Journal of Applied Statistics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/cjas20

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W. L. Pearn [a], M. H. Shu [b] & B. M. Hsu [c]

[a] Department of Industrial Engineering & Management, National Chiao Tung University

[b] Department of Industrial Engineering & Management, National Kaohsiung University of Applied Sciences

[c] Department of Industrial Engineering & Management, Cheng Shiu University, Taiwan

Published online: 20 Aug 2006.

To cite this article: W. L. Pearn, M. H. Shu & B. M. Hsu (2005) Testing process capability based on C_{pm} in the presence of random measurement errors, Journal of Applied Statistics, 32:10, 1003-1024, DOI: 10.1080/02664760500164951

To link to this article: http://dx.doi.org/10.1080/02664760500164951

Journal of Applied Statistics
Vol. 32, No. 10, 1003–1024, December 2005

Testing Process Capability Based on C_{pm} in the Presence of Random Measurement Errors

W. L. PEARN*, M. H. SHU** & B. M. HSU†

**Department of Industrial Engineering & Management, National Chiao Tung University, **Department of Industrial Engineering & Management, National Kaohsiung University of Applied Sciences, †Department of Industrial Engineering & Management, Cheng Shiu University, Taiwan*

ABSTRACT *Process capability indices have been widely used in the manufacturing industry providing numerical measures on process performance. The index C_p provides measures on process precision (or product consistency). The index C_{pm}, sometimes called the Taguchi index, meditates on process centring ability and process loss. Most research work related to C_p and C_{pm} assumes no gauge measurement errors. This assumption insufficiently reflects real situations even with highly advanced measuring instruments. Conclusions drawn from process capability analysis are therefore unreliable and misleading. In this paper, we conduct sensitivity investigation on process capability C_p and C_{pm} in the presence of gauge measurement errors. Due to the randomness of variations in the data, we consider capability testing for C_p and C_{pm} to obtain lower confidence bounds and critical values for true process capability when gauge measurement errors are unavoidable. The results show that the estimator with sample data contaminated by the measurement errors severely underestimates the true capability, resulting in imperceptible smaller test power. To obtain the true process capability, adjusted confidence bounds and critical values are presented to practitioners for their factory applications.*

KEY WORDS: Gauge measurement error, lower confidence bound, critical value, process capability analysis.

Introduction

Process capability indices, which establish the relationships between the actual process performance and the manufacturing specifications, have been the focus of recent research in quality assurance and process capability analysis. The first process capability index C_p, which was introduced outside of Japan by Juran *et al.* (1974) has been defined as

$$C_p = \frac{USL - LSL}{6\sigma} \tag{1}$$

Correspondence Address: M. H. Shu, Department of Industrial Engineering and Management, National Kaohsiung University of Applied Sciences, 415 Chien Kung Road, Seng-Min District, Kaohsiung 80778, Taiwan. Email: workman@cc.kuas.edu.tw

0266-4763 Print/1360-0532 Online/05/101003–22 © 2005 Taylor & Francis
DOI: 10.1080/02664760500164951

Table 1. Minimum proportion *NC* associated with various values of C_p

Amount of process data within specification range	C_p	Minimum % *NC*
6σ	1.00	0.27×10^{-2}
8σ	1.33	0.6334×10^{-4}
10σ	1.67	0.5733×10^{-6}
12σ	2.00	0.1973×10^{-8}

where *USL* is the upper specification limit, *LSL* is the lower specification limit, and σ is the process standard deviation. The numerator of C_p gives the size of the range over which the process measurements can vary, and the denominator gives the size of the range over which the process is actually varying. Obviously, it is desirable to have a C_p as large as possible. Small values of C_p would not be acceptable, since this indicates that the natural range of variation of the process does not fit within the tolerance band. Under the assumption of that process data are normal, independent, and in control, Kocherlakota (1992) developed a general guideline for the percentage *NC* (non-conforming units) associated with C_p, assuming that the process is perfectly centred at the midpoint of the specification range (see Table 1). Mizuno (1988) presented detailed criteria for C_p, which had been widely used in US industries. Clearly, the index C_p only measures process potential to reproduce acceptable product and does not take into account whether the process is centred.

The index C_{pm}, sometimes called the Taguchi index, adequately reveals the ability of the process to cluster around the target, which reflects the degrees of process targeting (centring). The index C_{pm} incorporates with the variation of production items with respect to the target value and the specification limits preset in the factory (see Hsiang & Taguchi, 1985; Chan *et al*., 1988; Kotz & Johnson, 1993; Kotz & Lovelace, 1998). The index C_{pm} is defined in the following:

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} \tag{2}$$

where μ denotes the process mean and T refers to the target value often set to the midpoint of the specification limits ($T = m = (USL + LSL)/2$). The capability index C_{pm} is not primarily designed to provide an exact measure on the number of conforming items, but considers the process departure $(\mu - T)^2$ (rather than 6σ alone) in the denominator of the definition to reflect process targeting (Hsiang & Taguchi, 1985; Chan *et al*., 1988). We note that $\sigma^2 + (\mu - T)^2 = E(X - T)^2$ which is the major part of the denominator of C_{pm}. Since $E(X - T)^2$ is the expected loss of the characteristic, X (missing the target) is assumed to be based on the well approximated symmetric squared error loss function, $loss(X) = k(X - T)^2$, the capability index C_{pm} has been referred to as a loss-based index.

Process Capability with Gauge Measurement Errors

Most research works related to C_p and C_{pm} have assumed no gauge measurement errors. For examples, Kane (1986), Kocherlakota (1992), Mizuno (1988), Marcucci & Beazley (1988), Boyles (1991), Pearn *et al.* (1992), Zimmer & Hubele (1997), Zimmer *et al.* (2001), and Pearn & Shu (2003). Such assumption, however, does not accommodate

Table 2. Guidelines for gauge capabilities

Gauge capability	Result
$\lambda < 10\%$	Gauge system O.K.
$10\% < \lambda < 30\%$	May be acceptable based on importance of application, cost of gauge, cost of repair, and so on.
$30\% < \lambda$	Gauge system needs improvement; make every effort to identify the problems and have them corrected.

closely real situations even with highly advanced measuring instruments. Any measurement error has some impacts on the determination of process capability. Montgomery & Runger (1993, 1993b) noted that quality of the collected data relies very much on the gauge accuracy. Clearly, conclusions about process capability based on the empirical index values are not reliable. To analyse the effects of measurement errors on true capability measure, Mittag (1994, 1997) and Levinson (1995) quantified the percentage error on process capability indices evaluation with the presence of measurement errors.

Suppose that the measurement errors can be described as a random variable $M \sim N(0, \sigma_M^2)$, Montgomery & Runger (1993) expressed the gauge capability as

$$\lambda = \frac{6\sigma_M}{USL - LSL} \times 100\% \tag{3}$$

For the measurement system to be deemed acceptable, the measurement variability due to the measurement system must be less than a predetermined percentage of the engineering tolerance. The automotive industry action group recommended the following guidelines (Table 2) for gauge acceptance.

In this paper, we consider sensitivity of the indices C_p and C_{pm} with gauge measurement errors. Because of the random variations in the data, we present some statistical analysis to obtain reliable lower confidence bounds and critical values for capability estimation and testing purposes.

Testing C_p with Gauge Measurement Errors

Considering the process capability in the measurement error system, we denote $X \sim N(\mu, \sigma^2)$ the relevant quality characteristic of a manufacturing process. Because of measurement errors, the observed variable $G \sim N(\mu_G = \mu, \sigma_G^2 = \sigma^2 + \sigma_M^2)$ is measured with X and M stochastically independent, instead of measuring the true variable X. The empirical process capability index C_P^G is obtained after substituting σ_G for σ, and we have the relationship between the true process capability C_P and the empirical process capability C_P^G stated below.

$$C_p^G = \frac{C_p}{\sqrt{1 + \lambda^2 C_p^2}} \tag{4}$$

Since the variation of data observed is larger than the variation of the original data, the true process capability will be under-estimated. Table 3 lists some process capabilities with $\lambda =$ 0.05(0.05)0.50 for various true process capability indices $C_p = 0.50$, 1.00, 1.33, 1.50, 1.67, 2.00, and 2.50. Obviously, the gauge becomes more important as the true capability improves (Levinson, 1995).

Table 3. Process capability with $\lambda = 0.05(0.05)0.50$ for various C_p

	λ									
C_p	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.50	0.50	0.50	0.50	0.50	0.50	0.49	0.49	0.49	0.49	0.49
1.00	1.00	1.00	0.99	0.98	0.97	0.96	0.94	0.93	0.91	0.89
1.33	1.33	1.32	1.30	1.29	1.26	1.24	1.21	1.17	1.14	1.11
1.50	1.50	1.48	1.46	1.44	1.40	1.37	1.33	1.29	1.24	1.20
1.67	1.66	1.65	1.62	1.58	1.54	1.49	1.44	1.39	1.34	1.28
2.00	1.99	1.96	1.92	1.86	1.79	1.71	1.64	1.56	1.49	1.41
2.50	2.48	2.43	2.34	2.24	2.12	2.00	1.88	1.77	1.66	1.56

Sampling Distribution of $\hat{C}_p^G$

Suppose that $\{X_i, i = 1, \ldots, n\}$ denotes the random sample of size n from the quality characteristics X. To estimate the precision index C_p, we consider the natural estimator $\hat{C}_P$ defined below, where $S = [\sum_{i=1}^{n} (X_i - \bar{X})/(n-1)]^{1/2}$ is the conventional estimator of σ, which may be obtained from a stable process,

$$\hat{C}_p = \frac{USL - LSL}{6S} \tag{5}$$

Chou & Owen (1989) have shown the probability density function (PDF) of $\hat{C}_P$ can be expressed as:

$$f_{\hat{C}_p}(x) = 2\frac{(\sqrt{(n-1)/2}C_p)^{n-1}}{\Gamma[(n-1)/2]}(x)^{-n}\exp[-(n-1)C_p^2(2x^2)^{-1}] \tag{6}$$

By adding the well-known correction factor

$$\Delta_{n-1} = \Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n-2}{2}\right)^{-1}\sqrt{\frac{2}{n-1}} \tag{7}$$

to $\hat{C}_P$, such as $\tilde{C}_P = \Delta_{n-1}\hat{C}_P$, Pearn *et al.* (1998) showed that $\tilde{C}_P$ is the uniformly minimum variance unbiased estimator (UMVUE) of C_p. In real applications, the sample observations are not $\{X_i, i = 1, \ldots, n\}$ but $\{G_i, i = 1, \ldots, n\}$. The estimator of C_p becomes

$$\tilde{C}_p^G = \Delta_{n-1}\left(\frac{USL - LSL}{6S_G}\right) \tag{8}$$

where $S_G = [\sum_{i=1}^{n} (G_i - \bar{G})/(n-1)]^{1/2}$. Based on the same arguments used in Chou & Owen (1989) and Pearn *et al*. (1998), the PDF of $\tilde{C}_P^G$ can be expressed as below

$$f_{\hat{C}_p^G}(x) = 2\frac{\left(\sqrt{(n-1)/2}C_P/\sqrt{1+\lambda^2 C_p^2}\right)^{n-1}}{\Gamma[(n-1)/2]}(x)^{-n}\exp\left[\frac{-(n-1)C_p^2(2x^2)^{-1}}{1+\lambda^2 C_p^2}\right] \tag{9}$$

Note that it can be shown that $Var(\tilde{C}_P^G) < Var(\tilde{C}_P)$.

Lower Confidence Bound Based on $\hat{C}_p^G$

The $100\gamma\%$ lower confidence bounds of C_p, L_p, can be established as

$$P(L_p \le C_p) = p\left(K \ge L_p^2\left[\frac{\Delta_{n-1}\sqrt{n-1}}{\tilde{C}_p}\right]^2\right) = \gamma$$

where the statistic $K = (n-1)S^2/\sigma^2$ is distributed as χ^2_{n-1}, a chi-square with $n-1$ degrees of freedom. Thus, the lower confidence bounds L_p can be obtained as:

$$L_p = \frac{\tilde{C}_p}{\Delta_{n-1}}\sqrt{\frac{\chi^2_{n-1,\gamma}}{n-1}} \tag{10}$$

where $\chi^2_{n-1,\gamma}$ is the upper $100\gamma th$ percentile of the χ^2_{n-1} distribution. However, while gauge measurement errors are unavoidable, $\tilde{C}_P^G$ taken as an estimator of C_p, the lower confidence bounds with measurement errors, L_p^{G}, are

$$L_p^G = \frac{\tilde{C}_p^G}{\Delta_{n-1}}\sqrt{\frac{\chi^2_{n-1,\gamma}}{n-1}} \tag{11}$$

and the confidence coefficient γ_G (the probability that the confidence interval contains the actual C_p value with gauge measurement errors) is

$$\gamma_G = p\left(\frac{\tilde{C}_p^G}{\Delta_{n-1}}\sqrt{\frac{\chi^2_{n-1,\gamma}}{n-1}} \le C_p\right) = p\left(K \ge \frac{1}{1+\lambda^2 C_p^2}\chi^2_{n-1,\,\gamma}\right)$$

Because of the measurement errors, the confidence coefficients become small. For instance, when $C_p = 2.00$, $n = 100$, and $\lambda = 0.50$, the confidence coefficient is 0.26%, which is much smaller than the stated confidence coefficient 95%.

Testing Process Capability Based on $\hat{C}_p^G$

To determine whether a given process meets the present capability requirement and runs under the desired quality condition. We can consider the following statistical testing hypothesis, H_0:$C_p \le c$ versus H_1:$C_p > c$. Process fails to meet the capability requirement if $C_p \le c$, and meets the capability requirement if $C_p > c$. The critical value c_0 can be determined by the following with α-risk $\alpha(c_0) = \alpha$ (the chance of incorrectly judging an incapable process as capable), $P(\tilde{C}_p \ge c_0 | C_p = c) = \alpha$, and c_0 can be obtained as:

$$c_0 = c\Delta_{n-1}\sqrt{\frac{n-1}{\chi^2_{n-1,\gamma}}} \tag{12}$$

Meanwhile, the power of the test (the chance of correctly judging a capable process as capable) can be computed as

$$\pi(C_p) = p(\tilde{C}_p > c_0 | C_p) = p\left(K < \frac{C_p^2 \Delta_{n-1}^2 (n-1)}{c_0^2}\right)$$

In the presence of measurement errors, however, the α -risk (denoted by α_G) and the power of the test (denoted by π_G) are as follows:

$$\alpha_G = p(\tilde{C}_p^G \geq c_0 | C_p = c) = p\left(K_G \leq \frac{\chi_{n-1,\gamma}^2}{1 + \lambda^2 C_p^2}\right), \text{ and}$$

$$\pi_G(C_p) = p(\tilde{C}_p^G > c_0 | C_p) = p\left(K_G < \frac{C_p^2 \chi_{n-1,\gamma}^2}{(1 + \lambda^2 C_p^2) c^2}\right)$$

where $K_G = (n-1)S_G^2/\sigma_G^2$ is distributed as χ_{n-1}^2. Since the process capability index is estimated by using $\tilde{C}_P^G$ instead of $\tilde{C}_P$, the true capability of the process is underestimated. The probability of $\tilde{C}_P^G$ being greater than c_0 will be less than that of using $\tilde{C}_P$. Thus, the α -risk using $\tilde{C}_P^G$ to estimate C_p is less than that of using $\tilde{C}_P$ when estimating C_p ($\alpha_G \leq \alpha$), and the power using $\tilde{C}_P^G$ in testing C_p is also less than the power using $\tilde{C}_P$ ($\pi_G \leq \pi$).

Adjusted Confidence Bounds and Critical Values of C_p

We showed earlier that the confidence intervals do not meet the stated confidence coefficients. We also showed that both the α-risk and the test power decrease when the gauge measurement error increases. If the producers do not take account of the gauge measurement errors, capability estimation and testing results would be misleading, thus result in serious loss. In that case, the producers cannot anymore affirm that their processes meet the capability requirement even if their processes are sufficiently capable. The producers may incur a lot of cost because quantities of qualified product units are incorrectly rejected. Improving the gauge measurement accuracy and training the operators by proper education are essential for reducing the measurement errors. Nevertheless, measurement errors may be unavoidable in most manufacturing processes. In the following, we adjust the confidence intervals and critical values in order to ensure the intervals have the desired confidence coefficients and improve the power of the test with appropriate α-risk. Suppose that the desired confidence coefficient is γ, the adjusted confidence interval of C_p with lower confidence bounds L_p^A, can be established as

$$P(L_p^A \leq C_p) = p\left(L_p^A \leq \frac{\tilde{C}_p^G}{\sqrt{(n-1)\Delta_{n-1}^2 K_G^{-1} - (\lambda \tilde{C}_p^G)^2)}}\right)$$

$$= p\left(K_G \geq (L_p^A)^2 \left[\frac{(n-1)\Delta_{n-1}^2}{(\tilde{C}_p^G)^2 (1 + (\lambda L_p^A)^2}\right]\right) = \gamma$$

By some simplification, the $100\gamma\%$ adjusted lower confidence bound can be written as

$$L_p^A = \frac{\sqrt{\chi^2_{n-1,\gamma}}\tilde{C}_p^G}{\sqrt{(n-1)\Delta^2_{n-1} - (\lambda \tilde{C}_p^G)^2 \chi^2_{n-1,\gamma}}} \tag{13}$$

With adjusted confidence bounds, we can ensure the interval would have the desired confidence coefficient. Moreover, in order to improve the power of the test, the adjusted critical values (denoted by c_0^A) are proposed to be satisfied $c_0^A < c_0$. Since $c_0^A < c_0$, the probability of $\tilde{C}_P^G$ being greater than c_0^A will be more than the probability of that $\tilde{C}_P^G$ being greater than c_0. In addition, both the α -risk and the power increase as c_0^A are taken to be adjusted critical values for testing hypothesis. Suppose that the α-risk by adjusted critical values c_0^A is α_A, the revised critical c_0^A can be introduced by

$$\alpha_A = p(\tilde{C}_p^G \geq c_0^A | C_p = c) = p\left(K_G \leq \frac{c^2 \Delta^2_{n-1}(n-1)}{(c_0^A)^2(1+\lambda^2 c^2)}\right).$$

To ensure that the α-risk is within the preset magnitude, we let $\alpha_A = \alpha$, thus c_0^A and the power (denoted by π_A) can be obtained as

$$\begin{aligned} c_0^A &= c\Delta_{n-1}\sqrt{\frac{n-1}{(1+\lambda^2 c^2)\chi^2_{n-1,\gamma}}} \\ \pi_A(C_P) &= p(\tilde{C}_P^G > c_0^A | C_p) = p\left[K_G < \left(\frac{C_p}{c}\right)^2 \left(\frac{1+\lambda^2 c^2}{1+\lambda^2 C_p^2}\right)\chi^2_{n-1,\gamma}\right] \end{aligned} \tag{14}$$

With adjusted critical values, the α-risk within the preset magnitude is ensured and a certain degree of power is improved. For the results to be practical and easily used, the tables of adjusted critical values for some commonly used capability requirements are tabulated in Tables 4 (a)–(d). Using those tables, the practitioner may skip the complex calculation and directly select the proper critical values for capability testing.

Extension to Multiple Samples

Many of the existing manufacturing factories have implemented a daily-based production control plan for monitoring/controlling process stability. A routine-basis data collection procedure is executed to run $\bar{X}$ and S control charts (for moderate sample sizes). The past ‘in control’ data consisting of multiple samples of m_s groups, with variable sample size $n_i = (X_{i1}, X_{i2}, \ldots, X_{in_i})$, are then analysed to compute the manufacturing capability. Thus, manufacturing information regarding the product quality characteristic is derived from multiple samples rather than one single sample. Under the assumption that these samples are taken from the normal distribution $N(\mu, \sigma^2)$, we consider the following estimators of process mean and process standard deviation,

$$\bar{X}_i = \sum_{j=1}^{n_i} X_{ij}/n_i, \quad S_i = \sqrt{\sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2/(n_i - 1)}$$

Table 4. Adjusted critical values of C_p

	λ									
n	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
(a) Adjusted critical values c_0^A for $C_p = 1.00$, with $n = 10(10)100$, $\lambda = 0.05(0.05)0.50$, and $\gamma = 0.95$										
10	1.502	1.496	1.487	1.474	1.459	1.440	1.419	1.396	1.371	1.345
20	1.314	1.309	1.301	1.290	1.276	1.260	1.242	1.221	1.200	1.177
30	1.245	1.240	1.232	1.222	1.209	1.194	1.176	1.157	1.137	1.115
40	1.207	1.202	1.195	1.185	1.172	1.157	1.140	1.122	1.102	1.081
50	1.182	1.177	1.170	1.160	1.148	1.133	1.117	1.099	1.079	1.058
60	1.164	1.160	1.152	1.143	1.131	1.116	1.100	1.082	1.063	1.042
70	1.150	1.146	1.139	1.129	1.117	1.103	1.087	1.069	1.050	1.030
80	1.140	1.135	1.128	1.119	1.107	1.093	1.077	1.059	1.041	1.021
90	1.131	1.127	1.120	1.110	1.098	1.085	1.069	1.051	1.033	1.013
100	1.124	1.119	1.112	1.103	1.091	1.077	1.062	1.044	1.026	1.006
(b) Adjusted critical values c_0^A for $C_p = 1.33$, with $n = 10(10)100$, $\lambda = 0.05(0.05)0.50$, and $\gamma = 0.95$										
10	1.995	1.982	1.961	1.932	1.898	1.857	1.813	1.765	1.716	1.665
20	1.746	1.734	1.716	1.691	1.660	1.625	1.586	1.545	1.501	1.457
30	1.654	1.643	1.626	1.602	1.573	1.540	1.503	1.463	1.422	1.380
40	1.603	1.593	1.576	1.553	1.525	1.492	1.457	1.419	1.379	1.338
50	1.570	1.560	1.543	1.521	1.493	1.462	1.427	1.389	1.350	1.310
60	1.547	1.536	1.520	1.498	1.471	1.440	1.405	1.368	1.330	1.291
70	1.529	1.519	1.502	1.480	1.454	1.423	1.389	1.352	1.314	1.276
80	1.514	1.504	1.488	1.467	1.440	1.410	1.376	1.340	1.302	1.264
90	1.503	1.493	1.477	1.455	1.429	1.399	1.365	1.330	1.292	1.254
100	1.493	1.483	1.467	1.446	1.420	1.390	1.356	1.321	1.284	1.246
(c) Adjusted critical values c_0^A for $C_p = 1.50$, with $n = 10(10)100$, $\lambda = 0.05(0.05)0.50$, and $\gamma = 0.95$										
10	2.249	2.230	2.200	2.160	2.112	2.057	1.997	1.934	1.869	1.804
20	1.968	1.951	1.925	1.890	1.848	1.799	1.747	1.692	1.635	1.579
30	1.864	1.849	1.824	1.791	1.750	1.705	1.655	1.603	1.549	1.496
40	1.807	1.792	1.768	1.736	1.697	1.653	1.604	1.554	1.502	1.450
50	1.770	1.755	1.732	1.700	1.662	1.619	1.571	1.522	1.471	1.420
60	1.743	1.729	1.705	1.674	1.637	1.594	1.548	1.499	1.449	1.398
70	1.723	1.709	1.686	1.655	1.618	1.576	1.530	1.482	1.432	1.382
80	1.707	1.693	1.670	1.639	1.603	1.561	1.515	1.468	1.419	1.369
90	1.694	1.680	1.657	1.627	1.590	1.549	1.504	1.456	1.408	1.359
100	1.683	1.669	1.646	1.616	1.580	1.539	1.494	1.447	1.399	1.350
(d) Adjusted critical values c_0^A for $C_p = 2.00$, with $n = 10(10)100$, $\lambda = 0.05(0.05)0.50$, and $\gamma = 0.95$										
10	2.992	2.949	2.880	2.792	2.690	2.578	2.463	2.348	2.235	2.126
20	2.618	2.580	2.520	2.443	2.353	2.256	2.155	2.054	1.956	1.860
30	2.480	2.444	2.387	2.314	2.229	2.137	2.042	1.946	1.853	1.762
40	2.404	2.369	2.314	2.243	2.161	2.072	1.979	1.887	1.796	1.709
50	2.355	2.320	2.267	2.197	2.117	2.029	1.939	1.848	1.759	1.673
60	2.319	2.286	2.232	2.164	2.085	1.999	1.909	1.820	1.732	1.648
70	2.292	2.259	2.207	2.139	2.060	1.975	1.887	1.799	1.712	1.629
80	2.271	2.238	2.186	2.119	2.041	1.957	1.870	1.782	1.696	1.614
90	2.253	2.221	2.169	2.103	2.026	1.942	1.855	1.768	1.683	1.601
100	2.239	2.206	2.155	2.089	2.012	1.929	1.843	1.757	1.672	1.591

for the $i-th$ sample mean and the sample standard deviation, respectively. Then, $S_p^2 = \sum_{i=1}^{m_s}(n_i - 1)S_i^2 / \sum_{i=1}^{m_s}(n_i - 1)$ are used for calculating the manufacturing capability C_p. For cases with multiple samples the natural estimator of C_p can be expressed below. The sensitivity investigation, capability testing, and adjusted confidence bounds and critical values for process capability C_p in the presence of gauge measurement errors based on multiple samples can be performed using the same techniques for cases with one single sample, although the derivations and calculations may be more tedious and complicated.

$$\tilde{C}_p^M = \Delta_{\sum_{i=1}^{m_s}(n_i-1)} \frac{USL - LSL}{6S_p}$$

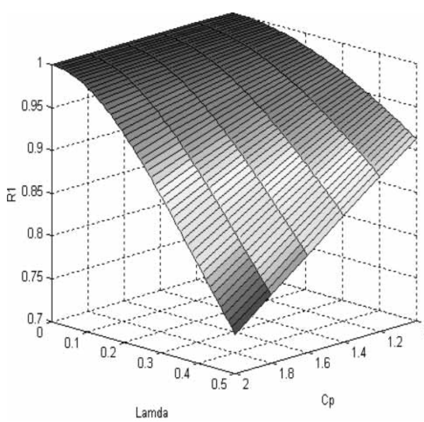
Testing C_{pm} with Gauge Measurement Errors

Similarly, in practice, the empirical process capability index C_{pm}^G is obtained after substituting σ_G for σ. The relationship between the true process capability C_{pm} and the empirical process capability C_{pm}^G can be expressed below, where $\xi = (\mu - T)/\sigma$.

$$\frac{C_{pm}^G}{C_{pm}} = \frac{\sqrt{1+\xi^2}}{\sqrt{1+\lambda^2 C_p^2 + \xi^2}} \tag{15}$$

Since the variation of data observed is larger than the variation of the original data, the denominator of the index C_{pm} becomes larger, and the true capability of the process is understated if calculation of the process capability index is based on empirical data G.

Figure 1(a) displays the surface plot of the ratio $R_1 = C_{pm}^G]/C_{pm}$ for λ in [0, 0.5] for $C_p \in [1, 2]$ with $\xi = 0.5$. Figure 1(b) plots the ratio R_1 versus λ for $C_p = 1.0(0.2)2.0$ with $\xi = 0.0$. Those figures show that the measurement errors result in a downward distortion of the index C_{pm}. Small process variation has the same effect as the presence of measurement error does. Since R_1 would be small if λ becomes large, the gauge becomes more important as the true capability improves. For instance, if $\lambda =$ 0.5, $C_p = 2$, and $\xi = 0.5$ (the ratio $R_1 = 0.7454$), $C_{pm}^G = 0.7454$ with $C_{pm} = 1$ (shrinks by about 25.46%), and $\lambda =$ 0.5, $C_p = 2$, and $\xi = 0.0$ (the ratio $R_1 = 0.7071$),



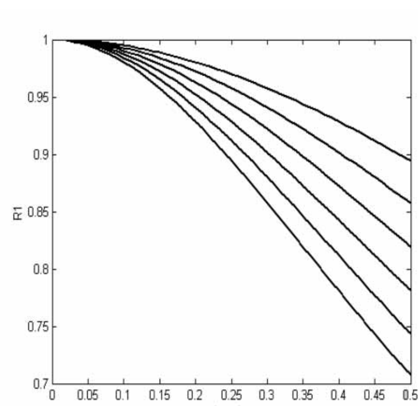


Figure 1. (a) Surface Plot of R_1 versus λ in [0,0.5] for $C_p = 1.0(0.2)2.0$ with $\xi = 0.5$.; (b) Plots of R_1 versus λ in [0,0.5] for $C_p = 1.0(0.2)2.0$ (top to bottom) with $\xi = 0.0$.

$C_{pm}^{G} = 1.7678$ with $C_{pm} = 2.50$ (shrinks by about 29.29%). The empirical process capability diverges more from the true process capability with large measurement errors.

Sampling Distribution of $\hat{C}_{pm}^{G}$

In practice, sample data must be collected in order to estimate the empirical process capability C_{pm}^{G}. The maximum likelihood estimator (MLE) of C_{pm}^{G} is defined as the following:

$$\hat{C}_{pm}^{G} = \frac{USL - LSL}{6\sqrt{\tilde{S}_n^2 + (\bar{G} - T)^2}} = \frac{d}{3\sqrt{\tilde{S}_n^2 + (\bar{G} - T)^2}} \tag{16}$$

where $\bar{G} = \Sigma_{i=1}^{n} G_i$, $\tilde{S}_n^2 = \Sigma_{i=1}^{n}(G_i - \bar{G})^2/n$, and $d = (USL - LSL)/2$. We note that $\bar{G}$ and $\tilde{S}_n^2$ are MLEs of μ and σ_G^2 respectively. Hence the estimated index $\hat{C}_{pm}^{G}$ is the MLE of C_{pm}^{G}. Furthermore, the term $\tilde{S}_n^2 + (\bar{G} - T)^2 = \Sigma_{i=1}^{n}(G_i - T)^2/n$ in the denominator of $\hat{C}_{pm}^{G}$ is the UMVUE of $\sigma_G^2 + (\mu - T)^2 = E(G - T)^2$ in the denominator of C_{pm}^{G}. Obviously, if the $\sigma_M = 0$, then the empirical process capability C_{pm}^{G} reduces to the basic index C_{pm}. As with Boyles (1991), the MLE of C_{pm} can be expressed in equation (17), where $\bar{X} = \Sigma_{i=1}^{n} X_i/n$ and $S_n^2 = \Sigma_{i=1}^{n}(X_i - \bar{X})^2/n$.

$$\hat{C}_{pm} = \frac{d}{3\sqrt{S_n^2 + (\bar{X} - T)^2}} \tag{17}$$

From equation (16), it is easy to show that $\hat{C}_{pm}^{G}$ is distributed as:

$$\hat{C}_{pm}^{G} \sim \frac{d}{3\sigma_G}\sqrt{\frac{n}{\chi^2_{n,\,\delta_G^2}}} = \hat{C}_{pm}^{G} \sim \frac{d}{3\sigma_G}\sqrt{\frac{n}{\chi^2_{n,\,\delta_G^2}}} = C_p^G\sqrt{\frac{n}{\chi^2_{n,\delta_G^2}}} = C_{pm}^{G}\sqrt{1 + \frac{\delta_G^2}{n}}\sqrt{\frac{n}{\chi^2_{n,\delta_G^2}}}$$

where χ^2_{n,δ_G^2} denotes the non-central Chi-square distribution with n degrees of freedom and non-centrality parameter $\delta_G^2 = n\xi_G^2$ where $\xi_G = (\mu - T)/\sigma_G$. We apply the method similarly to that used in Pearn *et al.* (1992), Vännman (1995), and Chen (1998), the cumulative distribution function (CDF) of $\hat{C}_{pmk}^{G}$ can be expressed in terms of a mixture of the Chi-square distribution and the normal distribution

$$F_{\hat{C}_{pm}^{G}}(x) = 1 - \int_0^{b_G\sqrt{n}/(3x)} F_K\left(\frac{(b_G\sqrt{n})^2}{9x^2} - t^2\right)[\phi(t + \xi_G\sqrt{n}) + \phi(t - \xi_G\sqrt{n})]dt \tag{18}$$

for $x > 0$ where $b_G = d/\sigma_G = 3C_p^G$. $F_K(\bullet)$ is the cumulative distribution function of the ordinary central Chi-square distribution χ^2_{n-1} and $\phi(\bullet)$ is the PDF of the standard normal distribution $N(0,1)$, where

$$\xi_G = \frac{\mu - m}{\sigma_G} = \sqrt{\left(\frac{C_p^G}{C_{pm}^G}\right)^2 - 1},\ C_p^G = \frac{C_p}{\sqrt{1 + \lambda^2 C_p^2}},\ \text{and } C_{pm}^G = \frac{C_{pm}\sqrt{1 + \xi^2}}{\sqrt{1 + \lambda^2 C_p^2 + \xi^2}}.$$

Obviously, if the $\sigma_M = 0$, then the CDF of $\hat{C}_{pm}$ can be easily obtained as:

$$F_{\hat{C}_{pm}}(x) = 1 - \int_0^{b\sqrt{n}/(3x)} F_K\left(\frac{(b\sqrt{n})^2}{9x^2} - t^2\right)[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})]dt \qquad (19)$$

for $x > 0$, where $b = d/\sigma = 3C_p$ and $\xi = \frac{\mu - m}{\sigma} = \sqrt{\left(\frac{C_p}{C_{pm}}\right)^2 - 1}$.

Lower Confidence Bound Based on $\hat{C}_{pm}^G$

The lower confidence bounds estimate the minimum process capability based on sample data. To find reliable $100\gamma\%$ lower confidence bound L_{pm} for C_{pm}, Pearn & Shu (2003) solved equation (20). Note that the term b can be expressed as $b = 3C_p = 3L\sqrt{1 + \xi^2}$. Since the process parameters μ and σ are unknown, then the distribution characteristic parameter $\xi = (\mu - m)/\sigma$ is also unknown. To eliminate the need for further estimating the distribution characteristic parameter ξ, Pearn & Shu (2003) investigated the behaviour of the lower confidence bound L_{pm} against the parameter ξ. They performed extensive calculations to obtain the lower confidence bound values L for $\xi = 0(0.05)3.00$, $\hat{C}_{pm} = 0.7(0.1)3.0$, $n =$ 10(5)200 with confidence coefficient $\gamma = 0.95$. They found that the lower confidence bound L obtains its minimum at $\xi = 0.0$ in all cases. Thus, for practical purposes they recommended solving equation (20) with $\xi = \hat{\xi} = 0.0$ to obtain the required lower confidence bounds, without having to further estimate the parameter ξ.

$$\int_0^{b\sqrt{n}/(3\hat{C}_{pm}} F_K\left(\frac{(b\sqrt{n})^2}{9\hat{C}_{pm}^2} - t^2\right)[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})]dt = 1 - \gamma \qquad (20)$$

In practice, sample data are observed measurements contaminated with errors to estimate the empirical process capability. Thus, $\hat{C}_{pm}^G$ is substituted into equation (18) and $\xi_G = 0.0$ to obtain the confidence bounds, which can be written as (we denote the bound originated from $\hat{C}_{pm}^G$ as L_{pm}^G) follows,

$$2\int_0^{b_G\sqrt{n}/(3\hat{C}_{pm}^G)} F_K\left(\frac{(b_G\sqrt{n})^2}{9(\hat{C}_{pm}^G)^2} - t^2\right)\phi(t)dt = 1 - \gamma$$

where $b_G = 3C_p^G = 3L_{pm}^G$. The confidence coefficient of the lower confidence bound L_{pm}^G (denoted by γ_G) is the following.

$$\gamma_G = 1 - 2\int_0^{b_G\sqrt{n}/(3\hat{C}_{pm})} F_K\left(\frac{(b_G\sqrt{n})^2}{9(\hat{C}_{pm})^2} - t^2\right)\phi(t)dt \qquad (21)$$

The γ_G is always not less than γ

Figures 2 (a), (b) and Figures 3(a), (b) plot L_{pm}^G versus $\lambda \in [0, 0.5]$ with $n =$ 30, 50, 70, 100, 150 for $\hat{C}_{pm} = 1.00$, 1.50 and $\hat{C}_p = \hat{C}_{pm} + R_3$, $R_3 = 0.33$ and 0.67 with 95% confidence level. It is noted that for sufficiently large sample size n, we have $\hat{C}_{pm}^{\hat{G}} = \hat{C}_{pm}/\sqrt{(1 + \lambda^2 C_p^2)}$. Therefore, we set $\hat{C}_{pm}^G = \hat{C}_{pm}/\sqrt{(1 + \lambda^2 C_p^2)}$ to obtain $\hat{C}_{pm}^G$ in Figures 2(a), (b) and Figures 3(a), (b). We see that in Figures 2(a), (b) and Figures 3(a), (b), L_{pm}^G decreases in λ, especially for large $\hat{C}_p$ values, and the decrement of L_{pm}^G is

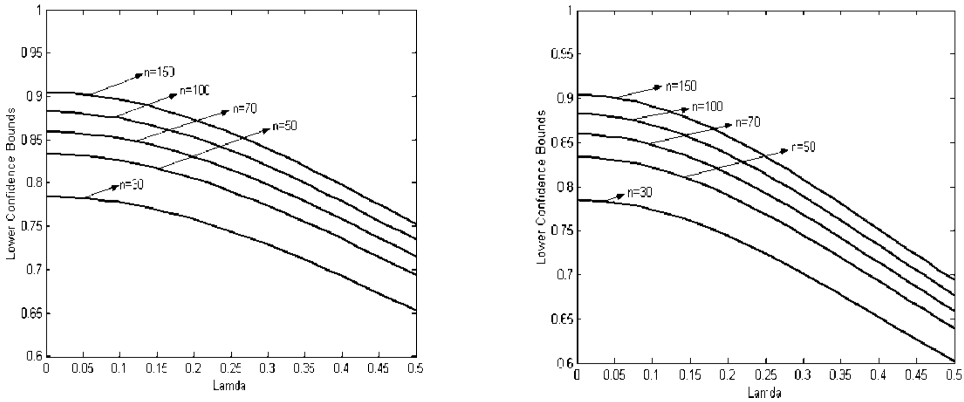


Figure 2. (a) Plots of L_{pm}^{G} versus λ with $n = 30, 50, 70, 100, 150$ for $\hat{C}_p = 1.33$ and $\hat{C}_{pm} = 1.00$; (b) Plots of L_{pm}^{G} versus λ with $n = 30, 50, 70, 100, 150$ for $\hat{C}_p = 1.67$ and $\hat{C}_{pm} = 1.00$.

more significant for large $\hat{C}_{pm}$. A large measurement error results in significantly underestimating the true process capability.

In current practice, a process is called 'inadequate' if $C_{pm} < 1.00$, 'marginally capable' if $1.00 \le C_{pm} < 1.33$, 'satisfactory' if $1.33 \le C_{pm} < 1.50$, 'excellent' if $1.50 \le C_{pm} < 2.00$, and 'super' if $2.00 \le C_{pm}$. In fact, Ruczinski (1996) showed that Yield $\ge 2\Phi(3C_{pm}) - 1$, or the fraction of non-conformities $\le 2\Phi(-3C_{pm})$. For example, if a process has capability with $C_{pm} \ge 1.25$, then the production yield would be at least 99.982%. If capability measures do not include the measurement errors, significant underestimation of the true process capability may result in high production cost, losing the power of competition. For instance, suppose that a process has a 95% lower confidence bound, 1.250 ($\hat{C}_{pm} = 1.50$) with $n = 50$, which meets the threshold of an 'excellent' process. But the bound may be calculated as 0.985 with measurement errors $\lambda = 0.36$ and the process is determined as 'inadequate'.

Testing Process Capability Based on $\hat{C}_{pm}^{G}$

To determine if a given process meets the preset capability requirement, we could consider the statistical testing with null hypothesis H_0:$C_{pm} \le c$ (process is not capable) and

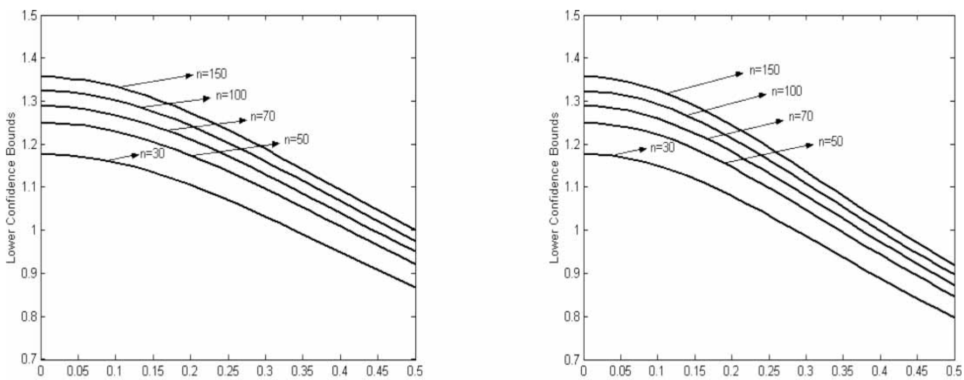


Figure 3. (a) Plots of L_{pm}^{G} versus λ with $n = 30, 50, 70, 100, 150$ for $\hat{C}_p = 1.83$ and $\hat{C}_{pm} = 1.50$; (b) Plots of L_{pm}^{G} versus λ with $n = 30, 50, 70, 100, 150$ for $\hat{C}_p = 2.17$ and $\hat{C}_{pm} = 1.50$.

alternative hypothesis $H_1{:}C_{pm} > c$ (process is capable), where c is the required process capability. Given values of capability requirement c, sample size n, and risk α, the critical value c_0 can be obtained by solving equation (2), using the available numerical methods.

$$\int_0^{b\sqrt{n}/(3c_0)} F_K\left(\frac{(b\sqrt{n})^2}{9c_0^2} - t^2\right)[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})]dt = \alpha \tag{22}$$

where $b = 3c\sqrt{1+\xi^2}$. Then, the test power can be expressed as the following,

$$\pi(C_{pm}) = P(\hat{C}_{pm} \geq c_0 | C_{pm} > c)$$
$$= \int_0^{b\sqrt{n}(3c_0)} F_K\left(\frac{(b\sqrt{n})^2}{9c_0^2} - t^2\right)[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})]dt$$

where $b = 3C_{pm}\sqrt{1+\xi^2}$.

To eliminate the need for estimating the characteristic parameter ξ, we follow the method of Pearn & Lin (2004) to examine the behaviour of the critical values c_0 against the parameter ξ. Extensive calculations to obtain the critical values c_0 for $\xi = 0(0.01)3$, $c = 1.00$, 1.33, 1.50, 1.67, 2.00, 2.5, and 3.0, $n = 10(50)300$, and $\alpha = 0.05$ are performed. The critical value c_0 obtains its maximum at $\xi = 0.0$ in all cases. For practice purposes, solving equation (22) with $\xi = 0.0$ to obtain the required critical values is recommended, without having to further estimate the parameter ξ. In practice, sample data are contaminated with measurement errors to estimate the empirical process capability. Thus, the α-risk corresponding to the test using the sample estimate $\hat{C}^G_{pm}$ becomes $P(\hat{C}^G_{pm} \geq c_0 | C_{pm} \leq c) = \alpha_G$, or

$$2\int_0^{b_G\sqrt{n}/(3c_0)} F_K\left(\frac{(b_G\sqrt{n})^2}{9c_0^2} - t^2\right)\phi(t)dt = \alpha_G \tag{23}$$

where

$$b_G = \frac{3c}{\sqrt{1+\lambda^2 C_p^2}} \text{ and } C_p = c.$$

The test power (denoted by π_G) is: $\pi_G(C_{pm}) = P(\hat{C}^G_{pm} \geq c_0 | C_{pm} > c)$. Thus,

$$\pi_G(C_{pm}) = 2\int_0^{b_G\sqrt{n}/(3c_0)} F_K\left(\frac{(b_G\sqrt{n})^2}{9c_0^2} - t^2\right)\phi(t)dt \tag{24}$$

where

$$b_G = \frac{3C_{pm}}{\sqrt{1+\lambda^2 C_p^2}} \text{ and } C_p = C_{pm}$$

Earlier discussions indicate that the true process capability would be severely underestimated if $\hat{C}^G_{pm}$ is used. The probability of $\hat{C}^G_{pm}$ being greater than c_0 would be less than that of using $\hat{C}_{pm}$. Thus, the α-risk using $\hat{C}^G_{pm}$ is, α_G, less than the α-risk if using $\hat{C}_{pm}$, α, when hypothesis testing C_{pm}. The test power if using $\hat{C}^G_{pm}$ is also less than the test power of using $\hat{C}_{pm}$. That is $\pi_G < \pi$. Figures 4(a), (b) are the plots of α_G with $n = 30, 50, 70, 100, 150$, $\lambda \in [0, 0.5]$ for $c = 1.00$, 1.50, and $\alpha = 0.05$. Figures 5(a), 5(b) plot π_G versus λ with $n = 50$, $\alpha = 0.05$, for $c = 1.00$, 1.50, and $C_{pm} = (c + 0.2)(0.20)(c + 1)$. Note that for

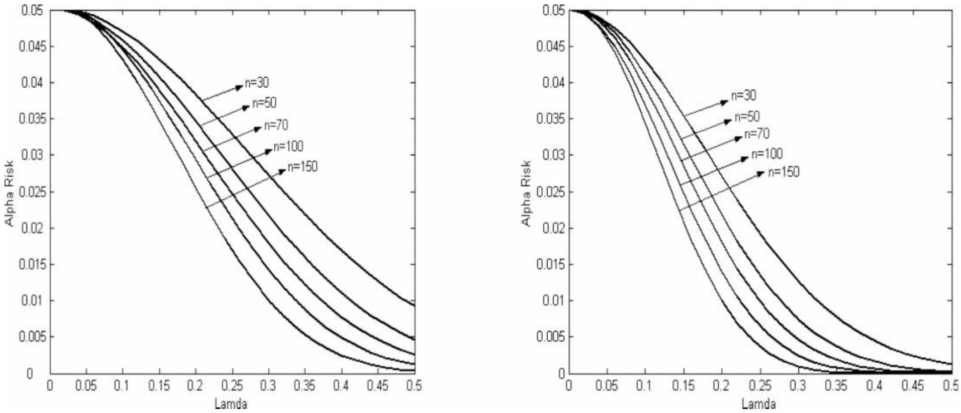


Figure 4. (a) Plots of α_G with $n = 30, 50, 70, 100, 150$ and λ in [0,0.5] for $c = 1.00$ and $\alpha = 0.05$; (b) Plots of α_G with $n = 30, 50, 70, 100, 150$ and λ in [0,0.5] for $c = 1.50$ and $\alpha = 0.05$.

$\lambda = 0$, $\alpha_G = \alpha$ and $\pi_G = \pi$. In Figures 4(a), (b), α_G decreases as λ or n increases, and the decreasing rate is more significant with large c. In fact, for large λ, α_G is smaller than 10^{-2}. In Figures 5(a), (b), π_G decreases as λ increases, but increases as n increases. The decrement of π_G in λ is more significant for large c. In the presence of measurement errors, π_G decreases. For instance, in Figure 9(b), later, the π_G values ($c = 1.50$, $n = 50$) for $C_{pm} = 2.1$ is $\pi_G = 0.9556$ if there is no measurement error ($\lambda = 0$). But, when $\lambda = 0.5$, π_G decreases to 0.0257, the decrement of the power is 0.9299.

Adjusted Confidence Bounds and Critical Values of C_{pm}

In this section, we consider the adjustment of confidence bounds and critical values of C_{pm} to provide better capability assessment. Suppose that the desired confidence coefficient is $100\gamma\%$ and the adjusted confidence interval of $\hat{C}_{pm}^{G}$ with the adjusted lower confidence

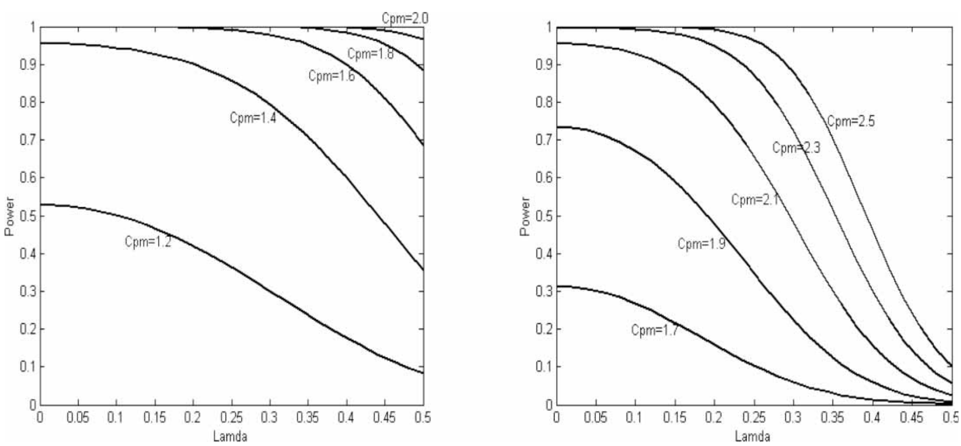


Figure 5. (a) Plots of π_G versus λ with $n = 50$, $\alpha = 0.05$ for $c = 1.00$, $C_{pm} = 1.2(0.20)2.00$; (b) Plots of π_G versus λ with $n = 50$, $\alpha = 0.05$ for $c = 1.50$, $C_{pm} = 1.70(0.20)2.50$.

bound L^A_{pm} can be established as:

$$P(\hat{C}^G_{pm} > L^A_{pm}) = \gamma$$
$$1 - 2\int_0^{b_A\sqrt{n}/(3\hat{C}^G_{pm})} F_K\left(\frac{(b_A\sqrt{n})^2}{9(\hat{C}^G_{pm})^2}\right)\phi(t)dx = \gamma \qquad (25)$$

where $b_A = 3L^A_{pm}/\sqrt{1+\lambda^2 C_p^2}$ and C_p can be obtained by solving $C^G_p = C^G_{pm}$, thus, $C_p = L_{pm}$. Figures 6 and 7, are the comparisons among L_{pm}, L^G_{pm}, and L^A_{pm} for $\hat{C}_{pm} = 1.00$, 1.50 with $n = 50$, where L_{pm} is the 95% lower confidence bound using $\hat{C}_{pm}$, L^G_{pm} is the 95% lower confidence bound using $\hat{C}^G_{pm}$, and L^A_{pm} is the adjusted 95% lower confidence bound using $\hat{C}^G_{pm}$. It can be noted that $\hat{C}^G_{pm} = \hat{C}_{pm}/\sqrt{1+\lambda^2\hat{C}_p^2}$ is used to obtain $\hat{C}^G_{pm}$ in equation (4). In this case, the probability that the lower confidence interval with bound L^G_{pm} contains the actual C_{pm} value is greater than that of the interval with the bound L_{pm} or L^A_{pm}, while the probability that the lower confidence interval with bound L_{pm} or L^A_{pm} contains the actual C_{pm} value is 0.95. From Figures 6 and 7, we see that the magnitude of lower confidence bounds remained underestimated even if it is adjusted. But the magnitude of underestimation using the adjusted confidence bound is significantly reduced.

In order to improve the test power, we revise the critical values c^A_0 to satisfy $c^A_0 < c_0$. Thus, the probability $P(\hat{C}^G_{pm} > c^A_0)$ is greater than $P(\hat{C}^G_{pm} > c_0)$. Both the α-risk and the test power increase when we use c^A_0 as a new critical value in the testing. Suppose that the α-risk using the revised critical value c^A_0 is α_A, the revised critical values c^A_0 can be determined by $P(\hat{C}^G_{pm} \geq c^A_0 | C_{pm} \leq c) = \alpha_A$,

$$2\int_0^{b_G\sqrt{n}/(3c^A_0)} F_K\left(\frac{(b_G\sqrt{n})^2}{9(c^A_0)^2} - t^2\right)\phi(t)dt = \alpha_A \qquad (26)$$

where $b_G = 3c/\sqrt{1+\lambda^2 C_p^2}$ and C_p can be obtained by solving equation $C^G_p = C^G_{pm}$, thus, $C_p = \text{c}$. To ensure that the α-risk is within the preset magnitude, we let $\alpha_A = \alpha$ and solve

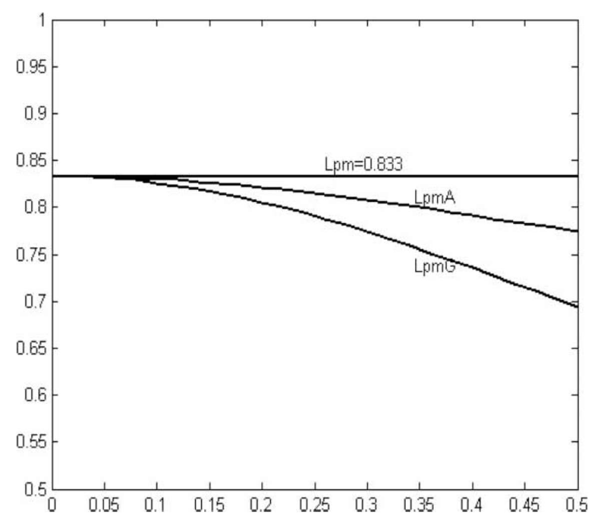


Figure 6. Plots of L_{pm}, L^A_{pm}, and L^G_{pm} versus λ with $n = 50$ and for $\hat{C}_{pm} = 1.00$ and $\hat{C}_p = 1.33$.

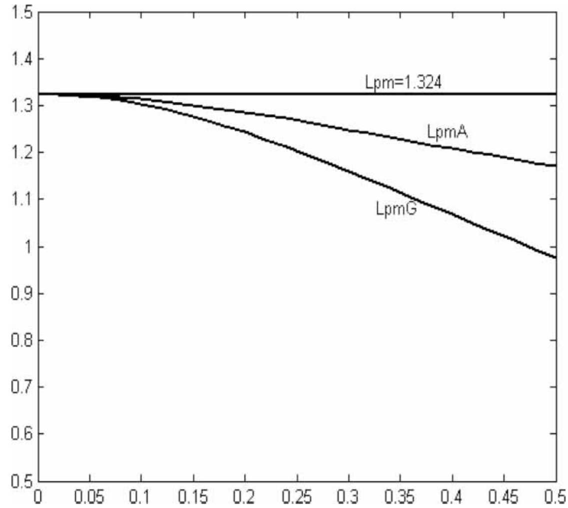


Figure 7. Plots of L_{pm}, L_{pm}^{A}, and L_{pm}^{G} versus λ with $n = 100$ and for $\hat{C}_{pm} = 1.50$ and $\hat{C}_p = 1.83$.

the equation to obtain c_0^A. The power (denoted by π_A) can be calculated as the following:

$$\pi_A(C_{pm}) = P(\hat{C}_{pm}^{G} \geq c_0^A | C_{pm} > c)$$

$$\pi_A(C_{pm}) = 2\int_0^{b_G\sqrt{n}/(3c_0^A)} F_K\left(\frac{(b_G\sqrt{n})^2}{9(c_0^A)^2} - t^2\right)\phi(t)dt \tag{27}$$

where

$$b_G = \frac{3C_{pm}}{\sqrt{1+\lambda^2 c^2}}.$$

Figures 8(a), (b) are plots of π_A versus λ with $n = 50$, $\alpha = 0.05$, for $c = 1.00$, 1.50 and $C_{pm} = (c + 0.2)(0.20)(c + 1)$. From those figures, we see that the powers corresponding to

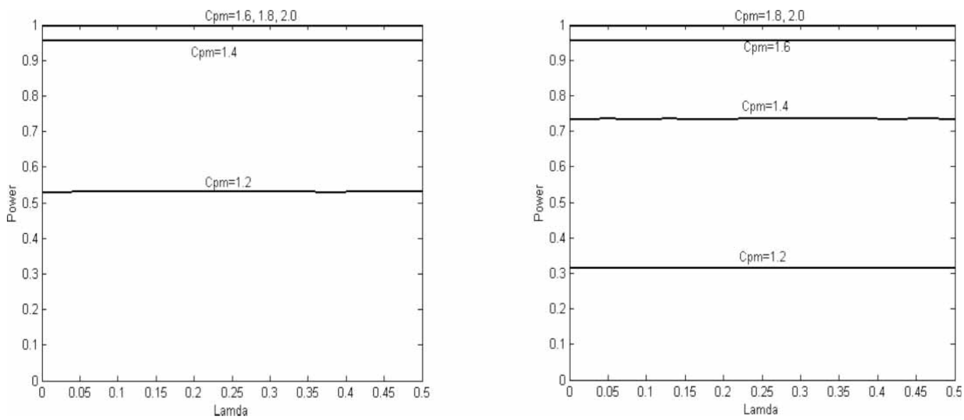


Figure 8. (a) Plots of π_A versus λ with $n = 50$, $\alpha = 0.05$ for $c = 1.00$, $C_{pm} =$ 1.2(0.2)2.00 (bottom to top); (b) Plots of π_A versus λ with $n = 50$, $\alpha = 0.05$ for $c = 1.50$, $C_{pm} =$ 1.70(0.20)2.50 (bottom to top).

Table 5. Adjusted critical values of C_{pm}

λ n	30	50	70	100	120	150
(a) Adjusted critical values c_0^A for $n = 30, 50, 70, 100, 120, 150$, $\lambda = 0.05(0.05)0.50$, $C_{pm} = 1.00$, and $\gamma = 0.95$						
0.00	1.273	1.199	1.163	1.132	1.119	1.105
0.02	1.273	1.199	1.162	1.132	1.119	1.105
0.04	1.272	1.198	1.162	1.131	1.118	1.104
0.06	1.271	1.197	1.161	1.130	1.117	1.103
0.08	1.269	1.195	1.159	1.129	1.116	1.102
0.10	1.267	1.193	1.157	1.127	1.114	1.100
0.12	1.264	1.190	1.154	1.124	1.111	1.097
0.14	1.261	1.187	1.151	1.121	1.108	1.095
0.16	1.257	1.184	1.148	1.118	1.105	1.091
0.18	1.253	1.180	1.144	1.114	1.102	1.088
0.20	1.248	1.175	1.140	1.110	1.098	1.084
0.22	1.243	1.171	1.135	1.106	1.093	1.079
0.24	1.238	1.166	1.131	1.101	1.088	1.075
0.26	1.232	1.160	1.125	1.096	1.083	1.070
0.28	1.226	1.154	1.120	1.090	1.078	1.064
0.30	1.219	1.148	1.114	1.085	1.072	1.059
0.32	1.213	1.142	1.107	1.078	1.066	1.053
0.34	1.205	1.135	1.101	1.072	1.060	1.046
0.36	1.198	1.128	1.094	1.065	1.053	1.040
0.38	1.190	1.121	1.087	1.058	1.046	1.033
0.40	1.182	1.113	1.079	1.051	1.039	1.026
0.42	1.174	1.105	1.072	1.044	1.032	1.019
0.44	1.165	1.097	1.064	1.036	1.024	1.012
0.46	1.157	1.089	1.056	1.029	1.017	1.004
0.48	1.148	1.081	1.048	1.021	1.009	0.996
0.50	1.139	1.072	1.040	1.013	1.001	0.988
(b) Adjusted critical values c_0^A for $n = 30, 50, 70, 100, 120, 150$, $\lambda = 0.05(0.05)0.50$, $C_{pm} = 1.33$, and $\gamma = 0.95$						
0.00	1.693	1.595	1.547	1.506	1.489	1.470
0.02	1.693	1.594	1.546	1.506	1.488	1.470
0.04	1.691	1.592	1.544	1.504	1.487	1.468
0.06	1.688	1.589	1.542	1.501	1.484	1.465
0.08	1.684	1.586	1.538	1.498	1.480	1.462
0.10	1.679	1.581	1.533	1.493	1.476	1.457
0.12	1.672	1.575	1.527	1.487	1.470	1.452
0.14	1.665	1.568	1.520	1.481	1.464	1.445
0.16	1.656	1.560	1.513	1.473	1.456	1.438
0.18	1.647	1.551	1.504	1.465	1.448	1.430
0.20	1.637	1.541	1.495	1.455	1.439	1.421
0.22	1.625	1.53	1.484	1.445	1.429	1.411
0.24	1.613	1.519	1.473	1.435	1.418	1.400
0.26	1.600	1.507	1.462	1.423	1.407	1.389
0.28	1.587	1.494	1.449	1.411	1.395	1.378
0.30	1.573	1.481	1.436	1.399	1.383	1.365
0.32	1.558	1.467	1.423	1.386	1.370	1.353

(*Table continued*)

Table 5. *Continued*

λ n	30	50	70	100	120	150
0.34	1.543	1.453	1.409	1.372	1.356	1.339
0.36	1.527	1.438	1.395	1.358	1.343	1.326
0.38	1.511	1.423	1.380	1.344	1.329	1.312
0.40	1.495	1.408	1.365	1.33	1.314	1.298
0.42	1.478	1.392	1.35	1.315	1.300	1.283
0.44	1.462	1.376	1.335	1.300	1.285	1.269
0.46	1.445	1.360	1.319	1.285	1.270	1.254
0.48	1.427	1.344	1.303	1.269	1.255	1.239
0.50	1.410	1.328	1.288	1.254	1.240	1.224

(c) Adjusted critical values c_0^A for $n = 30$, 50, 70,100,120,150, $\lambda = 0.05(0.05)0.50$, $C_{pm} = 1.50$, and $\gamma = 0.95$

0.00	1.910	1.798	1.744	1.699	1.679	1.658
0.02	1.909	1.798	1.743	1.698	1.678	1.657
0.04	1.907	1.795	1.741	1.696	1.676	1.655
0.06	1.902	1.791	1.737	1.692	1.672	1.651
0.08	1.896	1.786	1.732	1.687	1.667	1.646
0.10	1.889	1.779	1.725	1.680	1.661	1.640
0.12	1.880	1.770	1.717	1.672	1.653	1.632
0.14	1.869	1.760	1.707	1.662	1.643	1.623
0.16	1.857	1.749	1.696	1.652	1.633	1.612
0.18	1.844	1.736	1.684	1.640	1.621	1.601
0.20	1.829	1.723	1.671	1.627	1.608	1.588
0.22	1.814	1.708	1.656	1.613	1.595	1.575
0.24	1.797	1.692	1.641	1.598	1.580	1.560
0.26	1.779	1.675	1.625	1.583	1.564	1.545
0.28	1.761	1.658	1.608	1.566	1.548	1.529
0.30	1.742	1.640	1.591	1.549	1.531	1.512
0.32	1.722	1.621	1.572	1.531	1.514	1.495
0.34	1.701	1.602	1.554	1.513	1.496	1.477
0.36	1.681	1.582	1.535	1.495	1.477	1.459
0.38	1.659	1.562	1.515	1.476	1.459	1.440
0.40	1.638	1.542	1.496	1.457	1.440	1.422
0.42	1.616	1.522	1.476	1.437	1.421	1.403
0.44	1.594	1.501	1.456	1.418	1.401	1.384
0.46	1.572	1.480	1.436	1.398	1.382	1.365
0.48	1.550	1.459	1.415	1.378	1.363	1.345
0.50	1.528	1.439	1.395	1.359	1.343	1.326

(d) Adjusted critical values c_0^A for $n = 30$, 50, 70,100,120,150, $\lambda = 0.05(0.05)0.50$, $C_{pm} = 2.00$, and $\gamma = 0.95$

0.00	2.547	2.398	2.326	2.265	2.239	2.211
0.02	2.545	2.396	2.324	2.263	2.237	2.209
0.04	2.539	2.390	2.318	2.258	2.232	2.204
0.06	2.529	2.381	2.309	2.249	2.223	2.195
0.08	2.515	2.368	2.297	2.237	2.211	2.183
0.10	2.497	2.351	2.281	2.221	2.196	2.168
0.12	2.477	2.332	2.262	2.203	2.177	2.150

(*Table continued*)

Table 5. *Continued*

λ n	30	50	70	100	120	150
0.14	2.453	2.309	2.240	2.181	2.156	2.129
0.16	2.426	2.284	2.215	2.157	2.132	2.106
0.18	2.396	2.256	2.188	2.131	2.107	2.080
0.20	2.365	2.226	2.159	2.103	2.079	2.053
0.22	2.331	2.195	2.129	2.073	2.049	2.024
0.24	2.296	2.162	2.097	2.042	2.018	1.993
0.26	2.260	2.128	2.063	2.010	1.986	1.961
0.28	2.222	2.092	2.029	1.976	1.953	1.929
0.30	2.184	2.056	1.994	1.942	1.920	1.896
0.32	2.145	2.020	1.959	1.908	1.886	1.862
0.34	2.106	1.983	1.923	1.873	1.851	1.828
0.36	2.067	1.946	1.887	1.838	1.817	1.794
0.38	2.028	1.909	1.852	1.803	1.783	1.760
0.40	1.989	1.872	1.816	1.769	1.748	1.726
0.42	1.950	1.836	1.781	1.734	1.714	1.693
0.44	1.912	1.800	1.746	1.700	1.681	1.660
0.46	1.874	1.765	1.712	1.667	1.648	1.627
0.48	1.837	1.730	1.678	1.634	1.615	1.595
0.50	1.801	1.696	1.644	1.602	1.583	1.563

the adjusted critical values c_0^A remain stable in measurement error. We improve the test power to a certain degree. For instance, when we compare the π_G values in Figure 5(b) ($c = 1.50$, $n = 50$, $C_{pm} = 2.1$) to the π_A values in Figure 8(b) ($c = 1.50$, $n = 50$, $C_{pm} = 2.1$), we obtain that $\pi_G = 0.0257$ and $\pi_A = 0.9557$ with $\lambda = 0.5$. In this case, using the adjusted critical values c_0^A, we improve the test power by 0.930 (which is rather significant). For our results to be practical, we tabulate the adjusted critical values for some commonly used capability requirements in Tables 5(a)–(d). Using those tables, the practitioner may omit the complex calculation and simply select the proper critical values for capability testing.

Application Example on a pH Sensor

The product investigated is the pH sensor combining process-hardened pH electrodes, a double junction reference electrode, temperature compensation element, and a solution ground in a durable, reliable, high performance design. It is ideal for process control applications in the most aggressive streams found in traditional processing industries, including chemicals, paper, metals and mining, utilities, food, pharmaceutical, and others. For a wide range of measurements where high accuracy is required at either both extreme ends of the pH scales, or a spherical glass with minimal sodium error. For applications involving abrasive processes, a rugged glass with a thicker membrane is recommended. The rugged glass is most accurate in the range of 1 to 12 pH. High temperature construction allows the sensor to be used for process pH measurements at temperatures up to 120°C (250°F). An integral 100 platinum RTD (resistance temperature detector), compatible with many common pH transmitters and monitors, is a standard feature. The reference electrode utilizes a double junction design to inhibit silver ions from contacting the process solution, thereby preventing junction fouling from silver

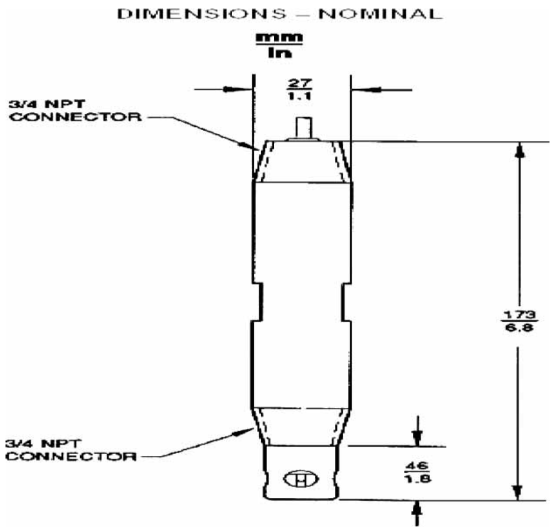


Figure 9. The pH sensor.

compound precipitates. To help facilitate a noise-free signal, the sensor incorporates a solution ground post of titanium metal. 3/4 NPT threads are provided at both sensor ends to allow connection to insertion or submersion type mountings. The pH sensor is depicted in Figure 9.

The measuring accuracy of the pH sensor is an important factor that has significant effect on the pH sensor quality. A type of the pH sensor has the specification limits, $T = 0.0$ pH, $USL = 0.05$ pH, and $LSL = -0.05$ pH. A total of 70 observations are collected and displayed in Table 6. Histogram and normal probability plots show that the collected data follow the normal distribution. The Shapiro–Wilk test is applied to further justify the assumption. To determine whether the process is 'excellent' ($C_{pm} > 1.33$) with unavoidable measurement errors $\lambda = 0.30$, we first determine that $c = 1.33$ and $\alpha = 0.05$. Then, based on the sample data of 70 observations, we obtain the sample mean $\bar{G} = 0.0200$, the sample standard deviation $\tilde{S}_n = 0.0109$, and the point estimator $\hat{C}^G_{pm} = 1.4629$. From Table 5(b), we obtain the critical value $c_0^A = 1.436$ based on α, λ and n. Since $\hat{C}^G_{pm} > c_0^A$, we therefore conclude that the process is 'excellent'. We also see that if we ignore the measurement errors and evaluate the critical value without any correction, the critical value may be calculated as $c_0 = 1.547$. In this case we would reject that the process is 'excellent' since $\hat{C}^G_{pm}$ is no

Table 6. 70 observations for the measuring accuracy (unit: pH)

0.0236	0.0433	0.032	0.0157	0.0129	0.0085	0.0406	0.0047	0.0047	0.0254
0.0257	0.0281	0.0246	0.0198	0.0335	0.0253	0.0115	0.0306	0.0097	0.0194
0.0304	0.0198	0.0172	0.0234	0.0311	0.0278	0.0035	0.0289	0.0044	0.0188
0.0289	0.0227	0.0186	0.0066	0.0265	0.0399	0.0168	0.0240	0.0040	0.0235
0.0057	−0.0018	0.0385	−0.0086	0.0178	0.0130	0.0240	0.0317	0.0200	0.0137
0.0188	−0.0046	0.0322	0.0136	0.0187	0.0249	0.0157	0.0250	0.0167	0.0083
0.0204	0.0153	0.0154	0.0413	0.0089	0.0344	0.0273	0.0236	0.0158	0.0185

greater than the uncorrected critical value 1.547. Moreover, input $T = 0.0$pH, $USL = 0.05$pH, and $LSL = -0.05$pH, 70 observations, $\lambda = 0.30$ (provided by the gauge manufacturing factory), and the desired confidence coefficient $\gamma = 0.95$ into the Matlab computer program (available upon request), the 95% lower confidence bound of the true process capability can be obtained as 1.415. We thus can ensure that the production yield is 99.9978%, and the number of the non-conformities is less than 21.78 PPM (Parts Per Million).

Conclusions

Gauge measurement errors have a significant impact on estimating and testing manufacturing reproduction capability. In this paper, we conducted the sensitivity study for process capability C_p and C_{pm} in the presence of gauge measurement errors. We investigated the statistical properties and capability testing of estimating C_p and C_{pm} to obtain lower confidence bounds and critical values for true process capability testing when gauge measurement errors are unavoidable. In estimating the capability, the estimator $\hat{C}_p^G$ and $\hat{C}_{pm}^G$ using the sample data contaminated with the measurement error severely underestimates the true capability in the presence of measurement errors. The statistical testing is performed to determine whether the process meets the capability requirement, the test power decreases in the presence of gauge measurement errors. Since the measurement errors are unavoidable in most industry applications, lower confidence bounds and critical values must be adjusted to improve the accuracy of capability assessment. For practical purposes, some adjusted critical values for C_p and C_{pm} are tabulated to the engineers for their factory applications.

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