Necessary and Sufficient Conditions for Rearrangeable $\mathrm{Log}_d(N, m, p)$

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***Abstract*—We extend previous results on sufficient conditions for rearrangeable $\mathbf{Log_2(N, m, p)}$ networks to $\mathbf{Log_d(N, m, p)}$ networks. We show that the original argument using path-intersecting graphs cannot be extended to $d > 2$, and we give a new argument. Further, we show our sufficient conditions are also necessary.**

***Index Terms*—Banyan networks, $\mathbf{Log(N, k, p)}$ networks, rearrangeable networks.**

I. Introduction

Lea [2] first proposed the $\mathrm{Log}_2(N, 0, p)$ (or a multi-$\mathrm{Log}_2 N$) network, which is obtained by vertically stacking up p copies of a binary inverse banyan network with N inputs and outputs (or a $\mathrm{Log}_2 N$ network) and identifying their inputs and outputs. One way of doing it is to create a new stage of N inputs and a new stage of N outputs, and to connect input (output) i of the new stage to input (output) i of each copy of the $\mathrm{Log}_2 N$ network (see Fig. 1).

Shyy and Lea [5] extended $\mathrm{Log}_2(N, 0, p)$ to $\mathrm{Log}_2(N, m, p)$ by replacing the inverse banyan-type network with an extra-stage inverse banyan-type network. Set $n = \log_2 N$. They used the extra-stage inverse banyan network $BY_2^{-1}(n, m)$, which has $n+m$ stages, with the m extra stages constituting a mirror image of the first m stages (see Fig. 2). $\mathrm{Log}_2(N, m, p)$ can also be extended to $\mathrm{Log}_d(N, m, p)$ by replacing inverse banyan networks with d-ary inverse banyan networks (i.e., $BY_d^{-1}(n, m)$).

Sufficient conditions for such networks to be either rearrangeable or strictly nonblocking have been well studied. On the strictly nonblocking side, Lea [2] gave sufficient conditions for $\mathrm{Log}_2(N, 0, p)$, Shyy and Lea [5] extended to $\mathrm{Log}_2(N, m, p)$, and Hwang [1] to $\mathrm{Log}_d(N, m, p)$. On the rearrangeable side, Lea [2] gave sufficient conditions for $\mathrm{Log}_2(N, 0, p)$, and Lea and Shyy [3] extended it to $\mathrm{Log}_2(N, m, p)$. However, an extension to the d-ary version has been missing, since the argument used in the binary case cannot be extended to $d > 2$.

It is rare that these sufficient conditions were proven to be also necessary. Lea [2] gave necessary conditions for the rearrangeable $\mathrm{Log}_2(N, 0, p)$ and Shyy and Lea [5] claimed the necessary conditions for the strictly nonblocking $\mathrm{Log}_2(N, m, p)$ to be obvious. In this letter, we give a new argument to obtain a sufficient condition for the rearrangeable $\mathrm{Log}_d(N, m, p)$, and also

Paper approved by T. T. Lee, the Editor for Switching Systems and Network Performance of the IEEE Communications Society. Manuscript received July 11, 2003; revised January 18, 2005. This work was supported in part by the Republic of China National Science Council under Grant NSC-91-2115-M-009-010 and in part by the Lee and MTI Center for Networking Research at National Chiao Tung University, Hsinchu, Taiwan, R.O.C.

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Digital Object Identifier 10.1109/TCOMM.2005.860073

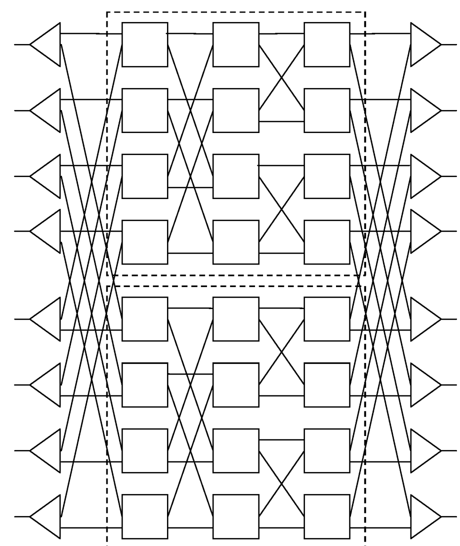

Fig. 1. $\mathrm{Log}_2(8, 0, 2)$ network.

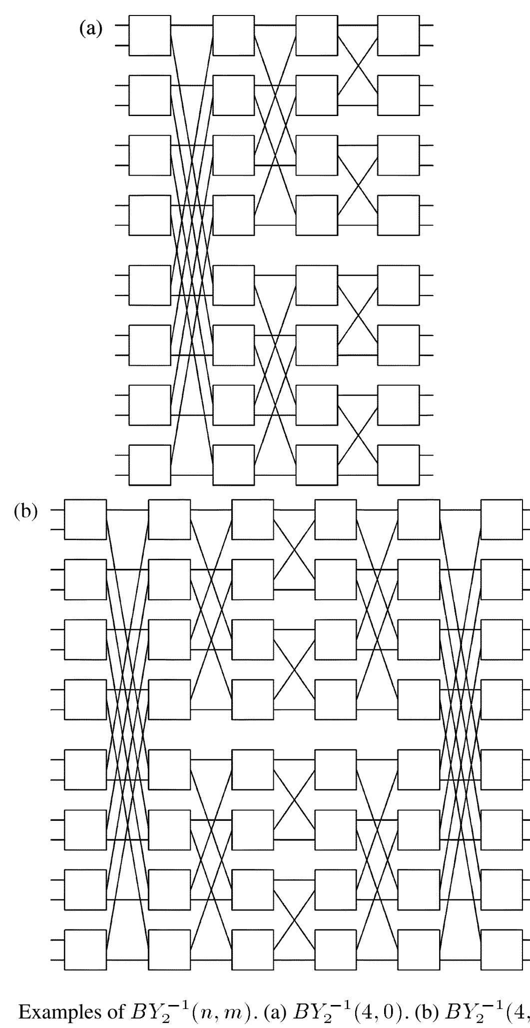

Fig. 2. Examples of $BY_2^{-1}(n, m)$. (a) $BY_2^{-1}(4, 0)$. (b) $BY_2^{-1}(4, 2)$.

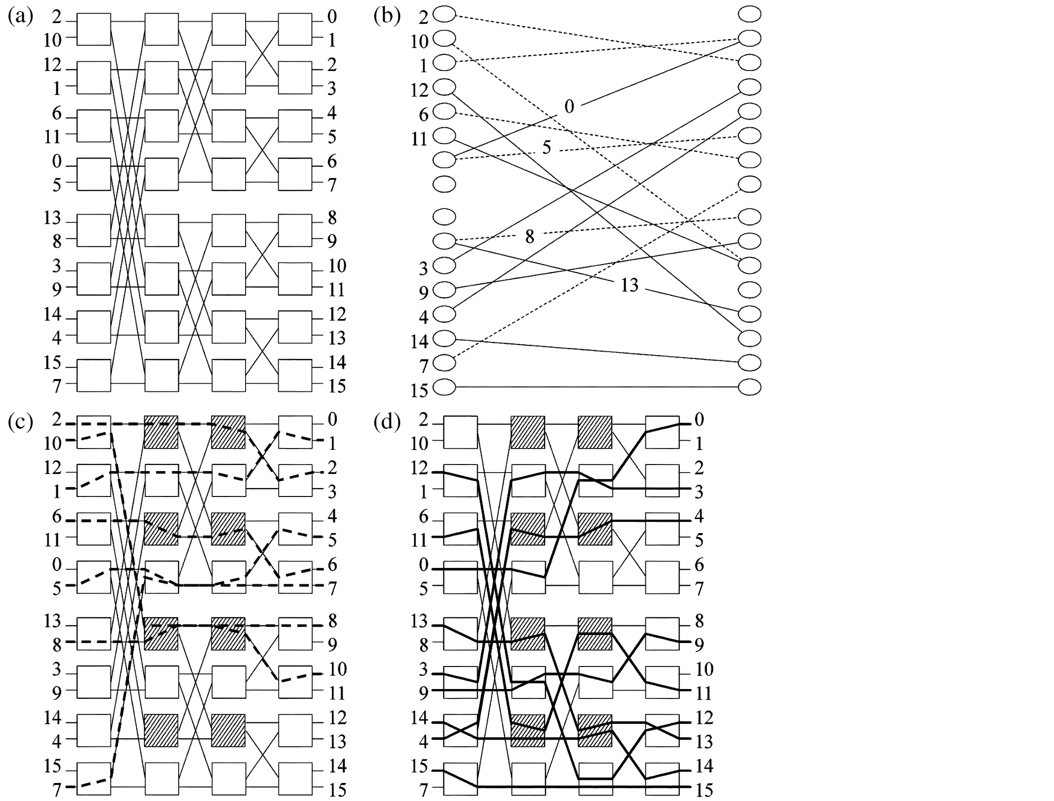

Fig. 3. (a) By mapping all copies of $BY_2^{-1}(4,0)$ to a single copy, we also map all requests of $\mathrm{Log}_2(16,0,4)$ to this copy. The first column indicates the destinations of the inputs. (b) By treating stage-1 (stage-4) outputs (inputs) as nodes, requests as edges, the resulting bipartite graph is two-edge-colorable. Edges (requests) are partitioned into two parts (dash lines and regular lines) according to the colors. (c), (d) The corresponding paths in $BY_2^{-1}(4,0)$ of the two parts.

prove it to be necessary, thus covering the necessary and sufficient conditions for the two special cases $\mathrm{Log}_d(N,0,p)$ and $\mathrm{Log}_2(N,m,p)$.

II. Main Results

We first give a general sketch of Lea's proof for the rearrangeable $\mathrm{Log}_2(N,0,p)$ to see why it can not be extended to the $d>2$ version.

Since $BY_2^{-1}(n,0)$ is a unique-path network, i.e., there is exactly one path for a given request, the given set of requests completely determines the set P of paths for all requests. Let link-stage i denote the set of links from stage i to stage $i+1$. Construct a path-intersection graph G_1 by taking the paths in P as vertices, and inserting an edge (u,v) if the two corresponding paths intersect at either link-stage 1 or link-stage $n-1$. Then G_1 has maximum degree two, since a 2×2 crossbar allows at most one path to intersect a given path at a given link-stage. Further, each vertex of degree two has one intersection at link-stage 1 and the other at link-stage $n-1$. Thus, G_1 is two-(vertex-)colorable.

Note that $BY_2^{-1}(n,0)$ is reduced to 2^2 copies of $BY_2^{-1}(n-2,0)$ if link-stages 1 and $n-1$ are stripped off. For each $BY_2^{-1}(n-2,0)$, we can construct a path-intersection graph G_2 by looking at the intersections at link-stages 2 and $n-2$. G_2 is two-colorable by an analogous reason as G_1. Similarly, we can define and two-color $G_3,\ldots,G_{\lfloor n/2\rfloor}$. Note that for even n, $G_{\lfloor n/2\rfloor}$ looks at intersections at the single link-stage $\lfloor n/2\rfloor$, and is also two-colorable. Therefore, all paths can be colored into $2^{\lfloor n/2\rfloor}$ colors, such that paths in the same color do not intersect each other.

By routing requests of the same color through one copy of $BY_2^{-1}(n,0)$, $2^{\lfloor n/2\rfloor}$ copies suffice to route all requests.

In the d-ary version, $d\geq 2$, we can still construct G_i. But the maximum degree in a G_i is $2(d-1)\geq d$, and equality assumes only when $d=2$. Therefore, a similar argument would not yield the result that $d^{\lfloor n/2\rfloor}$ copies suffice.

We now give a different argument to prove.

Theorem 1: $\mathrm{Log}_d(d^n,0,p)$ is rearrangeable if and only if $p\geq d^{\lfloor n/2\rfloor}$.

Proof: Sufficiency. Theorem 1 is obviously true for $n=1$ (the network is just a $d\times d$ crossbar) and $n=2$ (has a single link-stage where a link can be claimed by at most d requests). We prove the general $n\geq 3$ case by induction.

Our strategy is to reduce a $\mathrm{Log}_d(d^n,0,d^{\lfloor n/2\rfloor})$ to several $\mathrm{Log}_d(d^{n-2},0,d^{\lfloor (n/2)/2\rfloor})$ (with $\lfloor\,\rfloor$ as floor notation). For convenience, we map all copies of $BY_d^{-1}(n,0)$ (of $\mathrm{Log}_d(d^{\lfloor n/2\rfloor},0,p_2)$, with $\lfloor\,\rfloor$ as floor notation) to a single $BY_d^{-1}(n,0)$. In this $BY_d^{-1}(n,0)$, two requests (i.e., two paths, since $BY_d^{-1}(n,0)$ is a unique-path network) intersecting in some links means they must go through different copies of $BY_d^{-1}(n,0)$ (of $\mathrm{Log}_d(d^n,0,d^{\lfloor n/2\rfloor})$). Note that each stage-2 input (stage-$(n-1)$ output) is involved in at most d requests, because only d stage-1 inputs (stage-n outputs) could generate requests going through it.

Construct a bipartite graph by treating stage-1 outputs and stage-n inputs as the two parts, and add an edge (x,y) if there is a request going through both x and y. This bipartite graph is

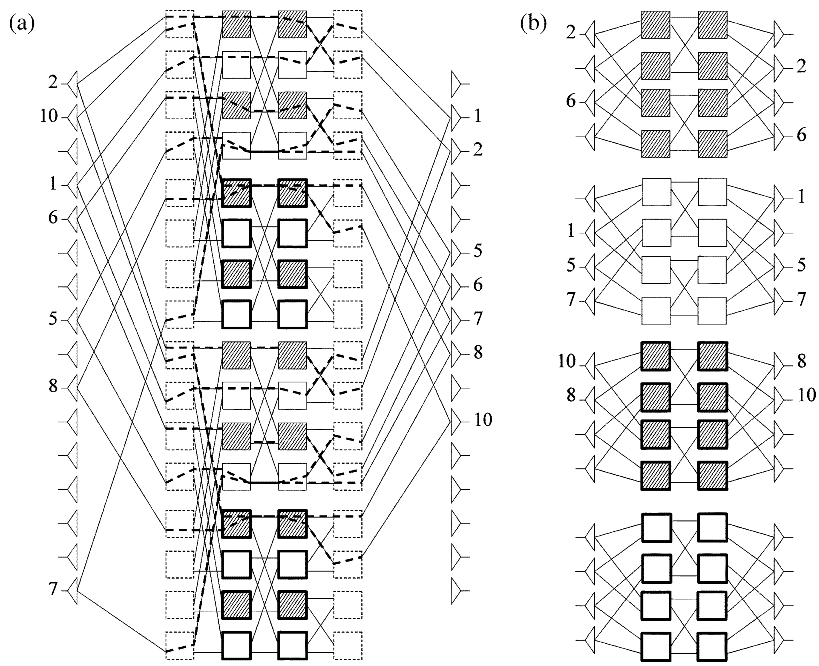

Fig. 4. (a) We present the routing of dash-line requests [Fig. 3(c)] in the two copies of $BY_2^{-1}(4,0)$ in $\mathrm{Log}_2(16,0,4)$ (two other copies omitted). The four copies of $BY_2^{-1}(2,0)$ in a $BY_2^{-1}(4,0)$ are drawn in different settings. (b) Each $BY_2^{-1}(2,0)$ and their related requests form $\mathrm{Log}_2(4,0,2)$ with a subset of original requests.

d-edge-colorable since its degree is bounded by d. Partition the requests into d parts according to the colors. Assign $d^{\lfloor (n-2)/2\rfloor}$ copies of $BY_d^{-1}(n,0)$ (of $\mathrm{Log}_d(d^n,0,d^{\lfloor n-2\rfloor}$, with $\lfloor\,\rfloor$ as floor notation) to route the requests of each color. Since the requests do not intersect in link-stage 1 and link-stage $n-1$, we could replace the first and last stages of crossbars by their assigned routes (i.e., the unique paths on the crossbars). The resulting network is a composition of several $\mathrm{Log}_d(d^{n-2},0,d^{\lfloor (n-2)/2\rfloor})$, each with a subset of original requests (see Fig. 4 for an example). By induction, each $\mathrm{Log}_d(d^{n-2},0,d^{\lfloor (n-2)/2\rfloor}))$ is rearrangeable for their requests; hence, all requests can be routed by $\mathrm{Log}_d(d^n,0,d^{\lfloor n/2\rfloor})$.

Necessity. Fix a link-stage $\lfloor n/2\rfloor$ (with $\lfloor\,\rfloor$ as floor notation) link x in $BY_d^{-1}(n,0)$. Let $X(Y)$ denote the set of inputs (outputs) of $BY_d^{-1}(n,0)$ which can access x. It is easy to verify that $|X| = d^{\lfloor n/2\rfloor}$ and $|Y| = d^{\lceil n/2\rceil}$. Since $BY_d^{-1}(n,0)$ is a unique-path network, a path starting from X to Y must go through x. By assigning $d^{\lfloor n/2\rfloor}$ requests from X to Y, we conclude that these requests are routable only if they can be spread to at least $d^{\lfloor n/2\rfloor}$ copies of $BY_d^{-1}(n,0)$. Q.E.D.

For easier understanding, we illustrate the concept of *Theorem 1* in Figs. 3 and 4 by using $\mathrm{Log}_2(16,0,4)$ as an example. The edge-coloring technique is also used differently in other kinds of rearrangeable networks (see [4] for examples), but we found that it is rare to treat links but not crossbars as vertices.

Theorem 2: $\mathrm{Log}_d(d^n,m,p)$ is rearrangeable (RNB) if and only if $p \geq d^{\lfloor (n-m)/2\rfloor}$.

Proof: Sufficiency. *Theorem 2* holds for $m=0$ (*Theorem 1*). We prove the general $m \geq 1$ case by induction.

We map all copies of $BY_d^{-1}(n,m)$ (of $\mathrm{Log}_d(d^n,m,p)$) to a single $BY_d^{-1}(n,m)$, called V, and remove its first and last stages. Then the network is reduced to d copies of $BY_d^{-1}(n-1,m-1)$, where each stage-1 and stage-n crossbar has a link to each $BY_d^{-1}(n-1,m-1)$.

Consider V and construct a bipartite graph G by taking the stage-1 crossbars and the stage-n crossbars as the two parts of vertices, and a request from vertex u to vertex ν as an edge. Then the maximum degree of G is bounded by d, and G can be d-edge-colored. Associate a color with each of the d copies of $BY_d^{-1}(n-1,m-1)$ mentioned in the last paragraph, and route the request through these $BY_d^{-1}(n-1,m-1)$ according to their colors. For a $BY_d^{-1}(n-1,m-1)$ of a given color, by induction, its requests can be routed if there are $d^{\lfloor (n-1)-(m-1)\rfloor}$ copies of it. Since x copies of $BY_d^{-1}(n,m)$ implies x copies of $BY_d^{-1}(n-1,m-1)$ for each color, $d^{\lfloor (n-1)-(m-1)\rfloor} = d^{\lfloor n-m\rfloor}$ copies of $BY_d^{-1}(n,m)$ suffice to route all requests in $BY_d^{-1}(n,m)$.

Necessity. By removing the first m stages and the last m stages of $BY_d^{-1}(n,m)$, the network is reduced to d^m copies of $BY_d^{-1}(n-m,0)$, denoted by V_i for $0 \leq i < d^m$, whose inputs and outputs will be referred as subinputs and suboutputs. Fix a link x of the link-stage $\lfloor (n-m)/2\rfloor$ in a $BY_d^{-1}(n-m,0)$. Let x_i be the link corresponding to x in V_i. Let X_i and Y_i (in V_i) assume the roles of X and Y in the necessity proof of *Theorem 1*. Then $|X_i| = d^{\lfloor (n-m)/2\rfloor}$ and $|Y_i| = d^{\lceil (n-m)/2\rceil}$. Define $f = d^{\lfloor (n-m)/2\rfloor}$. Let $X_i = (X_{i1},\ldots,X_{if})$ for $i=1,\ldots,d^m$. It is easily verified that X_{ij} has access to the same set I_j of d^m inputs for all i from 1 to d^m. Since an input has access to d^m subinputs, while X_{ij} already has d^m subinputs, it cannot have access to any other subinputs. This holds for all j from 1 to f. Similarly, we define $g = d^{\lceil (n-m)/2\rceil}$ and subsets O_j of outputs as the counterpart of I_j.

Suppose all inputs in $\bigcup_{j=1}^{f} I_j$ generate requests to $\bigcup_{j=1}^{g} O_j$. Then there are $f \times d^m = d^{\lfloor (n+m)/2\rfloor}$ requests going through X_i (to Y_i), $i=1,\ldots,d^m$. By *Theorem 1*, each request goes through x_i for some i. Since there are d^m x_i, there exists an x_i with at least $d^{\lfloor (n+m)/2 \rfloor}/d^m = f$ requests going through it. Thus, at least $f = d^{\lfloor (n-m)/2 \rfloor}$ copies of $BY_d^{-1}(n, m)$ are needed. Q.E.D.

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