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ORIGINAL ARTICLE

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Testing manufacturing performance based on capability index C_{pm}

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Abstract The loss-based process capability index C_{pm}, sometimes called the Taguchi index, has been proposed to the manufacturing industry to measure process performance. In this paper, we propose a method to assess the performance of a normally distributed process. We implement the theory of testing hypothesis using the natural estimator of C_{pm}, and provide an efficient program to calculate the *p*-values. We also provide tables of the critical values for some commonly used capability requirements. Based on the test, we develop a simple step-by-step procedure for in-plant applications.

Keywords *p*-value · Process capability index · Taguchi index · Testing hypothesis

1 Introduction

Process capability indices (PCIs) provide numerical measures on whether a manufacturing process meets the preset quality requirement. We note that PCIs have been the focus of recent research in quality assurance and process capability analysis. Three basic capability indices C_p, C_{pk}, and C_{pm}, which establish the relationship between the actual process performance and the manufacturing specifications, have been widely used in the manufacturing industry. These indices are explicitly defined as (see [1, 2]):

$$C_p = \frac{USL - LSL}{6\sigma},$$

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\},$$

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}},$$

where USL is the upper specification limit, LSL is the lower specification limit, μ is the process mean and σ is the process standard deviation, and T is the target value preset by the product designer or manufacturing engineer. We will focus on the situation in which the specification interval is two-sided with the target value T at $m = (USL + LSL)/2$, the midpoint of the specification interval (LSL, USL), which is most common in practice.

The index C_p measures the consistency of process quality characteristic relative to the manufacturing tolerance and, therefore, only reflects process potential (or process precision). The index C_{pk} takes into account the magnitudes of process variation as well as the degree of process centering, which measures manufacturing performance based on yield (proportion of conformities). Hence, the capability index C_{pk} is a yield-based index.

The index C_{pm} emphasizes measuring the ability of the process to cluster around the target, which therefore reflects the degrees of process targeting (centering). The index C_{pm} incorporates with the variation of production items with respect to the target value and the specification limits preset in the factory (see [3–5]). It is noted in some recent quality research and capability analysis works, that both of the C_{pk} and C_{pm} indices provide the same lower bounds on process yield, that is, Yield $\geq 2\Phi(3C_{pk}) - 1 = 2\Phi(3C_{pm}) - 1$. Pearn and Lin [6] investigated the behavior of the actual process yield for processes with fixed index value of $C_{pk} = C_{pm}$, but with different degrees of process centering. The results in Pearn and Lin [6] illustrate the advantage of using the index C_{pm} over the index C_{pk} in measuring manufacturing performance, as C_{pm} provides better protection to the customers.

In fact, the capability index C_{pm} is not primarily designed to provide an exact measure on the number of conforming items, *i.e.*, the process yield. But C_{pm} considers the process departure $(\mu - T)^2$ (rather than 6σ alone) in the denominator of the definition to reflect the degrees of process targeting (see [2, 3]). We note that $\sigma^2 + (\mu - T)^2 = E[(X - T)^2]$, which is the major part

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of the denominator of C_{pm}. Since $E[(X-T)^2]$ is the expected loss where we note that the loss of a characteristic X missing the target is often assumed to be well approximated by the symmetric squared error loss function, $loss(X) = (X-T)^2$. Hence, the capability index C_{pm} is a loss-based index.

In general the process mean μ and the process variance σ^2 are unknown. But, in practice μ and σ^2 can be estimated using the sample data. We then consider the estimated index $\hat{C}_{pm}$ the natural estimator of the index C_{pm}. In order to calculate the estimator, however, sample data must be collected, and a great degree of uncertainty may be introduced into capability assessments owing to the sampling errors. The approach by simply looking at the calculated values of the estimated indices and then make a conclusion on whether the given process is capable, is highly unreliable as the sampling errors have been ignored. As the use of the capability indices grows more widespread, users are becoming educated and sensitive to the impact of the estimators and their sampling distributions on constructing confidence intervals and performing hypothesis testing. For normally distributed processes, Cheng [7] has developed a hypothesis testing procedure where tables of the approximate p-values were provided for some commonly used capability requirements, using the natural estimator of C_{pm}. The practitioners can use the obtained results to determining if their process satisfies the targeted quality condition. But Cheng's approach requires further estimation of the distribution characteristic $(\mu - T)/\sigma$ when calculating the p-values, which introduces additional sampling errors thus making the decisions less reliable.

In this paper, we first obtain an explicit form of the cumulated distribution function of the maximal likelihood estimator (MLE) of C_{pm}, which can be expressed in terms of a mixture of the chi-square distribution and the normal distribution. We then implement the theory of testing hypothesis using the MLE of C_{pm}, and provide an efficient Maple program to calculate the p-values. To eliminate the need for estimating the distribution characteristic, the parameter $\xi = (\mu - T)/\sigma$, we examine the behaviors of the p-values as well as the critical values c_0 against the parameter ξ. Then, we develop a simple step-by-step procedure where tables of critical values are provided for some commonly used capability requirements. The practitioners can also use the proposed procedure, without having to run the Maple program, to determine whether their manufacturing process meets the preset capability requirement, and make reliable decisions.

2 Distribution of the estimated C_{pm}

The index C_{pm} can be rewritten as the following:

$$C_{pm} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}},$$

where $d = (USL - LSL)/2$ is half of the length of the specification interval. In generally, both the process mean μ and the process variance σ^2 are unknown, as pointed out earlier. The estimated index $\hat{C}_{pm}$ is obtained by replacing μ and σ^2 by their estimators. Chan et al. [2] and Boyles [8] proposed two different estimators of C_{pm} respectively defined as the following:

$$\hat{C}_{pm(CCS)} = \frac{d}{3\sqrt{S^2 + (\bar{X} - T)^2}}$$

and $$\hat{C}_{pm(B)} = \frac{d}{3\sqrt{S_n^2 + (\bar{X} - T)^2}},$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$, $S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1)$ and $S_n^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/n$. In fact, the two estimators, $\hat{C}_{pm(CCS)}$ and $\hat{C}_{pm(B)}$, are asymptotical equivalent. We note that $\bar{X}$ and S_n^2 are the MLEs of μ and σ^2, respectively. Hence, the estimated index $\hat{C}_{pm(B)}$ is the MLE of C_{pm}. Furthermore, the term $S_n^2 + (\bar{X} - T)^2$ in the denominator of $\hat{C}_{pm(B)}$ is the uniformly minimum variance unbiased estimator (UMVUE) of the term $\sigma^2 + (\mu - T)^2$ in the denominator of C_{pm}, where $S_n^2 + (\bar{X} - T)^2 = \sum_{i=1}^{n} (X_i - T)^2/n$ and $\sigma^2 + (\mu - T)^2 = E[(X - T)^2]$. Therefore, it is reasonable for reliability purpose, that we adapt the estimator $\hat{C}_{pm(B)}$ to evaluate the performances of normally distributed processes in this paper and define the estimated index $\hat{C}_{pm} = \hat{C}_{pm(B)}$.

Under the assumption of normality, Kotz and Johnson [4] obtained the rth moment, and calculated the first two moments, the mean, and the variance of $\hat{C}_{pm}$. Zimmer and Hubele [9] provided tables of exact percentiles for the sampling distribution of the estimator $\hat{C}_{pm}$. Zimmer et al. [10] proposed a graphical procedure to obtain exact confidence intervals for C_{pm}, where the parameter $(\mu - T)/\sigma$ is assumed to be a known constant, thus the results obtained is somewhat unreliable. On the other hand, using the method similar to that presented in Vännman [11], we may obtain an exact form of the cumulative distribution function of $\hat{C}_{pm}$. Under the assumption of normality, the cumulative distribution function of $\hat{C}_{pm}$ can be expressed in terms of a mixture of the chi-square distribution and the normal distribution:

$$F_{\hat{C}_{pm}}(x) = 1 - \int_0^{b\sqrt{n}/(3x)} G\left(\frac{b^2 n}{9x^2} - t^2\right)\left[\phi(t + \xi\sqrt{n}) + \phi(t - \xi\sqrt{n})\right] dt, \qquad (1)$$

for $x > 0$, where $b = d/\sigma$, $\xi = (\mu - T)/\sigma$, $G(\cdot)$ is the cumulative distribution function of the chi-square distribution χ^2_{n-1}, and $\phi(\cdot)$ is the probability density function of the standard normal distribution N(0, 1).

3 Testing process performance

To test whether a given process is capable, we may consider the following statistical testing hypotheses:

H_0: $C_{pm} \le C$ (process is not capable),

H_1: $C_{pm} > C$ (process is capable).

Based on a given $\alpha(c_0) = \alpha$, the chance of incorrectly concluding an incapable process ($C_{pm} \le C$) as capable ($C_{pm} > C$), the decision rule is to reject $H_0 (C_{pm} \le C)$ if $\hat{C}_{pm} > c_0$ and fails to reject H_0 otherwise.

For processes with target value setting on the middle of the specification limits ($T = m = (USL + LSL)/2$), which are fairly common situations, the index may be rewritten as: $C_{pm} = b/[3(1+\xi^2)^{1/2}]$. Given $C_{pm} = C$, $b = d/\sigma$ can be expressed as $b = 3C(1+\xi^2)^{1/2}$. Given a value of C (the capability requirement), the *p*-value corresponding to c^*, a specific value of $\hat{C}_{pm}$ calculated from the sample data, is (by Eq. 1):

$$P(\hat{C}_{pm} \geq c^* | C_{pm} = C) = \int_0^{b\sqrt{n}/(3c^*)} G\left(\frac{b^2 n}{9(c^*)^2} - t^2\right) \left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right] dt. \quad (2)$$

Given values of α and C, the critical value c_0 can be obtained by solving the equation $P(\hat{C}_{pm} \geq c_0 | C_{pm} = C) = \alpha$. Hence, given values of capability requirement C, parameter ξ, sample size n, and risk α, the critical value c_0 can be obtained by solving the following equation:

$$\int_0^{b\sqrt{n}/(3c_0)} G\left(\frac{b^2 n}{9c_0^2} - t^2\right) \left[\phi(t+\xi\sqrt{n}) + \phi(t-\xi\sqrt{n})\right] dt = \alpha\,. \quad (3)$$

It is noted that Eq. 2 is an even function of ξ for fixed values of c^*, n and C. Therefore, given values of n and C the *p*-values corresponding to a specific value c^* for $\xi = \xi_0$ and $\xi = -\xi_0$ are the same. Similarly, given values of C, n and α the critical values c_0 for $\xi = \xi_0$ and $\xi = -\xi_0$ are the same.

An efficient Maple program is developed for calculating Eq. 2 to obtain the *p*-value for given c^*. We note that similar programs can also be written using software "Mathematica" or "MatLab". The program is listed below, with input parameters set to $LSL = 2.40$, $USL = 3.40$, $T = 2.90$, $C = 1.00$, $n = 100$, $\bar{X} = 2.825$, $S_n = 0.125$ as an example. Here, we set $\xi = \hat{\xi} = (\bar{X} - T)/S_n$, since generally $\xi = (\mu - T)/\sigma$ is unknown. This approach is similar to one proposed by Cheng [7]. On the other hand, $c^* = \hat{C}_{pm}$ can be calculated from the sample data. The program gives $\hat{\xi} = -0.6$, $\hat{C}_{pm} = 1.1433$, and the corresponding *p*-value as 0.0338.

Maple program:

```
> # input parameter values LSL, USL, T, C, n, X_bar, Sn.
LSL:=2.40; USL:=3.40; T:=2.90; C:=1.00; n:=100;
X_bar:=2.825; Sn:=0.125;
d:=(USL - LSL)/2; ξ:=(X_bar - T)/Sn;
c1:=d/(3*(Sn^2 + (X_bar - T)^2)^0.5):
# Note that c1 = c* = Cpm_hat.
b:=3*C*(1 + ξ^2)^0.5:
G:=(c1,t)->stats[statevalf,cdf,chisquare[n - 1]]
((b^2*n)/(9*c1^2) - t^2):
h:=t->stats[statevalf,pdf,normald](t + ξ*n^0.5) +
stats[statevalf,pdf,normald](t - ξ*n^0.5):
pV:=c1->int(G(c1,t)*h(t),t=0..(b*n^0.5/(3*c1))):
Estimated_Cpm:=c1;
p_Value:=evalf(pV(c1));
```

The output is:

$LSL := 2.40$
$USL := 3.40$
$T := 2.90$
$C := 1.00$
$n := 100$
$X\_bar := 2.825$
$Sn := .125$
$d := .500000000$
$\xi := -.6000000000$
$Estimated\_Cpm := 1.143323901$
$p\_Value := .03376008156$

4 Parameter ξ and the critical value c_0

Since the process parameters μ and σ are unknown, then the distribution characteristic parameter $\xi = (\mu - T)/\sigma$ is also unknown, which has to be estimated in real applications, naturally by substituting μ and σ by $\bar{X}$ and S_n. Such approach introduces additional sampling errors from estimating ξ in calculating the *p*-values as well as in finding the critical values, and certainly would make our approach (and all existing methods) less reliable.

To eliminate the need for estimating the distribution characteristic parameter ξ, we examine the behavior of the *p*-values and the critical values c_0 against the parameter ξ. We find that the *p*-value for $\xi = 0$ is greater than all the *p*-values for other choices of ξ in all cases. Without having to estimate the unknown parameter ξ we can set $\xi = 0$ and calculate the corresponding *p*-value for an estimated index $\hat{C}_{pm} = c^*$ using the program provided in Sect. 3. Decisions made based on the conservative *p*-values are ensured to be more reliable than using the existing approaches. In fact, Zimmer et al. [10] noted that one can provide the largest confidence interval with $\xi = 0$ if one is unsure of the value of ξ. Hence, our result coincides with the confidence interval approach provided by Zimmer et al. [10]. In the example displayed in Sect. 3, if we set $\xi = 0$, then the program gives $\hat{C}_{pm} = 1.1433$ and the corresponding *p*-value as 0.0388 for the same input parameters: $LSL = 2.40$, $USL = 3.40$, $T = 2.90$, $C = 1.00$, $n = 100$, $\bar{X} = 2.825$, and $S_n = 0.125$.

We perform extensive calculations to obtain the critical values c_0 for $\xi = 0(0.05)3.00$, $n = 10(50)300$, $C = 1.00$, 1.33, 1.50, 1.67, 2.00, and $\alpha = 0.01$, 0.025, and 0.05. Noting that parameter values we investigated, $\xi = 0(0.05)3.00$, cover a wide range of applications with process capability $C_{pm} \geq 0$. We find that the critical value c_0 (i) is decreasing in ξ for $\xi \geq 0$, (ii) is decreasing in n, (iii) obtains its maximum at $\xi = 0$ in all cases. Hence, for practical purpose we may solve Eq. 3 with $\xi = 0$ to obtain the required critical values, without having to estimate the parameter ξ. This approach ensures that the decisions made based on those critical values are more reliable than all existing methods.

Figures 1a–5a display the surface plots of c_0 for $0 \leq \xi \leq 3.00$ and $30 \leq n \leq 300$ with $\alpha = 0.05$ for $C = 1.00$, 1.33, 1.50, 1.67, and 2.00, respectively. Figures 1b–5b plot the curves of c_0 ver-

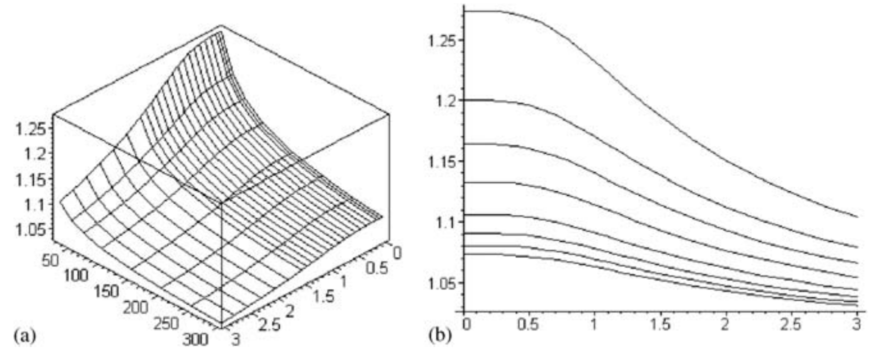

Fig. 1. a Surface plot of c_0 with $0 \leq \xi \leq 3$ and $30 \leq n \leq 300$ for $C_{pm} = 1.00$ and $\alpha = 0.05$ **b** Plots of c_0 versus ξ for $C_{pm} = 1.00$, $\alpha = 0.05$, and n = 30, 50, 70, 100, 150, 200, 250, 300 (top to bottom in plot)

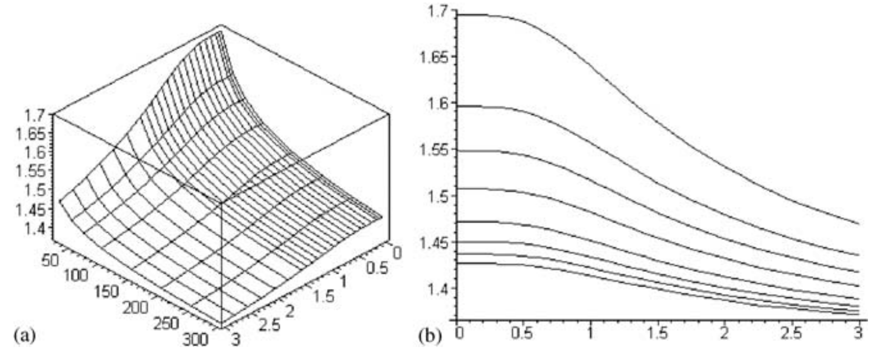

Fig. 2. a Surface plot of c_0 with $0 \leq \xi \leq 3$ and $30 \leq n \leq 300$ for $C_{pm} = 1.33$ and $\alpha = 0.05$ **b** Plots of c_0 versus ξ for $C_{pm} = 1.33$, $\alpha = 0.05$, and n = 30, 50, 70, 100, 150, 200, 250, 300 (top to bottom in plot)

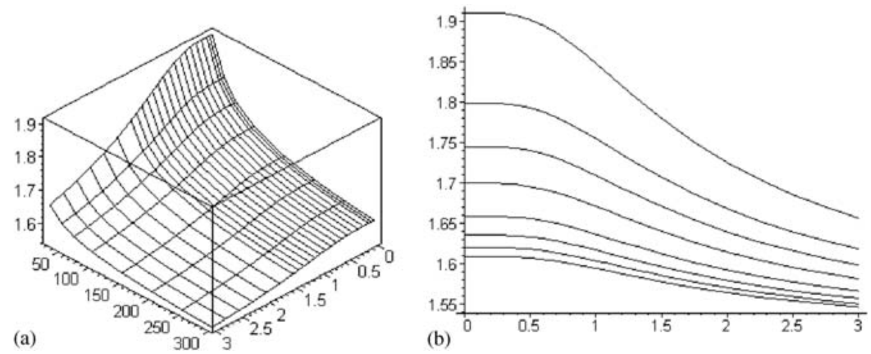

Fig. 3. a Surface plot of c_0 with $0 \leq \xi \leq 3$ and $30 \leq n \leq 300$ for $C_{pm} = 1.50$ and $\alpha = 0.05$ **b** Plots of c_0 versus ξ for $C_{pm} = 1.50$, $\alpha = 0.05$, and n = 30, 50, 70, 100, 150, 200, 250, 300 (top to bottom in plot)

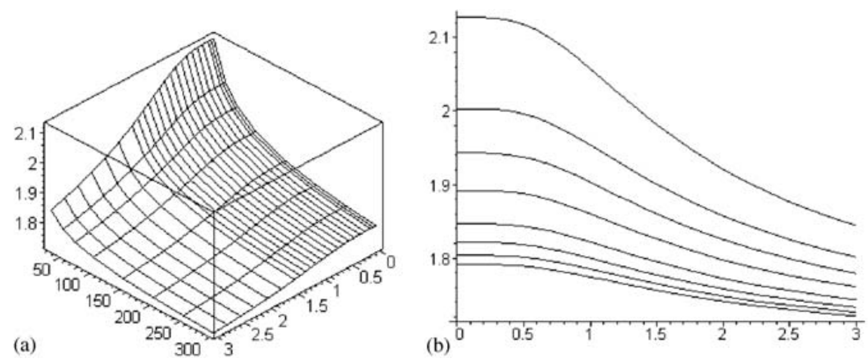

Fig. 4. a Surface plot of c_0 with $0 \leq \xi \leq 3$ and $30 \leq n \leq 300$ for $C_{pm} = 1.67$ and $\alpha = 0.05$ **b** Plots of c_0 versus ξ for $C_{pm} = 1.67$, $\alpha = 0.05$, and n = 30, 50, 70, 100, 150, 200, 250, 300 (top to bottom in plot)

sus the parameter ξ $(0 \leq \xi \leq 3.00)$ for sample size n = 30, 50, 70, 100(50)300 from top to bottom in plots, and C = 1.00, 1.33, 1.50, 1.67, 2.00 with $\alpha = 0.05$. We also investigated the behavior of the critical values c_0 versus the sample size n for other commonly used values of α (0.025 and 0.01), the results appear to be the same.

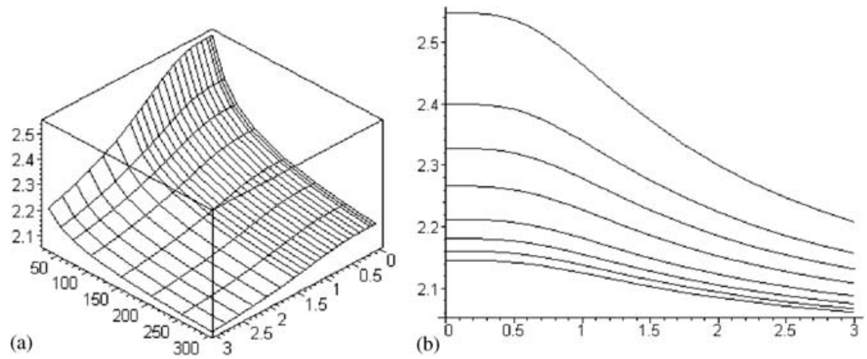

Fig. 5. a Surface plot of c_0 with $0 \le \xi \le 3$ and $30 \le n \le 300$ for $C_{pm} = 2.00$ and $\alpha = 0.05$ **b** Plots of c_0 versus ξ for $C_{pm} = 2.00$, $\alpha = 0.05$, and n = 30, 50, 70, 100, 150, 200, 250, 300 (top to bottom in plot)

5 Testing procedure and an application

The results obtained in last section show that we can set the parameter $\xi = (\mu - T)/\sigma$ to 0 for all cases in calculating the *p*-values and the critical values. Tables 1–5 display critical values c_0 for C = 1.00, 1.33, 1.50, 1.67, and 2.00, with sample sizes n = 10(5)405, and α-risk = 0.01, 0.025, 0.05. For practitioners who do not wish to run the proposed computer program to calculate the *p*-value, they could simply compare the estimated index $\hat{C}_{pm}$ with the corresponding critical value c_0 from the provided tables. To judge if a given process meets the capability requirement, we first determine the value of C (the capability requirement) and the α-risk. Checking the appropriate table from Tables 1–5, we may obtain the critical value c_0 based on given values of α-risk, C, and the sample size n. If the estimated value $\hat{C}_{pm}$ is greater than the critical value c_0 ($\hat{C}_{pm} > c_0$), then we conclude that the process meets the capability requirement ($C_{pm} > C$). Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In this case, we would believe that $C_{pm} \le C$.

Procedure:

STEP 1: Decide the definition of "capable" (set the value of C), and the α-risk (normally set to 0.01, 0.025, or 0.05), the chance of wrongly concluding an incapable process as capable.

STEP 2: Calculate the value of $\hat{C}_{pm}$ from the sample.

STEP 3: Check the appropriate table in Appendix, and find the critical value c_0 based on C, α-risk and n.

STEP 4: Conclude that the process is capable ($C_{pm} > C$) if $\hat{C}_{pm}$ value is greater than the critical value c_0 ($\hat{C}_{pm} > c_0$). Otherwise, we do not have enough information to conclude that the process is capable.

As an example, we consider a process manufacturing the chip resistors. Chip resistor is an electronic passive component commonly used on electronic circuits, providing the function of reducing the voltage, current and releasing the heat. We consider the following case taken from a factory located on an industrial park in Taiwan making chip resistors. One of the product characteristics is justified to be under statistical control, which is distributed as the normal distribution. The specification limits are set to LSL = 31.5Ω, T = 35.0Ω, and USL = 38.5Ω. The capability requirement in the company was set to C = 1.00 (capable). We note that d = (USL − LSL)/2 = 3.50. To test if the process meets the capability (quality) requirement, we must determine whether the process meets $C_{pm} > 1.00$. The α-risk is set to be 0.05. Thus, the chance of wrongly concluding an incapable process as capable is no greater than 5%. We take a sample of size n = 100, the sample mean is calculated as $\bar{X} = 35.58$, and the sample variance is calculated as $S_n^2 = 0.56$. We further calculate $\hat{C}_{pm} = d/\{3[S_n^2 + (\bar{X} - T)^2]^{1/2}\} = 1.232$. Checking Table 1 in the Appendix, we find the critical value $c_0 = 1.133$ based on C = 1.00, α-risk = 0.05 and n = 100. Because $\hat{C}_{pm} = 1.232$ is greater than the critical value $c_0 = 1.133$, we therefore conclude that the process meets the present capability requirement ($C_{pm} > 1.00$) and runs under the desired quality condition.

6 Conclusion

The Taguchi index C_{pm} has been proposed to the manufacturing industry to measure process performance. But all existing methods for testing C_{pm} require further estimation of the distribution parameters when calculating the *p*-values and critical values causing additional sampling errors, which is unreliable. In this paper, an efficient Maple computer program is provided to calculate the *p*-values. Extensive calculations were performed to examine the behavior of the *p*-values and the critical values c_0 against the distribution parameter $\xi = (\mu - T)/\sigma$. We find that the *p*-value for $\xi = 0$ is greater than all the *p*-values for other choices of ξ in all cases. Without having to estimate the unknown parameter ξ we can set $\xi = 0$ and calculate the corresponding *p*-value for an estimated index $\hat{C}_{pm} = c^*$ using the provided Maple program. At the same time, we find the critical value c_0 obtains its maximum at $\xi = 0$ in all cases. Hence, for practical purpose we may obtain the required critical values setting $\xi = 0$, without having to estimate the parameter ξ. This approach ensures that the decisions made based on those critical values are more reliable than all existing methods. Useful critical values for commonly used capability requirements are also tabulated. A simple but practical step-by-step hypothesis-testing procedure is developed for in-plant applications.

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References

1. Kane VE (1986) Process capability indices. J Qual Tech 18(1):41–52
2. Chan LK, Cheng SW, Spiring FA (1988) A new measure of process capability: C_{pm}. J Qual Tech 20(3):162–173
3. Hsiang TC, Taguchi G (1985) A tutorial on quality control and assurance—the Taguchi methods. ASA Annual Meeting, Las Vegas, NV
4. Kotz S, Johnson NL (1993) Process capability indices. Chapman & Hall, London
5. Kotz S, Lovelace CR (1998) Process capability indices in theory and practice. Arnold, London
6. Pearn WL, Lin PC (2004) Measuring process yield based on capability index C_{pm}. Int J Adv Manuf Technol 24(7):503–508
7. Cheng SW (1994) Practical implementation of the process capability indices. Qual Eng 7:239–259
8. Boyles RA (1991) The Taguchi capability index. J Qual Tech 23(1): 17–26
9. Zimmer LS, Hubele NF (1997) Quantiles of the sampling distribution of C_{pm}. Qual Eng 10:309–329
10. Zimmer LS, Hubele NF, Zimmer WJ (2001) Confidence intervals and sample size determination for C_{pm}. Qual Reliab Eng Int 17:51–68
11. Vännman K (1997) Distribution and moments in simplified form for a general class of capability indices. Comm St-Th 26(1):159–179

Appendix: Critical values for testing C_{pm}

Table 1. Critical values c_0 for $C = 1.00$, $n = 10(5)405$ and $\alpha = 0.01$, 0.025, 0.05

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
10	1.978	1.755	1.594	210	1.128	1.106	1.088
15	1.694	1.548	1.438	215	1.126	1.105	1.087
20	1.557	1.445	1.358	220	1.124	1.104	1.086
25	1.473	1.381	1.309	225	1.123	1.102	1.085
30	1.417	1.337	1.274	230	1.121	1.101	1.084
35	1.376	1.305	1.249	235	1.120	1.100	1.083
40	1.344	1.280	1.229	240	1.119	1.099	1.082
45	1.319	1.260	1.213	245	1.117	1.098	1.081
50	1.298	1.244	1.200	250	1.116	1.097	1.080
55	1.280	1.230	1.189	255	1.115	1.096	1.080
60	1.266	1.218	1.179	260	1.113	1.095	1.079
65	1.253	1.208	1.171	265	1.112	1.094	1.078
70	1.242	1.199	1.164	270	1.111	1.093	1.077
75	1.232	1.191	1.157	275	1.110	1.092	1.076
80	1.223	1.184	1.151	280	1.109	1.091	1.076
85	1.215	1.177	1.146	285	1.108	1.090	1.075
90	1.208	1.171	1.142	290	1.107	1.089	1.074
95	1.201	1.166	1.137	295	1.106	1.088	1.074
100	1.195	1.161	1.133	300	1.105	1.087	1.073
105	1.190	1.157	1.130	305	1.104	1.087	1.072
110	1.185	1.153	1.126	310	1.103	1.086	1.072
115	1.180	1.149	1.123	315	1.102	1.085	1.071
120	1.175	1.145	1.120	320	1.101	1.084	1.070
125	1.171	1.142	1.118	325	1.100	1.84	1.070
130	1.168	1.139	1.115	330	1.100	1.083	1.069
135	1.164	1.136	1.113	335	1.099	1.082	1.069
140	1.161	1.133	1.110	340	1.098	1.82	1.068
145	1.157	1.131	1.108	345	1.097	1.081	1.068
150	1.154	1.128	1.106	350	1.096	1.081	1.067
155	1.151	1.126	1.104	355	1.096	1.080	1.067
160	1.149	1.124	1.102	360	1.095	1.079	1.066
165	1.146	1.121	1.101	365	1.094	1.079	1.066
170	1.144	1.119	1.099	370	1.094	1.078	1.065
175	1.141	1.117	1.098	375	1.093	1.078	1.065
180	1.139	1.116	1.096	380	1.092	1.077	1.064
185	1.137	1.114	1.095	385	1.092	1.077	1.064
190	1.135	1.112	1.093	390	1.091	1.076	1.063
195	1.133	1.111	1.092	395	1.090	1.075	1.063
200	1.131	1.109	1.091	400	1.090	1.075	1.063
205	1.129	1.108	1.089	405	1.089	1.074	1.062

Table 2. Critical values c_0 for $C = 1.33$, $n = 10(5)405$ and $\alpha = 0.01, 0.025, 0.05$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
10	2.630	2.335	2.119	210	1.500	1.471	1.447
15	2.253	2.059	1.912	215	1.497	1.469	1.446
20	2.070	1.921	1.806	220	1.495	1.468	1.444
25	1.959	1.836	1.740	225	1.493	1.466	1.443
30	1.884	1.778	1.694	230	1.491	1.464	1.442
35	1.829	1.735	1.661	235	1.489	1.463	1.440
40	1.787	1.702	1.634	240	1.488	1.461	1.439
45	1.754	1.676	1.613	245	1.486	1.460	1.438
50	1.726	1.654	1.596	250	1.484	1.458	1.437
55	1.703	1.635	1.581	255	1.482	1.457	1.436
60	1.683	1.620	1.568	260	1.481	1.456	1.434
65	1.666	1.606	1.557	265	1.479	1.454	1.433
70	1.651	1.594	1.548	270	1.478	1.453	1.432
75	1.638	1.584	1.539	275	1.476	1.452	1.431
80	1.626	1.574	1.531	280	1.475	1.451	1.430
85	1.616	1.566	1.524	285	1.473	1.449	1.429
90	1.606	1.558	1.518	290	1.472	1.448	1.429
95	1.597	1.551	1.512	295	1.471	1.447	1.428
100	1.589	1.544	1.507	300	1.469	1.446	1.427
105	1.582	1.538	1.502	305	1.468	1.445	1.426
110	1.575	1.533	1.498	310	1.467	1.444	1.425
115	1.569	1.528	1.494	315	1.466	1.443	1.424
120	1.563	1.523	1.490	320	1.464	1.442	1.423
125	1.558	1.519	1.486	325	1.463	1.441	1.423
130	1.553	1.514	1.483	330	1.462	1.440	1.422
135	1.548	1.511	1.480	335	1.461	1.439	1.421
140	1.543	1.507	1.477	340	1.460	1.439	1.420
145	1.539	1.503	1.474	345	1.459	1.438	1.420
150	1.535	1.500	1.471	350	1.458	1.437	1.419
155	1.531	1.497	1.469	355	1.457	1.436	1.418
160	1.528	1.494	1.466	360	1.456	1.435	1.418
165	1.524	1.491	1.464	365	1.455	1.435	1.417
170	1.521	1.489	1.462	370	1.454	1.434	1.417
175	1.518	1.486	1.460	375	1.453	1.433	1.416
180	1.515	1.484	1.458	380	1.453	1.432	1.415
185	1.512	1.481	1.456	385	1.452	1.432	1.415
190	1.509	1.479	1.454	390	1.451	1.431	1.414
195	1.507	1.477	1.452	395	1.450	1.430	1.414
200	1.504	1.475	1.450	400	1.449	1.430	1.413
205	1.502	1.473	1.449	405	1.448	1.429	1.412

Table 3. Critical values c_0 for $C = 1.50$, $n = 10(5)405$ and $\alpha = 0.01, 0.025, 0.05$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
10	2.966	2.633	2.390	210	1.691	1.659	1.632
15	2.541	2.322	2.156	215	1.689	1.657	1.631
20	2.335	2.167	2.037	220	1.686	1.655	1.629
25	2.210	2.071	1.963	225	1.684	1.653	1.627
30	2.125	2.006	1.911	230	1.682	1.651	1.626
35	2.063	1.957	1.873	235	1.680	1.650	1.624
40	2.016	1.920	1.843	240	1.678	1.648	1.623
45	1.978	1.890	1.819	245	1.676	1.646	1.622
50	1.947	1.865	1.799	250	1.674	1.645	1.620
55	1.920	1.844	1.783	255	1.672	1.643	1.619
60	1.898	1.827	1.769	260	1.670	1.642	1.618
65	1.879	1.811	1.756	265	1.668	1.640	1.617
70	1.862	1.798	1.745	270	1.666	1.639	1.615
75	1.847	1.786	1.736	275	1.665	1.637	1.614
80	1.834	1.775	1.727	280	1.663	1.636	1.613
85	1.822	1.766	1.719	285	1.662	1.635	1.612
90	1.811	1.757	1.712	290	1.660	1.633	1.611
95	1.802	1.749	1.706	295	1.659	1.632	1.610
100	1.793	1.742	1.700	300	1.657	1.631	1.609
105	1.784	1.735	1.694	305	1.656	1.630	1.608
110	1.777	1.729	1.689	310	1.654	1.629	1.607
115	1.770	1.723	1.685	315	1.653	1.628	1.606
120	1.763	1.718	1.680	320	1.652	1.626	1.605
125	1.757	1.713	1.676	325	1.650	1.625	1.604
130	1.751	1.708	1.672	330	1.649	1.624	1.604
135	1.746	1.704	1.669	335	1.648	1.623	1.603
140	1.741	1.699	1.665	340	1.647	1.622	1.602
145	1.736	1.696	1.662	345	1.646	1.621	1.601
150	1.731	1.692	1.659	350	1.644	1.621	1.600
155	1.727	1.688	1.656	355	1.643	1.620	1.600
160	1.723	1.685	1.653	360	1.642	1.619	1.599
165	1.719	1.682	1.651	365	1.641	1.618	1.598
170	1.715	1.679	1.648	370	1.640	1.617	1.597
175	1.712	1.676	1.646	375	1.639	1.616	1.597
180	1.709	1.673	1.644	380	1.638	1.615	1.596
185	1.705	1.671	1.642	385	1.637	1.615	1.595
190	1.702	1.668	1.640	390	1.636	1.614	1.595
195	1.699	1.666	1.638	395	1.635	1.613	1.594
200	1.697	1.663	1.636	400	1.634	1.612	1.594
205	1.694	1.661	1.634	405	1.633	1.611	1.593

Table 4. Critical values c_0 for $C = 1.67$, $n = 10(5)405$ and $\alpha = 0.01, 0.025, 0.05$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
10	3.302	2.931	2.661	210	1.883	1.847	1.817
15	2.829	2.585	2.401	215	1.880	1.845	1.815
20	2.599	2.412	2.268	220	1.877	1.843	1.813
25	2.460	2.306	2.185	225	1.875	1.840	1.812
30	2.366	2.233	2.128	230	1.872	1.838	1.810
35	2.297	2.179	2.085	235	1.870	1.836	1.808
40	2.244	2.137	2.052	240	1.868	1.835	1.807
45	2.202	2.104	2.025	245	1.865	1.833	1.805
50	2.167	2.076	2.003	250	1.863	1.831	1.804
55	2.138	2.053	1.985	255	1.861	1.829	1.802
60	2.113	2.034	1.969	260	1.859	1.828	1.801
65	2.092	2.017	1.955	265	1.857	1.826	1.800
70	2.073	2.001	1.943	270	1.855	1.824	1.798
75	2.057	1.988	1.932	275	1.853	1.823	1.797
80	2.042	1.976	1.923	280	1.852	1.821	1.796
85	2.029	1.966	1.914	285	1.850	1.820	1.795
90	2.017	1.956	1.906	290	1.848	1.818	1.794
95	2.006	1.947	1.899	295	1.846	1.817	1.792
100	1.996	1.939	1.892	300	1.845	1.816	1.791
105	1.986	1.931	1.886	305	1.843	1.814	1.790
110	1.978	1.925	1.881	310	1.842	1.813	1.789
115	1.970	1.918	1.875	315	1.840	1.812	1.788
120	1.963	1.912	1.870	320	1.839	1.811	1.787
125	1.956	1.907	1.866	325	1.837	1.810	1.786
130	1.949	1.901	1.862	330	1.836	1.808	1.785
135	1.943	1.897	1.858	335	1.835	1.807	1.784
140	1.938	1.892	1.854	340	1.833	1.806	1.783
145	1.932	1.888	1.850	345	1.832	1.805	1.783
150	1.927	1.883	1.847	350	1.831	1.804	1.782
155	1.923	1.880	1.844	355	1.829	1.803	1.781
160	1.918	1.876	1.841	360	1.828	1.802	1.780
165	1.914	1.872	1.838	365	1.827	1.801	1.779
170	1.910	1.869	1.835	370	1.826	1.800	1.778
175	1.906	1.866	1.833	375	1.825	1.799	1.778
180	1.902	1.863	1.830	380	1.824	1.798	1.777
185	1.899	1.860	1.828	385	1.823	1.797	1.776
190	1.895	1.857	1.825	390	1.822	1.797	1.775
195	1.892	1.854	1.823	395	1.821	1.796	1.775
200	1.889	1.852	1.821	400	1.819	1.795	1.774
205	1.886	1.849	1.819	405	1.818	1.794	1.773

Table 5. Critical values c_0 for $C = 2.00$, $n = 10(5)405$ and $\alpha = 0.01, 0.025, 0.05$

n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$	n	$\alpha = 0.01$	$\alpha = 0.025$	$\alpha = 0.05$
10	3.955	3.510	3.187	210	2.255	2.212	2.176
15	3.388	3.096	2.875	215	2.252	2.209	2.174
20	3.113	2.889	2.716	220	2.248	2.207	2.172
25	2.946	2.761	2.617	225	2.245	2.204	2.170
30	2.833	2.674	2.548	230	2.242	2.202	2.168
35	2.751	2.609	2.497	235	2.239	2.199	2.166
40	2.687	2.560	2.457	240	2.237	2.197	2.164
45	2.637	2.520	2.425	245	2.234	2.195	2.162
50	2.595	2.487	2.399	250	2.231	2.193	2.160
55	2.560	2.459	2.377	255	2.229	2.191	2.159
60	2.531	2.435	2.358	260	2.226	2.189	2.157
65	2.505	2.415	2.341	265	2.224	2.187	2.155
70	2.483	2.397	2.327	270	2.222	2.185	2.154
75	2.463	2.381	2.314	275	2.219	2.183	2.152
80	2.445	2.367	2.302	280	2.217	2.181	2.151
85	2.429	2.354	2.292	285	2.215	2.179	2.149
90	2.415	2.342	2.283	290	2.213	2.178	2.148
95	2.402	2.332	2.274	295	2.211	2.176	2.147
100	2.390	2.322	2.266	300	2.209	2.174	2.145
105	2.379	2.313	2.259	305	2.207	2.173	2.144
110	2.369	2.305	2.252	310	2.206	2.171	2.143
115	2.359	2.297	2.246	315	2.204	2.170	2.141
120	2.350	2.290	2.240	320	2.202	2.168	2.140
125	2.342	2.283	2.235	325	2.200	2.167	2.139
130	2.335	2.277	2.229	330	2.199	2.166	2.138
135	2.327	2.271	2.225	335	2.197	2.164	2.137
140	2.321	2.266	2.220	340	2.195	2.163	2.136
145	2.314	2.261	2.216	345	2.194	2.162	2.135
150	2.308	2.256	2.212	350	2.192	2.161	2.134
155	2.302	2.251	2.208	355	2.191	2.159	2.133
160	2.297	2.247	2.204	360	2.189	2.158	2.132
165	2.292	2.242	2.201	365	2.188	2.157	2.131
170	2.287	2.238	2.198	370	2.187	2.156	2.130
175	2.282	2.234	2.195	375	2.185	2.155	2.129
180	2.278	2.231	2.192	380	2.184	2.154	2.128
185	2.274	2.227	2.189	385	2.183	2.153	2.127
190	2.270	2.224	2.186	390	2.181	2.151	2.126
195	2.266	2.221	2.183	395	2.180	2.150	2.125
200	2.262	2.218	2.181	400	2.179	2.149	2.125
205	2.258	2.215	2.178	405	2.178	2.148	2.124