \Delta^2+2\Delta-3$ when G is 3-connected and $\lambda(G) \leq \Delta^2$ when G is of diameter two. Griggs and Yeh conjectured that $\lambda(G) \leq \Delta^2$ in general. To study this conjecture, Sakai [18] considered the class of chordal graphs. She showed that $\lambda(G) \leq (\Delta+3)^2/4$ for any chordal graph G. For a unit interval graph G, which is a very special chordal graph, she also proved that $2\chi(G)-2 \leq \lambda(G) \leq 2\chi(G)$.

The purpose of this paper is to study Griggs and Yeh's conjectures. We also study $L(2,1)$-labeling numbers of the union and the join of two graphs to generalize results on the n-wheel that is the join of C_n and K_1. For this purpose and a further reason that will become clear in §3, we introduce a related problem, which we call the $L'(2,1)$-labeling problem. The definitions of an *$L'(2,1)$-labeling* f, a *k-$L'(2,1)$-labeling* f, and the *$L'(2,1)$-labeling number* $\lambda'(G)$ are the same as those of an $L(2,1)$-labeling f, a k-$L(2,1)$-labeling f, and the $L(2,1)$-labeling number $\lambda(G)$, respectively, except that the function f is required to be one-to-one. There is a natural connection between $\lambda'(G)$ and the path partition number $p_v(G^c)$ of the complement G^c of G. For any graph G, the *path partition number* $p_v(G)$ is the minimum number k such that $V(G)$ can be partitioned into k paths.

The rest of this paper is organized as follows. Section 2 gives general properties of $\lambda(G)$ and $\lambda'(G)$. Section 3 studies $\lambda(G\cup H)$, $\lambda(G+H)$, $\lambda'(G\cup H)$, and $\lambda'(G+H)$. Section 4 proves that $\lambda(G) \leq \Delta^2+\Delta$ for a general graph G of maximum degree Δ. This result improves on Griggs and Yeh's result $\lambda(G) \leq \Delta^2+2\Delta$. However, there is still a gap in the conjecture $\lambda(G) \leq \Delta^2$. Section 5 studies the upper bounds for subclasses of chordal graphs. Section 6 presents a polynomial time algorithm to determine $\lambda(T)$ of a tree T.

A referee points out that Georges, Mauro, and Whittlesey [8] also solved $\lambda(G+H)$ and $p_v(G+H)$ by a different approach. They actually gave the solutions without introducing the notion of λ'; see the remarks after Lemmas 2.3 and 3.4.

2. Basic properties of λ and λ'.

LEMMA 2.1. *$\lambda(G) \leq \lambda(H)$ and $\lambda'(G) \leq \lambda'(H)$ for any subgraph G of a graph H.*

LEMMA 2.2. *$\lambda(G) \leq \lambda'(G)$ for any graph G. $\lambda(G) = \lambda'(G)$ if G is of diameter at most two.*

LEMMA 2.3. *$p_v(G) = \lambda'(G^c) - |V(G)| + 2$ for any graph G.*

Proof. Suppose f is a $\lambda'(G^c)$-$L'(2,1)$-labeling of G^c. Note that for any two vertices x and y in $V(G)$, if $f(x) = f(y)+1$, then $(x,y) \notin E(G^c)$ and so $(x,y) \in E(G)$. Consequently, a subset of vertices whose labels form a consecutive segment of integers form a path in G. However, there are at most $\lambda'(G^c) - |V(G)| + 2$ such consecutive segments of integers. Thus $p_v(G) \leq \lambda'(G^c) - |V(G)| + 2$.

On the other hand, suppose $V(G)$ can be partitioned into $k \equiv p_v(G)$ paths in G, say, $(v_{i,1}, v_{i,2}, \ldots, v_{i,n_i})$ for $1 \leq i \leq k$. Consider a dummy path $(v_{0,1})$ and define f by

$$f(v_{i,j}) = \begin{cases} -2, & \text{if } i=0 \text{ and } j=1; \\ f(v_{i-1,n_{i-1}})+2, & \text{if } 1 \leq i \leq k \text{ and } j=1; \\ f(v_{i,j-1})+1, & \text{if } 1 \leq i \leq k \text{ and } 2 \leq j \leq n_i. \end{cases}$$

It is straightforward to check that f is a $(k+|V(G)|-2)$-$L'(2,1)$-labeling of G^c. Hence $\lambda'(G^c) \leq k + |V(G)| - 2$; i.e., $p_v(G) \geq \lambda'(G^c) - |V(G)| + 2$. □

Remark. Georges, Mauro, and Whittlesey [8, Thm. 1.1] proved that for any graph G of n vertices the following two statements hold.

(i) $\lambda(G) \leq n-1$ if and only if $p_v(G^c) = 1$.

(ii) Suppose r is an integer greater than 1. $\lambda(G) = n + r - 2$ if and only if $p_v(G^c) = r$.

Note that an $L(2,1)$-labeling is precisely a proper vertex coloring with some extra conditions on all vertex pairs of distance at most two. So, $\lambda(G)$ has a natural relation with the chromatic number $\chi(G)$.

For any fixed positive integer k, the *kth power* of a graph G is the graph G^k whose vertex set $V(G^k) = V(G)$ and edge set $E(G^k) = \{(x,y) : 1 \leq d_G(x,y) \leq k\}$.

LEMMA 2.4. *$\chi(G) - 1 \leq \lambda(G) \leq 2\chi(G^2) - 2$ for any graph G.*

Proof. $\chi(G) - 1 \leq \lambda(G)$ follows from definitions. $\lambda(G) \leq 2\chi(G^2) - 2$ follows from the fact that for any proper vertex coloring f of G^2, $2f-2$ is an $L(2,1)$-labeling of G. □

The *neighborhood* $N(x)$ of a vertex x is the set of all vertices y adjacent to x. The *closed neighborhood* $N[x]$ of x is $\{x\} \cup N(x)$.

LEMMA 2.5 (see [10]). *$\lambda(G) \geq \Delta + 1$ for any graph G of maximum degree Δ. If $\lambda(G) = \Delta + 1$, then $f(v) = 0$ or $\Delta + 1$ for any $\lambda(G)$-$L(2,1)$-labeling f and any vertex v of maximum degree Δ. In this case, $N[x]$ contains at most two vertices of degree Δ for any $x \in V(G)$.*

LEMMA 2.6. *$\lambda'(C_3) = \lambda'(C_4) = 4$ and $\lambda'(C_n) = n - 1$ for $n \geq 5$.*

Proof. The cases of C_3 and C_4 are easy to verify. For $n \geq 5$, $\lambda'(G) \geq n-1$ by definition. Let $v_0, v_1, \ldots, v_{n-1}$ be vertices of C_n such that v_i is adjacent to v_{i+1} for $0 \leq i \leq n-1$, where $v_n \equiv v_0$. Consider the following labeling:

$$f(v_i) = \begin{cases} i/2, & \text{if } 0 \leq i \leq n-1 \text{ and } i \text{ is even;} \\ \lceil n/2 \rceil + \lceil i/2 \rceil - 1, & \text{if } 0 \leq i \leq n-1 \text{ and } i \text{ is odd.} \end{cases}$$

It is straightforward to check that f is an $(n-1)$-$L'(2,1)$-labeling of C_n. So $\lambda'(C_n) \leq n-1$. □

LEMMA 2.7. *$\lambda'(P_1) = 0$, $\lambda'(P_2) = 2$, $\lambda'(P_3) = 3$, and $\lambda'(P_n) = n-1$ for $n \geq 4$.*

Proof. The cases of P_1, P_2, P_3, and P_4 are easy to verify. For $n \geq 5$, $\lambda'(P_n) \geq n-1$ by definition. Last, $\lambda'(P_n) \leq \lambda'(C_n) = n-1$ by Lemmas 2.1 and 2.6. □

3. Union and join of graphs. Suppose G and H are two graphs with disjoint vertex sets. The *union* of G and H, denoted by $G \cup H$, is the graph whose vertex set is $V(G) \cup V(H)$ and edge set is $E(G) \cup E(H)$. The *join* of G and H, denoted by $G + H$, is the graph obtained from $G \cup H$ by adding all edges between vertices in $V(G)$ and vertices in $V(H)$.

LEMMA 3.1. *$\lambda(G \cup H) = \max\{\lambda(G), \lambda(H)\}$ for any two graphs G and H.*

Proof. $\lambda(G \cup H) \geq \max\{\lambda(G), \lambda(H)\}$ follows from Lemma 2.1 and the fact that G and H are subgraphs of $G \cup H$. On the other hand, an $L(2,1)$-labeling of G together with an $L(2,1)$-labeling of H makes an $L(2,1)$-labeling of $G \cup H$. Hence $\lambda(G \cup H) \leq \max\{\lambda(G), \lambda(H)\}$. □

LEMMA 3.2. *$\lambda'(G \cup H) = \max\{\lambda'(G), \lambda'(H), |V(G)| + |V(H)| - 1\}$ for any two graphs G and H.*

Proof. $\lambda'(G \cup H) \geq \max\{\lambda'(G), \lambda'(H)\}$ follows from Lemma 2.1 and the fact that G and H are subgraphs of $G \cup H$. $\lambda'(G \cup H) \geq |V(G)| + |V(H)| - 1$ follows from the definition of λ'.

Assume f is a $\lambda'(G)$-$L'(2,1)$-labeling of G. There are no two consecutive integers $x < y$ in $[0, \lambda'(G)]$ that are not labels of vertices of G; otherwise we can "compact" the function f to get a $(\lambda'(G)-1)$-$L'(2,1)$-labeling f' of G defined by

$$f'(v) = \begin{cases} f(v), & \text{if } f(v) < x; \\ f(v) - 1, & \text{if } f(v) > x. \end{cases}$$

For the case where $\lambda'(G) \geq |V(G)| + |V(H)| - 1$, there are at least $|V(H)|$ pairwise nonconsecutive integers in $[0, \lambda'(G)]$ that are not labels of vertices of G. We can use them to label the vertices of H. This yields a $\lambda'(G)$-$L'(2,1)$-labeling of $G \cup H$. For the case where $\lambda'(H) \geq |V(G)| + |V(H)| - 1$, similarly, there exists a $\lambda'(H)$-$L'(2,1)$-labeling of $G \cup H$. For the case where $\max\{\lambda'(G), \lambda'(H)\} \leq |V(G)| + |V(H)| - 1$, without loss of generality, we may assume that $|V(G)| \geq |V(H)|$. Let f be a k-$L'(2,1)$-labeling of G such that $k \leq |V(G)| + |V(H)| - 1$ and there are no two consecutive integers in $[0, k]$ that are not labels of vertices of G. Such an f exists for $k = \lambda'(G)$. If $k \leq |V(G)| + |V(H)| - 3$, then $k \leq 2|V(G)| - 3$ and so there exist two consecutive labels $x < y$. In this case, we can "separate" f to get a $(k+1)$-$L'(2,1)$-labeling f' defined by

$$f'(v) = \begin{cases} f(v), & \text{if } f(v) \leq x; \\ f(v) + 1, & \text{if } f(v) \geq y. \end{cases}$$

Continuing this process, we obtain a k-$L'(2,1)$-labeling such that $|V(G)|+|V(H)|-2 \leq k \leq |V(G)| + |V(H)| - 1$ and there are no two consecutive integers in $[0, k]$ that are not labels of vertices of G. Using $|V(H)|$ nonlabels in $[0, |V(G)| + |V(H)| - 1]$ to label the vertices in H, we get a $(|V(G)| + |V(H)| - 1)$-$L'(2,1)$-labeling of $G \cup H$. By the conclusions of the above three cases, $\lambda'(G \cup H) \leq \max\{\lambda'(G), \lambda'(H), |V(G)| + |V(H)| - 1\}$. □

LEMMA 3.3. *$p_v(G \cup H) = p_v(G) + p_v(H)$ for any two graphs G and H.*

Proof. The proof is obvious. □

LEMMA 3.4. *$\lambda(G + H) = \lambda'(G + H) = \lambda'(G) + \lambda'(H) + 2$ for any two graphs G and H.*

Proof. $\lambda(G + H) = \lambda'(G + H)$ follows from Lemma 2.2 and the fact that $G + H$ is of diameter at most two. Also,

$\lambda'(G + H)$
$= p_v((G + H)^c) + |V(G + H)| - 2$ (by Lemma 2.3)
$= p_v(G^c \cup H^c) + |V(G)| + |V(H)| - 2$
$= p_v(G^c) + p_v(H^c) + |V(G)| + |V(H)| - 2$ (by Lemma 3.3)
$= \lambda'(G) + \lambda'(H) + 2$ (by Lemma 2.3). □

Remark. Georges, Mauro, and Whittlesey [8, Cor. 4.6] proved that $\lambda(G + H) = \max\{|V(G)| - 1, \lambda(G)\} + \max\{|V(H)| - 1, \lambda(H)\} + 2$.

LEMMA 3.5. *$p_v(G + H) = \max\{p_v(G) - |V(H)|, p_v(H) - |V(G)|, 1\}$ for any two graphs G and H.*

Proof.

$p_v(G + H)$
$= \lambda'((G + H)^c) - |V(G + H)| + 2$ (by Lemma 2.3)
$= \lambda'(G^c \cup H^c) - |V(G)| - |V(H)| + 2$
$= \max\{\lambda'(G^c), \lambda'(H^c), |V(G)| + |V(H)| - 1\} - |V(G)| - |V(H)| + 2$ (by Lemma 3.2)
$= \max\{\lambda'(G^c) - |V(G)| + 2 - |V(H)|, \lambda'(H^c) - |V(H)| + 2 - |V(G)|, 1\}$
$= \max\{p_v(G) - |V(H)|, p_v(H) - |V(G)|, 1\}$ (by Lemma 2.3). □

Cographs are defined recursively by the following rules.

(1) A vertex is a cograph.

(2) If G is a cograph, then so is its complement G^c.

(3) If G and H are cographs, then so is their union $G \cup H$.

Note that the above definition is the same as one with (2) replaced by the following.

(4) If G and H are cographs, then so is their join $G + H$.

There is a linear time algorithm to identify whether a graph is a cograph (see [3]). In the case of a positive answer, the algorithm also gives a *parsing tree.* Therefore, we have the following consequences.

THEOREM 3.6. *There is a linear time algorithm to compute $\lambda(G)$, $\lambda'(G)$, and $p_v(G)$ for a cograph G.*

4. Upper bound of λ in terms of maximum degree.

For any fixed positive integer k, a *k-stable set* of a graph G is a subset S of $V(G)$ such that every two distinct vertices in S are of distance greater than k. Note that 1-stability is the usual stability.

THEOREM 4.1. *$\lambda(G) \leq \Delta^2 + \Delta$ for any graph G with maximum degree Δ.*

Proof. Consider the following labeling scheme on $V(G)$. Initially, all vertices are unlabeled. Let $S_{-1} = \emptyset$. When S_{i-1} is determined and not all vertices in G are labeled, let

$$F_i = \{x \in V(G) : x \text{ is unlabeled and } d(x,y) \geq 2 \text{ for all } y \in S_{i-1}\}.$$

Choose a *maximal* 2-stable subset S_i of F_i; i.e., S_i is a 2-stable subset of F_i but S_i is not a proper subset of any 2-stable subset of F_i. Note that in the case where $F_i = \emptyset$, i.e., for any unlabeled vertex x there exists some vertex $y \in S_{i-1}$ such that $d(x,y) < 2$, $S_i = \emptyset$. In any case, label all vertices in S_i by i. Then increase i by one and continue the above process until all vertices are labeled. Assume k is the maximum label used, and choose a vertex x whose label is k. Let

$I_1 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) = 1 \text{ for some } y \in S_i\}$,

$I_2 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) \leq 2 \text{ for some } y \in S_i\}$,

$I_3 = \{i : 0 \leq i \leq k-1 \text{ and } d(x,y) \geq 3 \text{ for all } y \in S_i\}$.

It is clear that $|I_2|+|I_3| = k$. Since the total number of vertices y with $1 \leq d(x,y) \leq 2$ is at most $\deg(x) + \sum\{\deg(y) - 1 : (y,x) \in E(G)\} \leq \Delta + \Delta(\Delta-1) = \Delta^2$, we have $|I_2| \leq \Delta^2$. Also, there exist only $\deg(x) \leq \Delta$ vertices adjacent to x, so $|I_1| \leq \Delta$. For any $i \in I_3$, $x \notin F_i$; otherwise $S_i \cup \{x\}$ is a 2-stable subset of F_i, which contradicts the choice of S_i. That is, $d(x,y) = 1$ for some vertex y in S_{i-1}; i.e., $i - 1 \in I_1$. So, $|I_3| \leq |I_1|$. Then,

$$\lambda(G) \leq k = |I_2| + |I_3| \leq |I_2| + |I_1| \leq \Delta^2 + \Delta. \qquad \square$$

Jonas [12] proved that $\lambda(G) \leq \Delta^2 + 2\Delta - 4$ if $\Delta(G) \geq 2$. For the case of $\Delta = 3$, this bound improves the bound in Theorem 4.1 from 12 to 11.

5. Subclasses of chordal graphs.

A graph is *chordal* (or *triangulated*) if every cycle of length greater than three has a *chord*, which is an edge joining two nonconsecutive vertices of the cycle. Chordal graphs have been extensively studied as a subclass of perfect graphs (see [9]). For any graph G, $\chi(G)$ denotes the chromatic number of G and $\omega(G)$ the maximum size of a clique in G. It is easy to see that $\omega(G) \leq \chi(G)$ for any graph G. A graph G is *perfect* if $\omega(H) = \chi(H)$ for any vertex-induced subgraph H of G. In conjunction with the domination theory in graphs, the following subclasses of chordal graphs have been studied (see [1, 2, 6]). An *n-sun* is a chordal graph with a Hamiltonian cycle $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_1)$ in which each

x_i is of degree exactly two. An SF-chordal (resp., OSF-chordal, 3SF-chordal) graph is a chordal which contains no n-sun with $n \geq 3$ (resp. odd $n \geq 3$, $n = 3$) as an induced subgraph. SF-chordal graphs are also called *strongly chordal* graphs by Farber (see [6]). Strongly chordal graphs include directed path graphs, interval graphs, unit interval graphs, block graphs, and trees. A vertex x is *simple* if $N[y] \subseteq N[z]$ or $N[z] \subseteq N[y]$ for any two vertices $y, z \in N[x]$. Consequently, for any simple vertex x, $N[x]$ is a clique and x has a *maximum neighbor* $m \in N[x]$; i.e., $N[y] \subseteq N[m]$ for any $y \in N[x]$. Farber [6] proved that G is a strongly chordal graph if and only if every vertex-induced subgraph of G has a simple vertex.

THEOREM 5.1. *$\lambda(G) \leq 2\Delta$ for any OSF-chordal graph G with maximum degree Δ.*

Proof. First, $\lambda(G) \leq 2\chi(G^2) - 2$ by Lemma 2.4. By Corollary 3.11 of [2], G^2 is perfect and so $\chi(G^2) = \omega(G^2)$. Since G is OSF-chordal, it is 3SF-chordal. By Theorem 3.8 of [1], $\omega(G^2) = \Delta + 1$. The above inequality and equalities imply that $\lambda(G) \leq 2\Delta$. □

THEOREM 5.2. *$\lambda(G) \leq \Delta + 2\chi(G) - 2$ for any strongly chordal graph G with maximum degree Δ.*

Proof. We shall prove the theorem by induction on $|V(G)|$. The theorem is obvious when $|V(G)| = 1$. Suppose $|V(G)| > 1$. Choose a simple vertex v of G. Since $G - v$ is also strongly chordal, by the induction hypothesis,

$$\lambda(G - v) \leq \Delta(G - v) + 2\chi(G - v) - 2 \leq \Delta + 2\chi(G) - 2.$$

Let f be a $\lambda(G - v)$-$L(2, 1)$-labeling of $G - v$. Note that v is adjacent to $\deg(v)$ vertices, which form a clique in G. Let m be the maximum neighbor of v. Since every vertex of distance two from v is adjacent to m, there are $\deg(m) - \deg(v)$ vertices that are of distance two from v. Therefore, there are at most $3 \deg(v) + \deg(m) - \deg(v) \leq \Delta + 2\omega(G) - 2 = \Delta + 2\chi(G) - 2$ numbers used by f to be avoided by v. Hence there is still at least one number in $[0, \Delta + 2\chi(G) - 2]$ that can be assigned to v in order to extend f into a $(\Delta + 2\chi(G) - 2)$-$L(2, 1)$-labeling. □

Although a strongly chordal graph is OSF-chordal, the upper bounds in Theorems 5.1 and 5.2 are incomparable. Theorem 5.2 is a generalization of the result that $\lambda(T) \leq \Delta + 2$ for any nontrivial tree of maximum degree Δ. We conjecture that $\lambda(G) \leq \Delta + \chi(G)$ for any strongly chordal graph G with maximum degree Δ.

6. A polynomial algorithm for λ on trees. For a tree T with maximum degree Δ, Griggs and Yeh [10] proved that $\lambda(T) = \Delta + 1$ or $\Delta + 2$. They also conjectured that it is NP-complete to determine if $\lambda(T) = \Delta + 1$. On the contrary, this section gives a polynomial time algorithm to determine if $\lambda(T) = \Delta + 1$. Although not necessary, the following two preprocessing steps reduce the size of a tree before we apply the algorithm.

First, check if there is a vertex x whose closed neighborhood $N[x]$ contains three or more vertices of degree Δ. If the answer is positive, then $\lambda(T) = \Delta + 2$ by Lemma 2.5.

Next, check if there is a leaf x whose unique neighbor y has degree less than Δ. If there is such a vertex x, then $T - x$ also has maximum degree Δ. By Lemma 2.1 and precisely the same arguments as in the proof of Theorem 4.1 of [10], $\lambda(T - x) \leq \lambda(T) \leq \max\{\lambda(T - x), \deg(x) + 2\} \leq \lambda(T - x)$ and so $\lambda(T) = \lambda(T - x)$. Determining $\lambda(T)$ is then the same as determining $\lambda(T - x)$. Continue this process until any leaf of the tree is adjacent to a vertex of degree Δ.

Regardless of whether we apply the above two steps to reduce the tree size or not, from now on we assume that T' is a tree of at least two vertices and whose maximum degree is Δ. For any fixed positive integer k, the following algorithm determines if T' has a k-$L(2,1)$-labeling or not. We in fact only need to apply the algorithm for $k = \Delta + 1$.

For technical reasons, we may assume that T' is rooted at a leaf r', which is adjacent to r. Let $T = T' - r'$ be rooted at r. We can consider T' as the tree resulting from T by adding a new vertex r' that is adjacent to r only. For any vertex v in T, let $T(v)$ be the subtree of T rooted at v and $T'(v')$ be the tree resulting from $T(v)$ by adding a new vertex v' that is adjacent to v only. $T'(v')$ is considered to be rooted at the leaf v'. Note that $T(r) = T$ and $T'(r') = T'$. Denote

$$S(T(v)) = \{(a,b) : \text{ there is a } k\text{-}L(2,1)\text{-labeling } f \text{ on } T'(v') \text{ with } f(v') = a \text{ and } f(v) = b\}.$$

Note that $\lambda(T) \le k$ if and only if $S(T(r)) \ne \emptyset$. Now suppose $T(v) - v$ contains s trees $T(v_1)$, $T(v_2), \ldots, T(v_s)$ rooted at v_1, $v_2, \ldots, v_s$, respectively, where each v_i is adjacent to v in $T(v)$. Note that $T(v)$ can be considered as identifying v_1', $v_2', \ldots, v_s'$ to a vertex v on the disjoint union of $T'(v_1')$, $T'(v_2'), \ldots, T'(v_s')$.

For a system of sets $(A_i)_{i=1}^s \equiv (A_1, A_2, \ldots, A_s)$, a system of distinct representatives (SDR) is an s-tuple $(a_i)_{i=1}^s \equiv (a_1, a_2, \ldots, a_s)$ of s distinct elements such that $a_i \in A_i$ for $1 \le i \le s$.

THEOREM 6.1. *$S(T(v)) = \{(a,b) : 0 \le a \le k,\ 0 \le b \le k,\ |a-b| \ge 2$, and $(A_i)_{i=1}^s$ has an SDR, where $A_i = \{c : c \ne a$ and $(b,c) \in S(T(v_i))\}\}$.*

Proof. Denote by S the set on the right-hand side of the equality in the theorem.

Suppose $(a,b) \in S(T(v))$. There is a k-$L(2,1)$-labeling f of $T'(v')$ such that $f(v') = a$ and $f(v) = b$. Of course, $0 \le a \le k$, $0 \le b \le k$, and $|a-b| \ge 2$. Let f_i be the function f restricted on $T'(v_i')$ by viewing v_i' the same as v. Then f_i is a k-$L(2,1)$-labeling of $T'(v_i')$ with $f_i(v_i') = f(v) = b$ and $f_i(v_i) = f(v_i) \ne f(v') = a$, i.e., $(b, f(v_i)) \in S(T(v_i))$ and $f(v_i) \in A_i$. Thus $(f(v_i))_{i=1}^s$ is an SDR of $(A_i)_{i=1}^s$. This proves $S(T(v)) \subseteq S$.

On the other hand, suppose $(a,b) \in S$. Then $0 \le a \le k$, $0 \le b \le k$, $|a-b| \ge 2$, and $(A_i)_{i=1}^s$ has an SDR $(c_i)_{i=1}^s$. Let f_i be a k-$L(2,1)$-labeling of $T'(v_i')$ such that $f_i(v_i') = b$ and $f(v_i) = c_i$. Consider the labeling f of T' defined by $f(x) = f_i(x)$ for $x \in V(T(v_i))$ and $f(v') = a$. It is straightforward to confirm that f is a k-$L(2,1)$-labeling of $T'(v')$ with $f(v') = a$ and $f(v) = b$; i.e., $(a,b) \in S(T(v))$. □

Our algorithm for determining if a tree has a k-$L(2,1)$-labeling recursively applies the above theorem with the initial condition that for any leaf v of T,

$$S(T(v)) = \{(a,b) : 0 \le a \le k, 0 \le b \le k, |a-b| \ge 2\}.$$

To decide if the tree T' has a k-$L(2,1)$-labeling, we calculate $S(T(v))$ for all vertices v of the tree T. The algorithm starts from the leaves and works toward r. For any vertex v, whose children are $v_1, v_2, \ldots, v_s$, we use $S(T(v_1)), \ldots, S(T(v_s))$ to calculate $S(T(v))$ by Theorem 6.1. More precisely, for any (a,b) with $0 \le a \le k$, $0 \le b \le k$, $|a-b| \ge 2$, we check if $(a,b) \in S(T(v))$ by the following method. Construct a bipartite graph $G = (X, Y, E)$ with

$$X = \{x_1, x_2, \ldots, x_s\}, \qquad Y = \{0, 1, \ldots, k\},$$

and

$$E = \{(x_i, c) : c \ne a \text{ and } (b,c) \in S(T(v_i))\}.$$

We can use any well-known algorithm to find a maximum matching of the bipartite graph G. Then $(a, b) \in S(T(v))$ if and only if G has a matching of size s. Note that for any vertex v we need to solve the bipartite matching problem $O(k^2)$ times. Therefore, the complexity of the above algorithm is $O(|V(T)|k^2 g(2k))$, where $g(n)$ is the complexity of solving the bipartite matching problem of n vertices. The well-known flow algorithm gives $g(n) = O(n^{2.5})$.

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