Generalization of the Lunin-Maldacena transformation on the $AdS_5 \times S^5$ background

R. C. Rashkov*
Department of Physics, Sofia University, 1164 Sofia, Bulgaria

K. S. Viswanathan†
Department of Physics, Simon Fraser University, Burnaby BC, Canada

Yi Yang‡
Department of Electrophysics, National Chiao Tung University, Hsinchu, Taiwan
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In this paper we consider a simple generalization of the method of Lunin and Maldacena for generating new string backgrounds based on TsT transformations. We study multishift $Ts\cdots sT$ transformations applied to backgrounds with at least two U(1) isometries. We prove that the string currents in any two backgrounds related by $Ts\cdots sT$ transformations are equal. Applying this procedure to the $AdS_5 \times S^5$, we find a new background and study some properties of the semiclassical strings.

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I. INTRODUCTION

In this paper we present a simple generalization of the method for obtaining deformed string backgrounds proposed by Lunin and Maldacena [1] and developed in detail by Frolov [2]. The method in the above papers is based on T-duality on one of the U(1) variables, shift of another U(1) variable, and T-duality back on the first U(1) variable (called TsT transformation).[1] Our method consists in multishifts at the second step which allows one to obtain new string backgrounds (we call this $Ts_1\cdots s_nT$ transformation). We prove also that the U(1) string currents in any two backgrounds related by $Ts_1\cdots s_nT$ transformations are equal. We present also an application of our method to string theory in $AdS_5 \times S^5$ background.

In the past few years, the main efforts in string theory were directed towards establishing string/gauge theory correspondence. The vast majority of papers were on qualitative and quantitative descriptions of $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with a SU(N) gauge group by making use of the string sigma model on $AdS_5 \times S^5$ [3–5]. The AdS/CFT correspondence implies that the energy of closed string states is equal to the anomalous dimensions of certain local SYM operators [6,7]. At supergravity level this correspondence has been checked in a number of cases (for review, see for instance [8]) but the match between the string energy and the anomalous dimensions beyond that approximation still remains a challenge.

The first important step in establishing AdS/CFT correspondence is to obtain the spectrum of the anomalous dimensions of the primary operators made up of local gauge fields. On the string theory side, it requires one not only to solve the theory at the classical level but also include its quantization.

The main challenge in quantizing string theory is that it is highly nonlinear and thus difficult to manage. The only option available so far is to look at the semiclassical region of large quantum numbers where the results are reliable. On the gauge theory side, the derivation of the anomalous dimensions is also a difficult task. A breakthrough in this direction has been the observation of Minahan and Zarembo that a one loop dilatation operator restricted to the bosonic sector of $N = 4$ SYM theory can be interpreted as the Hamiltonian of the integrable spin chain [9]. This observation raised the question about the dilatation operator in $N = 4$ SYM theory and integrability (for a recent review, see for instance [10] and references therein).

On the other hand, the question of reduction of the string sigma model to particular integrable systems and the question of integrability of string theory at the classical and the quantum level was considered in a number of papers [11–14]. The intensive study of "nearly" Bogomol'nyi-Prasad-Sommerfield saturated (BPS), or Berenstein-Maldacena-Nastase (BMN) type, quantum strings and non-BPS ones gives a remarkable match with the results from the gauge theory side at least at the first few loops [9,10,12–20]. This match however is not a coincidence. In the above papers it was suggested that certain spin chains should describe particular string sectors and thus should allow the comparison to the gauge theory computations. Subsequently, it has been found that the match between string theory and SYM theory in the examples discussed above lies in the Yangian symmetries responsible to a large extent for the integrability on both sides [21,22]. Since in this paper we will consider the string theory side, we refer the reader to the above papers for details on this connection.

*Electronic address: rash@phys.uni-sofia.bg
†Electronic address: kviswana@sfu.ca
‡Electronic address: yiyang@mail.nctu.edu.tw
[1]See the discussion in Sec. II.

From the picture emerging from the above studies, one can conclude that the integrable structures play an important role in establishing the AdS/CFT correspondence at the classical and hopefully at the quantum level as well.

Although we already have some understanding of string/gauge theory correspondence in the case of $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ SYM, much less is known in the case of theories with less than maximal supersymmetry. There have been some studies of AdS/CFT correspondence for less supersymmetric string backgrounds [23–44]. However, it is not quite clear how exactly to implement the correspondence. The main obstacles lie in knowing if and how Kaluza-Klein modes naturally present in such backgrounds contribute to the string energy, which corner in the space of gauge operators is described by these strings, and if these subsectors are closed under the renormalization group flow.

An important step towards a deeper understanding of AdS/CFT correspondence in its less supersymmetric sector was recently given by Lunin and Maldacena [1]. From the gauge theory point of view, the possible deformations of $\mathcal{N} = 4$ SYM gauge theory that break the supersymmetry were studied by Leigh and Strassler [45]. It should be mentioned that the deformations of $\mathcal{N} = 4$ SYM theory and integrable spin chains have been considered in some detail in [46]. In [1] Lunin and Maldacena found the gravity dual to the β-deformations of $\mathcal{N} = 4$ SYM theory studied in [45]. They demonstrated that a certain deformation of the $AdS_5 \times S^5$ background corresponds to a gauge theory with less supersymmetry classified in [45]. This deformation of the string background can be obtained applying two T-dualities accompanied by certain shift parametrized by β (TsT transformation). For real values of β, Frolov obtained the Lax operator for the deformed background which proves the integrability at classical level [2]. String theory in this background was studied in [47,48] and its pp-wave limit was investigated in [49,50]. The β-deformations of more complicated (non)supersymmetric backgrounds was considered also in [51,52].

The aim of this paper is to consider a simple extension of the transformations considered in [1,2] and to prove that, under TsT transformations applied to any background possessing U(1) symmetries, the corresponding currents before and after the transformation are equal.

The paper is organized as follows. In the next section we give a brief review of the β-deformations of the $\mathcal{N} = 4$ gauge theory and its gravity dual. In Sec. III we consider a general background with at least two U(1) isometries. We show that the U(1) currents are equal after $Ts_1 s_2 \cdots s_n T$ transformations where $s_1 \cdots s_n$ means multishifts along the remaining U(1) variables. In the next section, as an example for multishift procedure, we consider $AdS_5 \times S^5$ and find a new background parametrized by two real parameters. We show that the new background reduces to those found in [1,2] when one of the parameters vanishes. We also consider the limit of pointlike string which corresponds actually to the BMN limit. In the concluding section, we comment on the results found in the paper.

II. LUNIN-MALDACENA BACKGROUND

In this section we give a very brief review of the procedure of Lunin and Maldacena for obtaining the gravity dual of the β-deformed SYM theory considered in [45].

Let us consider the $\mathcal{N} = 4$ SYM gauge theory in terms of $\mathcal{N} = 1$ supersymmetry. The theory contains a vector multiplet V and three chiral multiplets Φ^i. The superpotential is given by the expression

$$W = g' \operatorname{Tr}[[\Phi^1, \Phi^2]\Phi^3]. \tag{2.1}$$

The action then can be written as

$$\begin{aligned} S = \operatorname{Tr}\bigg\{ & \int d^4x d^4\theta e^{-gV} \bar{\Phi}_i e^{gV} \Phi^i \\ & + \frac{1}{2g^2}\bigg[\int d^4x d^2\theta W^\alpha W_\alpha + \text{c.c.}\bigg]\frac{g'}{3!} \\ & \times \bigg[\int d^4 d^2\theta \varepsilon_{ijk}\Phi^i[\Phi^j, \Phi^k] + \text{c.c.}\bigg]\bigg\}. \end{aligned} \tag{2.2}$$

We note that the $\mathcal{N} = 4$ theory is conformal at any value of the complex coupling

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g_{YM}^2} \tag{2.3}$$

and the deformations that change this value are exactly marginal.

In [45] Leigh and Strassler considered deformations of the superpotential of the form

$$\begin{aligned} W = {} & h \operatorname{Tr}[e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2] \\ & + h' \operatorname{Tr}[\Phi_1^3 + \Phi_2^3 + \Phi_3^3]. \end{aligned} \tag{2.4}$$

Let us focus on the $h' = 0$ case. The symmetries are: one U(1) R-symmetry group and two global U(1) $\times$ U(1) groups acting as follows:

$$\begin{aligned} \mathrm{U}(1)_1: \quad & (\Phi_1, \Phi_2\Phi_3) \to (\Phi_1, e^{i\varphi_1}\Phi_2, e^{-i\varphi_1}\Phi_3) \\ \mathrm{U}(1)_2: \quad & (\Phi_1, \Phi_2\Phi_3) \to (e^{-i\varphi_2}\Phi_1, e^{i\varphi_2}\Phi_2, \Phi_3). \end{aligned} \tag{2.5}$$

Since the theory is periodic in β, one can think of β as living on a torus with complex structure τ_s and the SL(2, Z) duality group acts on it and β as follows:

$$\begin{gathered} \tau_s \to \frac{a\tau_s + b}{c\tau_s + d}; \qquad \beta \to \frac{\beta}{c\tau_s + d} \\ \beta \sim \beta + 1 \sim \beta + \tau_s. \end{gathered} \tag{2.6}$$

As a result of all of this, we end up with a $\mathcal{N} = 1$ supersymmetric conformal field theory.

The gravity dual for real β can be obtained in three steps [2]. Consider the S^5 part of the $AdS_5 \times S^5$ background. In the first step, we perform a T-duality with respect to one of the U(1) isometries parametrized by the angle φ_1.[2] The second step consists in performing a shift $\varphi_2 \to \varphi_2 + \gamma\varphi_1$ where φ_2 parametrizes another U(1) isometry of the background and γ is a real parameter. In the last step we T-dualize back on φ_1. The resulting geometry is described by

$$ds^2_{str} = R^2\left[ds^2_{AdS_5} + \sum(dr_i^2 + Gr_i^2 d\phi_i^2) + \tilde{\gamma}^2 r_1^2 r_2^2 r_3^2\left(\sum d\phi_i\right)^2\right], \tag{2.7}$$

where

$$G^{-1} = 1 + \gamma^2(r_1^2r_2^2 + r_2^2r_3^2 + r_1^2r_3^2); \qquad \tilde{\gamma} = R^2\gamma. \tag{2.8}$$

The other fields are correspondingly[3]

$$e^{2\phi} = e^{2\phi_0}G, \tag{2.9}$$

$$B^{NS} = \tilde{\gamma}^2 R^2 G(r_1^2r_2^2 d\phi_1 d\phi_2 + r_2^2r_3^2 d\phi_2 d\phi_3 + r_3^2r_1^2 d\phi_3 d\phi_1), \tag{2.10}$$

$$C_2 = -3\gamma(16\pi N)w_1 d\psi, \tag{2.11}$$

$$C_4 = (16\pi N)w_4 + Gw_1 d\phi_1 d\phi 2 d\phi_3), \tag{2.12}$$

$$F_5 = (16\pi N)(w_{AdS_5} + Gw_{S^5}). \tag{2.13}$$

Using the fact that the currents J_α before the TsT transformations are equal to the currents $\tilde{J}_\alpha$ after the transformations, Frolov obtained the Lax operator for the deformed geometry, thus proving the integrability at the classical level. The properties of string theory in this background were further studied in [47,48]. The Penrose limit of the Lunin-Maldacena background was investigated in [49,50].

III. U(1) CURRENTS AND TST TRANSFORMATION

As mentioned in the previous section, based on the observation that the string U(1) currents before and after TsT transformation are equal, Frolov was able to obtain the Lax operator of the theory in the deformed background. He also conjectured that the equality of the currents holds for any two backgrounds related by TsT transformation. Below we prove the following.

Proposition.—The U(1) currents of strings in any two backgrounds related by TsT transformation are equal.

We start with the general action

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}]. \tag{3.1}$$

We will assume that $G_{\mu\nu}$ and $B_{\mu\nu}$ do not depend on X^1 and X^2 allowing to perform TsT transformation.

In what follows we use the notations $\mu = 1, \ldots, d$, $i = 2, \ldots, d$, $a = 3, \ldots, d$. We will prove the statement in several steps.

Step 1: T-duality on X^1.

For completeness we write again the T-duality rules and relations[4]

$$\tilde{G}_{11} = \frac{1}{G_{11}}, \qquad \tilde{G}_{ij} = G_{ij} - \frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}},$$
$$\tilde{G}_{1i} = \frac{B_{1i}}{G_{11}}, \qquad \tilde{B}_{ij} = B_{ij} - \frac{G_{1i}B_{1j} - B_{1i}G_{1j}}{G_{11}}, \tag{3.2}$$
$$\tilde{B}_{1i} = \frac{G_{1i}}{G_{11}},$$

$$\epsilon^{\alpha\beta}\partial_\beta \tilde{X}^1 = \gamma^{\alpha\beta}\partial_\beta X^M G_{1M} - \epsilon^{\alpha\beta}\partial_\beta X^M B_{1M}, \tag{3.3}$$

$$\partial_\alpha \tilde{X}^1 = \gamma_{\alpha\sigma}\epsilon^{\sigma\rho}\partial_\rho X^\mu G_{1\mu} - \partial_\alpha X^\mu B_{1\mu}, \tag{3.4}$$

$$\partial_\alpha X^1 = \gamma_{\alpha\sigma}\epsilon^{\sigma\rho}\partial_\rho \tilde{X}^\mu \tilde{G}_{1\mu} - \partial_\alpha \tilde{X}^\mu \tilde{B}_{1\mu}, \tag{3.5}$$

$$\tilde{X}^i = X^i. \tag{3.6}$$

The T-dual action has the same form but with transformed background fields:

$$\tilde{S} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}\partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu \tilde{G}_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu \tilde{B}_{\mu\nu}]. \tag{3.7}$$

Step 2 consists in shift of $\tilde{X}^2$

$$\tilde{X}^2 = \tilde{x}^2 + \hat{\gamma}\tilde{x}^1, \qquad \tilde{X}^1 = \tilde{x}^1, \qquad \tilde{X}^a = \tilde{x}^a. \tag{3.8}$$

Note that the background remains independent of $\tilde{X}^1$ and $\tilde{X}^2$.

The shift described above produces the following transformations of the metric

$$\tilde{g}_{11} = \tilde{G}_{11} + 2\hat{\gamma}\tilde{G}_{12} + \hat{\gamma}^2\tilde{G}_{22},$$
$$\tilde{g}_{1i} = \tilde{G}_{1i} + \hat{\gamma}\tilde{G}_{2i}, \qquad \tilde{g}_{ij} = \tilde{G}_{ij}, \tag{3.9}$$

and for the $\tilde{B}_{\mu\nu}$ we get

$$\tilde{b}_{ij} = \tilde{B}_{ij}, \qquad \tilde{b}_{1i} \to \tilde{B}_{1i} + \hat{\gamma}\tilde{B}_{2i}. \tag{3.10}$$

The relations (3.3), (3.4), and (3.5) are also changed; for

[2]See the appendix for general U(1) T-duality.
[3]See [1,2] for details.
[4]See also the appendix.

instance, (3.5) becomes

$$\partial_\alpha X^1 = \gamma_{\alpha\sigma}\epsilon^{\sigma\rho}\partial_\rho \tilde{x}^\mu \tilde{G}_{1\mu} - \partial_\alpha \tilde{x}^\mu \tilde{B}_{1\mu} + \hat{\gamma}\gamma_{\alpha\sigma}\epsilon^{\sigma\rho}\partial_\rho \tilde{x}^1 \tilde{G}_{12} - \hat{\gamma}\partial_\alpha \tilde{x}^1 \tilde{B}_{12}. \tag{3.11}$$

Note that it is crucial that the background is independent of X^1 and X^2, otherwise we cannot perform a T-duality back on $\tilde{x}_1$.

In the new variables the action is given by

$$\tilde{\tilde{S}} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}\partial_\alpha \tilde{x}^\mu \partial_\beta \tilde{x}^\nu \tilde{g}_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha \tilde{x}^\mu \partial_\beta \tilde{x}^\nu \tilde{b}_{\mu\nu}]. \tag{3.12}$$

In step 3 we T-dualize back on $\tilde{x}^1$.

The action again has the standard form

$$\tilde{\tilde{\tilde{S}}} = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu b_{\mu\nu}], \tag{3.13}$$

where $g_{\mu\nu}$ and $b_{\mu\nu}$ are obtained from $\tilde{g}_{\mu\nu}$ and $\tilde{b}_{\mu\nu}$ by making use of the standard rule equations (3.2), (3.3), (3.4), and (3.5).

Now we will prove that the currents J^α_μ and j^α_μ obtained from (3.1) and (3.13) respectively are equal, i.e.,

$$J^\alpha_\mu = j^\alpha_\mu, \tag{3.14}$$

where

$$j^\alpha_\mu = -\sqrt{\lambda}\gamma^{\alpha\beta}\partial_\beta x^\nu g_{\mu\nu} + \sqrt{\lambda}\epsilon^{\alpha\beta}\partial_\beta x^\nu b_{\mu\nu}, \tag{3.15}$$

$$J^\alpha_\mu = -\sqrt{\lambda}\gamma^{\alpha\beta}\partial_\beta x^\nu G_{\mu\nu} + \sqrt{\lambda}\epsilon^{\alpha\beta}\partial_\beta x^\nu B_{\mu\nu}. \tag{3.16}$$

We will prove the statement directly, but in two steps.

(a) First we will prove the equality (3.14) for J^α_1 and j^α_1 and then for J^α_i and j^α_i

$$\begin{aligned}\frac{j^\alpha_1}{-\sqrt{\lambda}} &= \gamma^{\alpha\beta}\partial_\beta x^1 g_{11} + \gamma^{\alpha\beta}\partial_\beta x^i g_{1i} - \epsilon^{\alpha\beta}\partial_\beta x^i b_{1i} \\ &= \gamma^{\alpha\beta}\partial_\beta x^1 g_{11} + \gamma^{\alpha\beta}\partial_\beta \tilde{x}^i g_{1i} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^i b_{1i} \\ &= \frac{\gamma^{\alpha\beta}}{\tilde{g}_{11}}(\gamma_{\beta\sigma}\epsilon^{\sigma\rho}\partial_\rho \tilde{x}^\mu \tilde{g}_{1\mu} - \partial_\beta \tilde{x}^\mu \tilde{b}_{1\mu}) \\ &\quad + \gamma^{\alpha\beta}\partial_\beta \tilde{x}^i \frac{\tilde{b}_{1i}}{\tilde{g}_{11}} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^i \frac{\tilde{g}_{1i}}{\tilde{g}_{11}} \\ &= \gamma^{\alpha\beta}\gamma_{\beta\sigma}\epsilon^{\sigma\rho}\partial_\rho \tilde{x}^\mu \frac{\tilde{g}_{1\mu}}{\tilde{g}_{11}} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^i \frac{\tilde{g}_{1i}}{\tilde{g}_{11}} \\ &= \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^1.\end{aligned} \tag{3.17}$$

Now we use (3.3) and find

$$\frac{j^\alpha_1}{-\sqrt{\lambda}} = \gamma^{\alpha\beta}\partial_\beta X^\mu G_{1\mu} - \epsilon^{\alpha\beta}\partial_\beta X^\mu B_{1\mu} = \frac{J^\alpha_1}{-\sqrt{\lambda}}. \tag{3.18}$$

(b) We turn now to the case of J^α_i and j^α_i ($i = 2, \ldots, d$). In this case there are more transformations to be performed but all of them are based on (3.2), (3.3), (3.4), and (3.5)

$$\begin{aligned}\frac{j^\alpha_i}{-\sqrt{\lambda}} &= \gamma^{\alpha\beta}\partial_\beta x^\mu g_{i\mu} - \epsilon^{\alpha\beta}\partial_\beta x^\mu b_{i\mu} \\ &= \gamma^{\alpha\beta}\partial_\beta x^1 g_{i1} + \gamma^{\alpha\beta}\partial_\beta \tilde{x}^j g_{ij} - \epsilon^{\alpha\beta}\partial_\beta x^1 b_{1i} \\ &\quad - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^j b_{ij} \\ &= \gamma^{\alpha\beta}\partial_\beta \tilde{x}^1 \tilde{g}_{i1} + \gamma^{\alpha\beta}\partial_\beta \tilde{x}^j \tilde{g}_{ij} + \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^1 \tilde{b}_{1i} \\ &\quad - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^j \tilde{b}_{ij}.\end{aligned} \tag{3.19}$$

Now we go to the $\tilde{X}^\mu$ variables by making the inverse shift

$$\frac{j^\alpha_i}{-\sqrt{\lambda}} = \gamma^{\alpha\beta}\partial_\beta \tilde{X}^\mu \tilde{G}_{i\mu} - \epsilon^{\alpha\beta}\partial_\beta \tilde{X}^\mu \tilde{B}_{i\mu}. \tag{3.20}$$

Since $\tilde{X}^i = X^i$, we separate $\tilde{X}^1$ and $\tilde{X}^i$ dependent parts and find

$$\frac{j^\alpha_i}{-\sqrt{\lambda}} = \gamma^{\alpha\beta}\partial_\beta \tilde{X}^1 \tilde{G}_{i1} - \epsilon^{\alpha\beta}\partial_\beta \tilde{X}^1 \tilde{B}_{i1} + \gamma^{\alpha\beta}\partial_\beta \tilde{X}^j \tilde{G}_{ij} - \epsilon^{\alpha\beta}\partial_\beta \tilde{X}^j \tilde{B}_{ij} \tag{3.21}$$

$$= \gamma^{\alpha\beta}\partial_\beta X^1 G_{i1} - \epsilon^{\alpha\beta}\partial_\beta X^1 B_{i1} + \gamma^{\alpha\beta}\partial_\beta X^j G_{ij} - \epsilon^{\alpha\beta}\partial_\beta X^j B_{ij}. \tag{3.22}$$

Therefore

$$\frac{j^\alpha_i}{-\sqrt{\lambda}} = \gamma^{\alpha\beta}\partial_\beta X^\mu G_{i\mu} - \epsilon^{\alpha\beta}\partial_\beta X^\mu B_{i\mu} = \frac{J^\alpha_i}{-\sqrt{\lambda}}, \tag{3.23}$$

which proves the statement (3.14).

IV. $Ts_1 \cdots s_d T$ TRANSFORMATIONS

In this section we make a simple generalization of the TsT transformation. We proceed as follows. First we make a T-duality on X^1 after which the original action

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}] \tag{4.1}$$

becomes

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}\partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu \tilde{G}_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}^\nu \tilde{B}_{\mu\nu}], \tag{4.2}$$

where the tilde variables are defined in (3.2), with the relations (3.4) and (3.6) satisfied.

The second step consists in applying multishifts along the U(1) isometries unaffected by the T-duality in the previous step. This slightly generalizes the Maldacena-Lunin procedure described in the previous section,

$$\tilde{X}^i = \tilde{x}^i + \gamma^i \tilde{x}^1, \qquad \tilde{X}^1 = \tilde{x}^1, \tag{4.3}$$

or $\tilde{\boldsymbol{X}} = \boldsymbol{A}\tilde{\boldsymbol{x}}$ where

$$\tilde{\boldsymbol{X}} = \begin{pmatrix} \tilde{X}^1 \\ \vdots \\ \tilde{X}^N \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ \gamma^2 & 1 & & & \vdots \\ \vdots & & \ddots & & 0 \\ \gamma^N & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{4.4}$$

Under these multishifts the background fields take the form

$$\tilde{g}_{11} = \tilde{G}_{11} + 2\gamma^i \tilde{G}_{1i} + \gamma^i\gamma^j \tilde{G}_{ij}, \qquad \tilde{g}_{1i} = \tilde{G}_{1i} + \gamma^j \tilde{G}_{ij},$$
$$\tilde{g}_{ij} = \tilde{G}_{ij}, \qquad \tilde{b}_{1i} = \tilde{B}_{1i} + \gamma^j \tilde{B}_{ij}, \qquad \tilde{b}_{ij} = \tilde{B}_{ij}. \tag{4.5}$$

The last step consists in T-dualization back on $\tilde{x}^1$. The resulting action is

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu g_{\mu\nu} - \epsilon^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu b_{\mu\nu}]. \tag{4.6}$$

As in the case of TsT transformation, for the generalization described above we prove below.

Proposition.—The U(1) currents of strings in any two backgrounds related by $Ts_1 \cdots S_n T$ transformation are equal.

Proof.—One can first consider j_1^α and using the relations between the variables write them in terms of the original coordinates

$$\begin{aligned}\frac{j_1^\alpha}{-\sqrt{\lambda}} &= \gamma^{\alpha\beta}\partial_\beta x^\gamma g_{1\gamma} - \epsilon^{\alpha\beta}\partial_\beta x^i b_{1i} \\ &= \gamma^{\alpha\beta}\partial_\beta x^1 g_{11} + \gamma^{\alpha\beta}\partial_\beta \tilde{x}^i g_{1i} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^i b_{1i} \\ &= \frac{\gamma^{\alpha\beta}}{\tilde{g}_{11}}(\gamma_{\beta\sigma}\epsilon^{\sigma\rho}\partial_\rho \tilde{x}^\mu \tilde{g}_{1\mu} - \partial_\beta \tilde{x}^\mu \tilde{b}_{1\mu}) \\ &\quad + \gamma^{\alpha\beta}\partial_\beta \tilde{x}^i \frac{\tilde{b}_{1i}}{\tilde{g}_{11}} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^i \frac{\tilde{g}_{1i}}{\tilde{g}_{11}} \\ &= \gamma^{\alpha\beta}\gamma_{\beta\sigma}\epsilon^{\sigma\rho}\partial_\rho \tilde{x}^\mu \frac{\tilde{g}_{1\mu}}{\tilde{g}_{11}} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^i \frac{\tilde{g}_{1i}}{\tilde{g}_{11}}\, \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^1.\end{aligned} \tag{4.7}$$

But

$$\frac{J_1^\alpha}{-\sqrt{\lambda}} = \gamma^{\alpha\beta}\partial_\beta X^\mu G_{1\mu} - \epsilon^{\alpha\beta}\partial_\beta X^\mu B_{1\mu} = \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^1, \tag{4.8}$$

and therefore

$$j_1^\alpha = J_1^\alpha. \tag{4.9}$$

Let us show that this equality is satisfied for the other currents. One can easily show that

$$j_i^\alpha = \tilde{j}_i^\alpha. \tag{4.10}$$

Let us see how $\tilde{j}_i^\alpha$ is related to $\tilde{J}_i^\alpha$

$$\begin{aligned}\frac{j_i^\alpha}{-\sqrt{\lambda}} &= \gamma^{\alpha\beta}\partial_\beta \tilde{x}^\mu \tilde{g}_{i\mu} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^\mu \tilde{b}_{i\mu} \\ &= \gamma^{\alpha\beta}\partial_\beta \tilde{x}^1(\tilde{G}_{1i} + \gamma^j \tilde{G}_{ij}) - \gamma^{\alpha\beta}(\partial_\beta \tilde{x}^j - \gamma^j \partial_\beta \tilde{x}^1)\tilde{G}_{ij}\end{aligned} \tag{4.11}$$

$$+\epsilon^{\alpha\beta}\partial_\beta \tilde{x}^1(\tilde{B}_{1i} + \gamma^j \tilde{B}_{ij}) - \epsilon^{\alpha\beta}(\partial_\beta \tilde{x}^j - \gamma^j \partial_\beta \tilde{x}^1)\tilde{B}_{ij} \tag{4.12}$$

$$= \gamma^{\alpha\beta}\partial_\beta \tilde{x}^\mu \tilde{G}_{1\mu} - \epsilon^{\alpha\beta}\partial_\beta \tilde{x}^\mu \tilde{B}_{1\mu} = \frac{\tilde{J}_i^\alpha}{-\sqrt{\lambda}}. \tag{4.13}$$

Simple calculations now lead to $J_i^\alpha = \tilde{J}_i^\alpha$. This proves that

$$j_i^\alpha = J_i^\alpha. \tag{4.14}$$

Although the proof is straightforward, it may have important consequences. For instance, if the theory in the initial background is integrable, one can study integrability of the second theory by making use of the above relation. We will comment on this issue in the next section.

The equality between the currents in the $AdS_5 \times S^5$ background and its deformation relate the boundary conditions imposed on the fields in the initial and the transformed backgrounds. It remains to examine how the boundary conditions for x^μ and X^μ in our case are related. First we notice that the time component of J_μ^α, i.e. J_μ^0, is just the momentum conjugated to X^μ. The equality of j_μ^0 and J_μ^0 means that the two momenta are equal and constant (due to the isometry). Therefore this property, observed first in [2], continues to hold in the general case of TsT and multishift transformations. To examine the boundary conditions, we will use the relation

$$\partial_\alpha x^1 = \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma \tilde{x}^\mu \tilde{g}_{1\mu} - \partial_\alpha \tilde{x}^\mu \tilde{b}_{1\mu}. \tag{4.15}$$

To simplify the calculation, we choose the conformal gauge for the 2D metric $\gamma_{\alpha\beta} = \text{diag}(-1, 1)$ and $\epsilon^{01} = 1$. Let us compute the boundary conditions for x^1. To do this we need expressions for the metric components $\tilde{g}_{\mu\nu}$ in terms of the original metric $G_{\mu\nu}$. Using the transformation properties, we find

$$\tilde{g}_{11} = \frac{G}{G_{11}}, \tag{4.16}$$

$$\tilde{g}_{1i} = \frac{B_{1i} + \gamma^j(G_{ij}G_{11} - G_{1i}G_{1j} + B_{1i}B_{1j})}{G_{11}}, \tag{4.17}$$

and

$$\tilde{b}_{1i} = \frac{G_{1i} + \gamma^j(B_{ij}G_{11} - G_{1i}B_{1j} + B_{1i}G_{1j})}{G_{11}}, \quad (4.18)$$

where

$$G = 1 + 2\gamma^i B_{1i} + \gamma^i\gamma^j(G_{ij}G_{11} - G_{1i}G_{1j} + B_{1i}B_{1j}) \quad (4.19)$$

(all others are not changed by the shifts and are given in the appendix).

Substituting the above expressions for $\tilde{g}_{\mu\nu}$ and $\tilde{b}_{\mu\nu}$ in (4.15) and using the inverse transformations relating $\tilde{x}^\mu$ withX^μ, we find

$$\partial_1 x^1 = \partial_1 X^1 - \gamma^i J_i^0, \qquad i = 2, \ldots, N. \quad (4.20)$$

The boundary conditions for the other coordinates are easily obtained from

$$\partial_\alpha x^i = \partial_\alpha \tilde{x}^i - \gamma^i \partial_\alpha \tilde{x}^1. \quad (4.21)$$

Using the relation (3.11) and (4.3), we get

$$\partial_1 x^i = \partial_1 X^i + \gamma^i(\partial_0 x^\mu G_{1\mu} + \partial_1 x^j B_{1j}) = \partial_1 x^i + \gamma^i J_1^0. \quad (4.22)$$

Therefore, the boundary conditions for the fields in the deformed background are twisted as follows:

$$\partial_1 x^1 = \partial_1 X^1 - \gamma^i J_i^0, \quad (4.23)$$

$$\partial_1 x^i = \partial_1 X^i + \gamma^i J_1^0. \quad (4.24)$$

Integrating over σ we find

$$x^1(2\pi) - x^1(0) = 2\pi(n_1 - \gamma^i J_i), \quad (4.25)$$

$$x^i(2\pi) - x^i(0) = 2\pi(n_i + \gamma^i J_1), \quad (4.26)$$

where

$$X^\mu(2\pi) - X^\mu(0) = 2\pi n_\mu, \quad (4.27)$$

and the current

$$J_\mu = \int \frac{d\sigma}{2\pi} J_\mu^0. \quad (4.28)$$

In the next section we will apply these results to the $AdS_5 \times S^5$ background and analyze the implications of these transformations to string theory.

V. $(\hat{\gamma}_2, \hat{\gamma}_3)$-DEFORMATION

A. Supergravity solution

We start with the S^5 part of string action as in [2] with $i = 1, 2, 3$:

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}(\partial_\alpha r_i \partial_\beta r_i + g_{ij}\partial_\alpha \tilde{\tilde{\varphi}}_i \partial_\beta \tilde{\tilde{\varphi}}_j) + \Lambda(r_i^2 - 1)], \quad (5.1)$$

where the metric g_{ij} and the antisymmetric 2-form field b_{ij} are

$$g_{11} = r_2^2 + r_3^2, \qquad g_{22} = r_1^2 + r_2^2, \qquad g_{33} = 1,$$
$$g_{12} = r_2^2, \qquad g_{13} = r_2^2 - r_3^2, \qquad g_{23} = r_2^2 - r_1^2, \quad (5.2)$$
$$b_{ij} = 0.$$

and Λ is a Lagrangian multiplier which ensures the constraint

$$\sum r_i^2 = 1. \quad (5.3)$$

This action is related to the one used in [1] by the following change of the variables:

$$\tilde{\tilde{\varphi}}_1 = \tfrac{1}{3}(\hat{\varphi}_1 + \hat{\varphi}_2 - 2\hat{\varphi}_3),$$
$$\tilde{\tilde{\varphi}}_2 = \tfrac{1}{3}(-2\hat{\varphi}_1 + \hat{\varphi}_2 + \hat{\varphi}_3), \qquad \tilde{\tilde{\varphi}}_3 = \tfrac{1}{3}(\hat{\varphi}_1 + \hat{\varphi}_2 + \hat{\varphi}_3), \quad (5.4)$$

which leads to the following relations between the old and new angular momenta:

$$\tilde{\tilde{J}}_1 = \hat{J}_2 - \hat{J}_3, \quad (5.5)$$

$$\tilde{\tilde{J}}_2 = \hat{J}_2 - \hat{J}_1, \quad (5.6)$$

$$\tilde{\tilde{J}}_3 = \hat{J}_1 + \hat{J}_2 + \hat{J}_3. \quad (5.7)$$

We next make the T-duality transformation on the circle parametrized by φ_1; the action becomes

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}(\partial_\alpha r_i \partial_\beta r_i + \tilde{g}_{ij}\partial_\alpha \tilde{\varphi}_i \partial_\beta \tilde{\varphi}_j) - \epsilon^{\alpha\beta}\tilde{b}_{ij}\partial_\alpha \tilde{\varphi}_i \partial_\beta \tilde{\varphi}_j + \Lambda(r_i^2 - 1)], \quad (5.8)$$

where

$$\tilde{g}_{11} = \frac{1}{r_2^2 + r_3^2}, \qquad \tilde{g}_{22} = \frac{r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2}{r_2^2 + r_3^2},$$
$$\tilde{g}_{33} = -\frac{r_2^2 + r_3^2 - (r_2^2 - r_3^2)^2}{r_2^2 + r_3^2}, \qquad \tilde{g}_{12} = \tilde{g}_{13} = 0,$$
$$\tilde{g}_{23} = \frac{2r_2^2 r_3^2 - r_1^2 r_2^2 - r_1^2 r_3^2}{r_2^2 + r_3^2}, \qquad \tilde{b}_{12} = \frac{r_2^2}{r_2^2 + r_3^2},$$
$$\tilde{b}_{13} = \frac{r_2^2 - r_3^2}{r_2^2 + r_3^2}, \qquad \tilde{b}_{23} = 0. \quad (5.9)$$

The T-dual variables $\tilde{\varphi}_i$ are related to $\tilde{\tilde{\varphi}}_i$ as follows:

$$\partial_\alpha \tilde{\tilde{\varphi}}_1 = \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\tilde{\varphi}_1\tilde{g}_{11} - \partial_\alpha\tilde{\varphi}_i\tilde{b}_{1i}, \quad \tilde{\tilde{\varphi}}_2 = \tilde{\varphi}_2, \qquad \tilde{\tilde{\varphi}}_3 = \tilde{\varphi}_3. \tag{5.10}$$

Next, we make the following shift of the angle variables $\tilde{\varphi}_2$ and $\tilde{\varphi}_3$ simultaneously:

$$\tilde{\varphi}_2 \to \tilde{\varphi}_2 + \hat{\gamma}_2\tilde{\varphi}_1, \qquad \tilde{\varphi}_3 \to \tilde{\varphi}_3 + \hat{\gamma}_3\tilde{\varphi}_1, \tag{5.11}$$

where $\hat{\gamma}_2$ and $\hat{\gamma}_3$ are two arbitrary constants. The metric transforms in the following way under the above shift:

$$\begin{aligned}
\tilde{g}_{11} &\to \tilde{g}_{11} + \hat{\gamma}_2^2\tilde{g}_{22} + \hat{\gamma}_3^2\tilde{g}_{33} + 2\hat{\gamma}_2\tilde{g}_{12} + 2\hat{\gamma}_3\tilde{g}_{13} + 2\hat{\gamma}_2\hat{\gamma}_3\tilde{g}_{23},\\
\tilde{g}_{12} &\to \tilde{g}_{12} + \hat{\gamma}_2\tilde{g}_{22} + \hat{\gamma}_3\tilde{g}_{23},\\
\tilde{g}_{13} &\to \tilde{g}_{13} + \hat{\gamma}_2\tilde{g}_{23} + \hat{\gamma}_3\tilde{g}_{33},\\
\tilde{b}_{12} &\to \tilde{b}_{12} - \hat{\gamma}_3\tilde{b}_{23},\\
\tilde{b}_{13} &\to \tilde{b}_{13} + \hat{\gamma}_2\tilde{b}_{23},
\end{aligned} \tag{5.12}$$

and the variables $\tilde{\varphi}_i$ transform into

$$\begin{aligned}
\partial_\alpha\tilde{\tilde{\varphi}}_1 &= \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\tilde{\varphi}_1\tilde{g}_{11} - \partial_\alpha\tilde{\varphi}_i\tilde{b}_{1i} - \hat{\gamma}_2\partial_\alpha\tilde{\varphi}_1\tilde{b}_{12} - \hat{\gamma}_3\partial_\alpha\tilde{\varphi}_1\tilde{b}_{13},\\
\tilde{\tilde{\varphi}}_2 &= \tilde{\varphi}_2, \qquad \tilde{\tilde{\varphi}}_3 = \tilde{\varphi}_3.
\end{aligned} \tag{5.13}$$

Finally, we make the T-duality transformation on the circle parametrized by $\tilde{\varphi}_1$ again. After the TsT transformation, the $(\hat{\gamma}_2, \hat{\gamma}_3)$-deformed background becomes

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}(\partial_\alpha r_i\partial_\beta r_i + G_{ij}\partial_\alpha\varphi_i\partial_\beta\varphi_i) - \epsilon^{\alpha\beta}B_{ij}\partial_\alpha\varphi_i\partial_\beta\varphi_i + \Lambda(r_i^2 - 1)], \tag{5.14}$$

where

$$\begin{aligned}
G_{1i} &= Gg_{1i},\\
G_{22} &= G(g_{22} + 9\hat{\gamma}_3^2r_1^2r_2^2r_3^2),\\
G_{33} &= G(g_{33} + 9\hat{\gamma}_2^2r_1^2r_2^2r_3^2),\\
G_{23} &= G(g_{23} - 9\hat{\gamma}_2\hat{\gamma}_3r_1^2r_2^2r_3^2),\\
B_{12} &= G[\hat{\gamma}_2(r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2) + \hat{\gamma}_3(2r_2^2r_3^2 - r_1^2r_2^2 - r_1^2r_3^2)],\\
B_{13} &= G[\hat{\gamma}_2(2r_2^2r_3^2 - r_1^2r_2^2 - r_1^2r_3^2) + \hat{\gamma}_3(r_2^2 + r_3^2 - (r_2^2 - r_3^2)^2)],\\
B_{23} &= -G[\hat{\gamma}_2(2r_1^2r_2^2 - r_1^2r_3^2 - r_2^2r_3^2) + \hat{\gamma}_3(g_{13}g_{23} - g_{12})],
\end{aligned} \tag{5.15}$$

where

$$\begin{aligned}
G^{-1} = 1 &+ \hat{\gamma}_2^2(r_1^2r_2^2 + r_1^2r_3^2 + r_2^2r_3^2) + \hat{\gamma}_3^2[r_2^2 + r_3^2 - (r_2^2 - r_3^2)^2]\\
&+ 2\hat{\gamma}_2\hat{\gamma}_3(2r_2^2r_3^2 - r_1^2r_2^2 - r_1^2r_3^2),
\end{aligned} \tag{5.16}$$

and we have used the constraint (5.3).

The variables $\tilde{\varphi}_i$ are related to the T-dual variables φ_i as follows:

$$\partial_\alpha\tilde{\varphi}_1 = \gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\tilde{\varphi}_iG_{1i} - \partial_\alpha\tilde{\varphi}_iB_{1i}, \quad \tilde{\varphi}_2 = \varphi_2, \qquad \tilde{\varphi}_3 = \varphi_3. \tag{5.17}$$

Equations (5.10), (5.13), and (5.17) allow us to determine the following relations between the angle variables $\tilde{\tilde{\varphi}}_i$ and the TsT-transformed variables φ_i:

$$\begin{aligned}
\partial_\alpha\tilde{\tilde{\varphi}}_1 &= [\tilde{g}_{11}G_{1i} + (\hat{\gamma}_2\tilde{b}_{12} + \hat{\gamma}_3\tilde{b}_{13})B_{1i} - \tilde{b}_{1i}]\partial_\alpha\varphi_i\\
&\quad - [(\hat{\gamma}_2\tilde{b}_{12} + \hat{\gamma}_3\tilde{b}_{13})G_{1i} + \tilde{g}_{11}B_{1i}]\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\varphi_i,
\end{aligned} \tag{5.18}$$

$$\partial_\alpha\tilde{\tilde{\varphi}}_2 = \partial_\alpha\varphi_2 - \hat{\gamma}_2B_{1i}\partial_\alpha\varphi_i + \hat{\gamma}_2G_{1i}\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\varphi_i, \tag{5.19}$$

$$\partial_\alpha\tilde{\tilde{\varphi}}_3 = \partial_\alpha\varphi_3 - \hat{\gamma}_3B_{1i}\partial_\alpha\varphi_i + \hat{\gamma}_3G_{1i}\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\varphi_i, \tag{5.20}$$

which gives the boundary conditions

$$\tilde{\tilde{\varphi}}'_1 = \varphi'_1 + \hat{\gamma}_2J_2^0 + \hat{\gamma}_3J_3^0, \qquad \tilde{\tilde{\varphi}}'_2 = \varphi'_2 - \hat{\gamma}_2J_1^0, \qquad \tilde{\tilde{\varphi}}'_3 = \varphi'_3 - \hat{\gamma}_3J_1^0, \tag{5.21}$$

which are consistent with the boundary conditions (4.23) and (4.24). It is easy to see that, when $\hat{\gamma}_3 = 0$, the above background reduces to the Lunin-Maldacena background [1,2].

We can check that the Virasoro constraint,

$$g_{ij}(\dot{\tilde{\tilde{\varphi}}}_i\dot{\tilde{\tilde{\varphi}}}_j + \tilde{\tilde{\varphi}}'_i\tilde{\tilde{\varphi}}'_j) = G_{ij}(\dot{\varphi}_i\dot{\varphi}_j + \varphi'_i\varphi'_j), \tag{5.22}$$

is satisfied as expected.

B. The dual field theory

According to the AdS/CFT duality, string theory in the background (5.15) is dual to a field theory on the boundary of the AdS space. This field theory is a deformed theory from $\mathcal{N} = 4$ SYM theory by the deformation $(\hat{\gamma}_2, \hat{\gamma}_3)$, so we will call it $(\hat{\gamma}_2, \hat{\gamma}_3)$-deformed $\mathcal{N} = 4$ SYM theory. Now the question is: what is this dual field theory? To answer this question, let us look at the symmetries of the deformed background (5.15).

We try first to find how many supersymmetries are preserved in the dual field theory. To derive the background (5.15), we wrote the S^5 part of $AdS \times S^5$ as (5.1). The metric has manifestly a $U(1) \times U(1) \times U(1)$ isometry, of which a $U(1) \times U(1)$ preserves the Killing spinors. In the case of the Lunin-Maldacena background, a very special torus was chosen to compactify the 10D string theory. The TsT transformation only breaks the supersymmetry corresponding to the Killing spinor associated to $U(1) \times U(1)$ so that the deformed background preserves $1/4$ supersymmetries. The left $U(1)$ remains an R-symmetry in the dual $\mathcal{N} = 1$ SYM theory. In our case, $TssT$ transformation breaks all $U(1) \times U(1) \times U(1)$ isometry so that no Killing spinor is preserved. Therefore the dual field theory has *no* supersymmetry.

Next we try to learn more about the dual field theory from the gravity side. Let us recall the relation between the TsT transformation of the supergravity background and the star product of the dual field theory in the case of the Lunin-Maldacena background [1]. $SL(2, R)$ acts on the parameter

$$\tau = B_{12} + i\sqrt{g}, \tag{5.23}$$

as

$$\tau \to \tau' = \frac{\tau}{1 + \gamma\tau} \quad \text{or} \quad \frac{1}{\tau} \to \frac{1}{\tau'} = \frac{1}{\tau} + \gamma. \tag{5.24}$$

Schematically, $1/\tau$ can be written as [53]

$$\frac{1}{\tau} \sim \left(\frac{1}{\mathbf{g} + \mathbf{B}}\right)^{ij} = G^{ij}_{\mathrm{open}} + \theta^{ij}, \tag{5.25}$$

where G^{ij}_{open} is the open string metric and θ^{ij} is the noncommutative parameter. Then the result of the $SL(2, R)$ transformation (5.24) is just to introduce a noncommutativity parameter $\theta^{12} \sim \gamma$. This analogy can be seen more precisely if we define a 2×2 matrix γ as

$$\begin{aligned} \gamma &\equiv \left(\frac{1}{\mathbf{g}' + \mathbf{B}'}\right) - \left(\frac{1}{\mathbf{g} + \mathbf{B}}\right) \\ &= (\mathbf{G}'_{\mathrm{open}} - \mathbf{G}_{\mathrm{open}}) + (\theta' - \theta). \end{aligned} \tag{5.26}$$

It is easy to get the matrix

$$\gamma = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}. \tag{5.27}$$

Thus the TsT transformation of the supergravity background is equivalent to a shift of the noncommutative parameter by $\theta^{12} = -\gamma$ in the dual field theory.

Now let us look at the $(\hat{\gamma}_2, \hat{\gamma}_3)$-deformed background which we found in the previous section. We can similarly define a 3×3 matrix γ as in (5.26). Straightforward calculation leads to the following[5] γ:

$$\gamma = \begin{pmatrix} 0 & -\gamma^{12} & -\gamma^{13} \\ \gamma^{12} & 0 & 0 \\ \gamma^{13} & 0 & 0 \end{pmatrix}. \tag{5.28}$$

Thus in our case the TsT transformation of the supergravity background is equivalent to a shift of the noncommutative parameters by $\theta^{12} = -\gamma^{12}$ and $\theta^{13} = -\gamma^{13}$ in the dual field theory. Since the modification only affects the directions (ϕ_1, ϕ_2, ϕ_3), the action of the dual field theory will be the same as the one of the $\mathcal{N} = 4$ SYM theory except the superpotential term, which can be obtained from the undeformed one by replacing the usual product $\phi_i \phi_j$ by the associative star product $\phi_i * \phi_j$. Obviously, we will not be able to write down the action by using the $\mathcal{N} = 1$ superfields since all supersymmetries are broken in the process.

C. Semiclassical analysis

A classical solution of the sigma model associated with the background (5.15) is obtained as

$$\begin{aligned} &t = \tau, \qquad \rho = 0, \qquad \varphi_1 = \nu_1\tau, \qquad \varphi_2 = \nu_2\tau, \\ &\varphi_3 = \nu_3\tau, \qquad \alpha = \arccos\left(\sqrt{\frac{\hat{\gamma}_2 + 2\hat{\gamma}_3}{4\hat{\gamma}_3 - \hat{\gamma}_2}}\right), \qquad \theta = \frac{\pi}{4}, \end{aligned} \tag{5.29}$$

where

$$\nu_1 = -\tfrac{2}{3}, \qquad \nu_2 = \tfrac{4}{3}, \qquad \nu_3 = \tfrac{1}{3}. \tag{5.30}$$

The angular momenta and the energy corresponding to this state are

$$J_1 = 0, \tag{5.31}$$

$$J_2 = -\frac{3\hat{\gamma}_3}{\hat{\gamma}_2 - 4\hat{\gamma}_3}C, \tag{5.32}$$

$$J_3 = \frac{3\hat{\gamma}_2}{\hat{\gamma}_2 - 4\hat{\gamma}_3}C, \tag{5.33}$$

and

$$E = \nu_1 J_1 + \nu_2 J_2 + \nu_3 J_3 = 3C, \tag{5.34}$$

where $C(\propto N)$ is a constant. From the relations of angular momenta (5.5), (5.6), and (5.7), we can see that this solution is associated to the state with

$$(\hat{J}_1, \hat{J}_2, \hat{J}_3) = \left(\frac{\hat{\gamma}_2 + 2\hat{\gamma}_3}{\hat{\gamma}_2 - 4\hat{\gamma}_3}C, \frac{\hat{\gamma}_2 - \hat{\gamma}_3}{\hat{\gamma}_2 - 4\hat{\gamma}_3}C, \frac{\hat{\gamma}_2 - \hat{\gamma}_3}{\hat{\gamma}_2 - 4\hat{\gamma}_3}C\right). \tag{5.35}$$

It is easy to see that the above state reduces to the (J, J, J) state when $\hat{\gamma}_3 = 0$ and to the $(-J, 0, 0)$ state when $\hat{\gamma}_2 = \hat{\gamma}_3$ with $J = E/3$.

Next, let us consider the fluctuations around the above classical solution (5.29) with large 't Hooft coupling $\lambda = g_{\mathrm{YM}} N = R^4/\alpha'^2$ as

[5] Here we define new symbols $(\gamma^{12}, \gamma^{13})$ which are related to the symbols we used in the previous section as $\gamma^{12} \equiv \hat{\gamma}^2/R^2$ and $\gamma^{13} \equiv \hat{\gamma}^3/R^2$, where R is the radius of S^5.

$$
\begin{aligned}
t &= \tau + \frac{1}{\lambda^{1/4}}\tilde{t},\\
\rho &= \frac{1}{\lambda^{1/4}}\tilde{\rho},\\
\varphi_1 &= \nu_1\tau + \frac{1}{\lambda^{1/4}}\tilde{\varphi}_1,\\
\varphi_2 &= \nu_2\tau + \frac{1}{\lambda^{1/4}}\tilde{\varphi}_2,\\
\varphi_3 &= \nu_3\tau + \frac{1}{\lambda^{1/4}}\tilde{\varphi}_3,\\
\alpha &= \arccos\left(\sqrt{\frac{\hat{\gamma}_2+2\hat{\gamma}_3}{4\hat{\gamma}_3-\hat{\gamma}_2}}\right) + \frac{1}{\lambda^{1/4}}\tilde{\alpha},\\
\theta &= \frac{\pi}{4} + \frac{1}{\lambda^{1/4}}\sqrt{\frac{\hat{\gamma}_2-4\hat{\gamma}_3}{2(\hat{\gamma}_2-\hat{\gamma}_3)}}\tilde{\theta}.
\end{aligned}
\tag{5.36}
$$

where we have defined

$r_1 = \cos\alpha, \qquad r_2 = \sin\alpha\cos\theta, \qquad r_3 = \sin\alpha\sin\theta.$

The difference between energy and angular momenta is

$$E - (\nu_1 J_1 + \nu_2 J_2 + \nu_3 J_3) = \frac{1}{2}\int_0^{2\pi}\frac{d\sigma}{2\pi}\mathcal{H} \tag{5.37}$$

where the energy and angular momenta are defined as

$$E \equiv P_t = -\frac{\delta S}{\delta \dot{t}}, \tag{5.38}$$

$$J_i \equiv P_{\varphi_i} = \frac{\delta S}{\delta\dot{\varphi}_i}, \qquad i = 1, 2, 3, \tag{5.39}$$

and $\mathcal{H}$ is the corresponding Hamiltonian. By using the Virasoro constraints

$$T_{aa} = G_{mn}\partial_a X^m \partial_a X^n = 0, \tag{5.40}$$

and keeping the terms up to quadratic order, the transverse Hamiltonian can be obtained as

$$
\begin{aligned}
\mathcal{H} =& -\partial_a\tilde{t}\partial_a\tilde{t} + \eta_\mu^2 + \partial_a\eta_\mu\partial_a\eta_\mu + 4G(\hat{\gamma}_2+2\hat{\gamma}_3)(\hat{\gamma}_2-\hat{\gamma}_3)^2\tilde{\alpha}^2 + \partial_a\tilde{\alpha}\partial_a\tilde{\alpha} + 4G(\hat{\gamma}_2+2\hat{\gamma}_3)(\hat{\gamma}_2-\hat{\gamma}_3)^2\tilde{\theta}^2 + \partial_a\tilde{\theta}\partial_a\tilde{\theta}\\
&+ 2G(\hat{\gamma}_3-\hat{\gamma}_2)\partial_a\tilde{\varphi}_1\partial_a\tilde{\varphi}_1 + \frac{3G\hat{\gamma}_3}{(4\hat{\gamma}_3-\hat{\gamma}_2)^2}(3\hat{\gamma}_2^3\hat{\gamma}_3 + \hat{\gamma}_2^2 - 9\hat{\gamma}_2\hat{\gamma}_3^3 + 6\hat{\gamma}_3^4 - 8\hat{\gamma}_2\hat{\gamma}_3 + 16\hat{\gamma}_3^2)\partial_a\tilde{\varphi}_2\partial_a\tilde{\varphi}_2\\
&+ \frac{G}{(4\hat{\gamma}_3-\hat{\gamma}_2)^2}(9\hat{\gamma}_2^5 + 64\hat{\gamma}_3^3 + 18\hat{\gamma}_2^2\hat{\gamma}_3^3 + 12\hat{\gamma}_2^2\hat{\gamma}_3 - \hat{\gamma}_2^3 - 27\hat{\gamma}_2^3\hat{\gamma}_3^2 - 48\hat{\gamma}_2\hat{\gamma}_3^2)\partial_a\tilde{\varphi}_3\partial_a\tilde{\varphi}_3 + 2G(\hat{\gamma}_3-\hat{\gamma}_2)\partial_a\tilde{\varphi}_1\partial_a\tilde{\varphi}_2\\
&+ \frac{2G}{(4\hat{\gamma}_3-\hat{\gamma}_2)^2}(15\hat{\gamma}_2^2\hat{\gamma}_3 - 18\hat{\gamma}_2\hat{\gamma}_3^4 - 16\hat{\gamma}_3^3 + 27\hat{\gamma}_2^2\hat{\gamma}_3^3 - 2\hat{\gamma}_2^3 - 9\hat{\gamma}_2^4\hat{\gamma}_3 - 24\hat{\gamma}_2\hat{\gamma}_3^2)\partial_a\tilde{\varphi}_2\partial_a\tilde{\varphi}_3\\
&+ 2G(\hat{\gamma}_3-\hat{\gamma}_2)\sqrt{2(\hat{\gamma}_3-\hat{\gamma}_2)(\hat{\gamma}_2+2\hat{\gamma}_3)}\tilde{\alpha}(\tilde{\varphi}_2' + 2\tilde{\varphi}_1') + 2G(\hat{\gamma}_3-\hat{\gamma}_2)(\hat{\gamma}_2+2\hat{\gamma}_3)\sqrt{\frac{2(\hat{\gamma}_3-\hat{\gamma}_2)}{4\hat{\gamma}_3-\hat{\gamma}_2}}\tilde{\theta}(\tilde{\varphi}_2' - 4\tilde{\varphi}_3').
\end{aligned}
\tag{5.41}
$$

where we have made a change of coordinates $(\tilde{\rho}, \Omega_3) \rightarrow \eta_\mu$, $\mu = 1, 2, 3, 4$, and

$$G^{-1} = \hat{\gamma}_2^3 - \hat{\gamma}_2 - 3\hat{\gamma}_2\hat{\gamma}_3^2 + 2\hat{\gamma}_3^2 + 4\hat{\gamma}_3. \tag{5.42}$$

We diagonalize the Hamiltonian by making the following coordinate transformations:

$$\tilde{\varphi}_1 = \phi_1 - \frac{1}{2}\phi_2, \qquad \tilde{\varphi}_2 = \phi_2, \qquad \phi_3 - \frac{15\hat{\gamma}_2^2\hat{\gamma}_3 - 18\hat{\gamma}_2\hat{\gamma}_3^4 - 16\hat{\gamma}_3^3 + 27\hat{\gamma}_2^2\hat{\gamma}_3^3 - 2\hat{\gamma}_2^3 - 9\hat{\gamma}_2^4\hat{\gamma}_3 - 24\hat{\gamma}_2\hat{\gamma}_3^2}{9\hat{\gamma}_2^5 + 64\hat{\gamma}_3^3 + 18\hat{\gamma}_2^2\hat{\gamma}_3^3 + 12\hat{\gamma}_2^2\hat{\gamma}_3 - \hat{\gamma}_2^3 - 27\hat{\gamma}_2^3\hat{\gamma}_3^2 - 48\hat{\gamma}_2\hat{\gamma}_3^2}\phi_2.$$

Then

$$
\begin{aligned}
\mathcal{H} =& -\partial_a\tilde{t}\partial_a\tilde{t} + \eta_\mu^2 + \partial_a\eta_\mu\partial_a\eta_\mu + 4G(\hat{\gamma}_2+2\hat{\gamma}_3)(\hat{\gamma}_2-\hat{\gamma}_3)^2\tilde{\alpha}^2 + \partial_a\tilde{\alpha}\partial_a\tilde{\alpha} + 4G(\hat{\gamma}_2+2\hat{\gamma}_3)(\hat{\gamma}_2-\hat{\gamma}_3)^2\tilde{\theta}^2 + \partial_a\tilde{\theta}\partial_a\tilde{\theta}\\
&+ 2G(\hat{\gamma}_3-\hat{\gamma}_2)\partial_a\phi_1\partial_a\phi_1 + \frac{9G^2(\hat{\gamma}_3-\hat{\gamma}_2)(4\hat{\gamma}_3-\hat{\gamma}_2)}{9\hat{\gamma}_2^5 + 64\hat{\gamma}_3^3 + 18\hat{\gamma}_2^2\hat{\gamma}_3^3 + 12\hat{\gamma}_2^2\hat{\gamma}_3 - \hat{\gamma}_2^3 - 27\hat{\gamma}_2^3\hat{\gamma}_3^2 - 48\hat{\gamma}_2\hat{\gamma}_3^2}\partial_a\phi_2\partial_a\phi_2\\
&+ \frac{2G}{(4\hat{\gamma}_3-\hat{\gamma}_2)}(9\hat{\gamma}_2^5 + 64\hat{\gamma}_3^3 + 18\hat{\gamma}_2^2\hat{\gamma}_3^3 + 12\hat{\gamma}_2^2\hat{\gamma}_3 - \hat{\gamma}_2^3 - 27\hat{\gamma}_2^3\hat{\gamma}_3^2 - 48\hat{\gamma}_2\hat{\gamma}_3^2)\partial_a\phi_3\partial_a\phi_3\\
&+ 4G(\hat{\gamma}_2-\hat{\gamma}_3)(\hat{\gamma}_2-4\hat{\gamma}_3)\sqrt{2(\hat{\gamma}_2-\hat{\gamma}_3)(\hat{\gamma}_2+2\hat{\gamma}_3)}\tilde{\alpha}\phi_1' - 2G(\hat{\gamma}_2-\hat{\gamma}_3)(\hat{\gamma}_2+2\hat{\gamma}_3)\sqrt{2(\hat{\gamma}_2-\hat{\gamma}_3)(\hat{\gamma}_2-4\hat{\gamma}_3)}\\
&\times\tilde{\theta}\left(\frac{9\hat{\gamma}_2(4\hat{\gamma}_3-\hat{\gamma}_2)}{9\hat{\gamma}_2^5 + 64\hat{\gamma}_3^3 + 18\hat{\gamma}_2^2\hat{\gamma}_3^3 + 12\hat{\gamma}_2^2\hat{\gamma}_3 - \hat{\gamma}_2^3 - 27\hat{\gamma}_2^3\hat{\gamma}_3^2 - 48\hat{\gamma}_2\hat{\gamma}_3^2}\phi_2' + 4\phi_3'\right).
\end{aligned}
\tag{5.43}
$$

Since the coefficients are constants, the Hamiltonian can be quantized to get the string spectrum as discussed in [52,54]

VI. CONCLUSIONS

In this paper, we consider a deformation of the $AdS_5 \times S^5$ background of string theory. We propose a simple generalization of the Lunin-Maldacena procedure for obtaining a so-called beta deformed theory which, from the gauge theory side, corresponds to a deformation of Yang-Mills theory studied by Leigh and Strassler. For real deformation parameter $\beta = \gamma$, the Lunin-Maldacena background can be thought of as a T-duality on one of the angles ϕ_1 corresponding to one of the three U(1) isometries of the $AdS_5 \times S^5$ background , a shift on another isometry variable, followed by T-duality again of ϕ_1. It was proven in the original paper by Lunin and Maldacena that this procedure does not produce additional singularities except for only those in the original background. Our generalization consists in additional shifts on the other U(1) variables in the intermediate step. In this way, one can obtain a new deformed background which depends on more parameters $\gamma_1 \cdots \gamma_n$. Since this procedure consists only in additional shifts, the resulting background again contains only the singularities descended from the original one.

In Sec. II, we reviewed the Lunin-Maldacena background and the TsT transformation procedure. In the next section we have proved that the currents for any two backgrounds related by TsT transformations are equal (which was conjectured in [2]).

In the next section, we consider $Ts \cdots sT$ transformations. We find that due to these transformations the boundary conditions for the U(1) variables are twisted. We prove also that the U(1) currents in any two backgrounds related by $Ts \cdots sT$ transformations are equal. This property is important since, as it is discussed in [2], it means that the theory preserves the nice property of integrability. The integrability can be proved along the lines of the paper by Frolov [2].

In Sec. V, we apply the TssT transformation to $AdS_5 \times S^5$ background. The obtained background is new and the string theory on it is integrable. We argue that the supersymmetry is broken and the background is less supersymmetric than that of Lunin and Maldacena.

After short comments on the gauge theory side, we perform a semiclassical analysis of string theory in $\gamma_2 - \gamma_3$ deformed $AdS_5 \times S^5$ background. We study the theory in the BMN limit and obtain the corresponding conserved quantities important for AdS/CFT correspondence. It is important to note that for $\gamma_3 = 0$ the background and therefore string theory reduce to that studied by Lunin and Maldacena. In the appendix we give for completeness a detailed derivation of the T-duality transformations.

There are several ways to develop the results obtained in this paper. First of all, one can study multispin solutions in our background along the lines of [47]. To clarify the AdS/CFT correspondence, one must consider the gauge theory side in more detail. It would be interesting to see what kind of spin chain should describe the string and gauge theory in this case. One can use then the powerful Bethe ansatz technique to study the correspondence on both sides. We leave these questions for further study.

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APPENDIX: T-DUALITY TRANSFORMATIONS

In this appendix we give a detailed derivation of the T-duality transformation.

We start with the general string theory action:

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau \frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N G_{MN}(X^i) - \epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N B_{MN}(X^i)], \tag{A1}$$

where (a) $M, N = 1, \ldots, d-1$, $i = 2, \ldots, d-1$, and (b) the background fields G_{MN} and B_{MN} do not depend on X^1.

The equation of motion for X^1 tells us that there exists conserved current J^α:

$$\partial_\alpha J^\alpha = 0 \Leftrightarrow J^\alpha \equiv -\frac{\sqrt{\lambda}}{2\pi}\frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^1)}. \tag{A2}$$

Let us define p^α as

$$p^\alpha = \gamma^{\alpha\beta}\partial_\beta X^N G_{1N} - \epsilon^{\alpha\beta}\partial_\beta X^N B_{1N}. \tag{A3}$$

The action (5.3) can be rewritten in terms of p^α as follows:

$$\begin{aligned}
S &= -\sqrt{\lambda}\int d\tau\frac{d\sigma}{2\pi}\bigg[\frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha X^1\partial_\beta X^1 G_{11} + \gamma^{\alpha\beta}\partial_\alpha X^1\partial_\beta X^i G_{1i} - \epsilon^{\alpha\beta}\partial_\alpha X^1\partial_\beta X^N B_{1N} \\
&\quad + \frac{1}{2}(\gamma^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j G_{ij} - \epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j B_{ij})\bigg] \\
&= -\sqrt{\lambda}\int d\tau\frac{d\sigma}{2\pi}\bigg[\partial_\alpha X^1(\gamma^{\alpha\beta}\partial_\beta X^N G_{1N} - \epsilon^{\alpha\beta}\partial_\beta X^N B_{1N}) - \frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha X^1\partial_\beta X^1 G_{11} \\
&\quad + \frac{1}{2}(\gamma^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j G_{ij} - \epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j B_{ij})\bigg] \\
&= -\sqrt{\lambda}\int d\tau\frac{d\sigma}{2\pi}\bigg[p^\alpha\partial_\alpha X^1 - \frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha X^1\partial_\beta X^1 G_{11} + \frac{1}{2}(\gamma^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j G_{ij} - \epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j B_{ij})\bigg]. \qquad \text{(A4)}
\end{aligned}$$

Let us consider the second term in the above expression:

$$\frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha X^1\partial_\beta X^1 G_{11} = \partial_\alpha X^1 G_{11}\frac{\gamma^{\alpha\beta}}{2G_{11}}\partial_\beta X^1 G_{11}. \qquad \text{(A5)}$$

In order to perform T-duality, we have to eliminate X^1 which enters the action only through $\partial_\alpha X^1 G_{11}$. From the definition of p^α,

$$p^\alpha = \gamma^{\alpha\beta}\partial_\beta X^1 G_{11} + \gamma^{\alpha\beta}\partial_\beta X^i G_{1i} - \epsilon^{\alpha\beta}\partial_\beta X^i B_{1i}, \qquad \text{(A6)}$$

we find

$$\gamma^{\alpha\beta}\partial_\beta X^1 G_{11} = p^\alpha - \gamma^{\alpha\beta}\partial_\beta X^i G_{1i} + \epsilon^{\alpha\beta}\partial_\beta X^i B_{1i}. \qquad \text{(A7)}$$

Substituting for $\partial_\alpha X^1 G_{11}$ in (A5) we find

$$\begin{aligned}
\partial_\alpha X^1 G_{11}\frac{\gamma^{\alpha\beta}}{2G_{11}}\partial_\beta X^1 G_{11} &= \gamma^{\alpha\sigma}\partial_\sigma X^1 G_{11}\frac{\gamma_{\alpha\beta}}{2G_{11}}\gamma^{\beta\rho}\partial_\rho X^1 G_{11} \\
&= (p^\alpha - \gamma^{\alpha\sigma}\partial_\sigma X^i G_{1i} + \epsilon^{\alpha\sigma}\partial_\sigma X^i B_{1i})\frac{\gamma_{\alpha\beta}}{2G_{11}}(p^\beta - \gamma^{\beta\rho}\partial_\rho X^i G_{1i} + \epsilon^{\beta\rho}\partial_\rho X^i B_{1i}) \\
&= \frac{p^\alpha\gamma_{\alpha\beta}p^\beta}{2G_{11}} - p^\alpha\bigg(\partial_\alpha X^i\frac{G_{1i}}{G_{11}} - \gamma_{\alpha\beta}\epsilon^{\beta\rho}\partial_\rho X^i\frac{B_{1i}}{G_{11}}\bigg) + \frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\frac{G_{1i}G_{1j}}{G_{11}} \\
&\quad + \frac{1}{2}\epsilon^{\alpha\sigma}\gamma_{\alpha\beta}\epsilon^{\beta\rho}\partial_\sigma X^i\partial_\rho X^j\frac{B_{1i}B_{1j}}{G_{11}} + \frac{\epsilon^{\alpha\sigma}}{2}\gamma_{\alpha\beta}\gamma^{\beta\rho}\partial_\rho X^j\partial_\sigma X^i\frac{B_{1i}G_{1j}}{G_{11}} \\
&\quad + \frac{\epsilon^{\beta\rho}}{2}\gamma_{\alpha\beta}\gamma^{\alpha\sigma}\partial_\sigma X^i\partial_\rho X^j\frac{G_{1i}B_{1j}}{G_{11}} \\
&= \frac{p^\alpha\gamma_{\alpha\beta}p^\beta}{2G_{11}} - p^\alpha\bigg(\partial_\alpha X^i\frac{G_{1i}}{G_{11}} - \gamma_{\alpha\beta}\epsilon^{\beta\rho}\partial_\rho X^i\frac{B_{1i}}{G_{11}}\bigg) + \frac{1}{2}\bigg(\gamma^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}} \\
&\quad - \epsilon^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\frac{G_{1i}B_{1j} - G_{1j}B_{1i}}{G_{11}}\bigg). \qquad \text{(A8)}
\end{aligned}$$

Substitution of (A8) into (A4) gives

$$\begin{aligned}
S = -\sqrt{\lambda}\int d\tau\frac{d\sigma}{2\pi}\bigg[& p^\alpha\bigg(\partial_\alpha X^N\frac{G_{1N}}{G_{11}} - \gamma_{\alpha\beta}\epsilon^{\beta\rho}\partial_\rho X^N\frac{B_{1N}}{G_{11}}\bigg) - \frac{\gamma_{\alpha\beta}p^\alpha p^\beta}{2G_{11}} + \frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j\bigg(G_{ij} - \frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}}\bigg) \\
& - \frac{1}{2}\epsilon^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\bigg(B_{MN} - \frac{G_{1M}B_{1N} - G_{1N}B_{1M}}{G_{11}}\bigg)\bigg]. \qquad \text{(A9)}
\end{aligned}$$

We will use now the conservation of p^α:

$$\partial_\alpha p^\alpha = 0, \qquad \text{(A10)}$$

to write down the general solution to (A10) as

$$p^\alpha = \epsilon^{\alpha\beta}\partial_\beta\tilde{X}^1, \qquad \text{(A11)}$$

where $\tilde{X}^1$ is a scalar field which is the T-dual of X^1.

If we substitute for p^α from (A11) into its definition (A3), we find the relation

$$\epsilon^{\alpha\beta}\partial_\beta\tilde{X}^1 = \gamma^{\alpha\beta}\partial_\beta X^M G_{1M} - \epsilon^{\alpha\beta}\partial_\beta X^M B_{1M}. \tag{A12}$$

Now we can derive the T-dual action by substituting for p^α the expression (A11).

Let us consider the different terms separately.

Obviously (ij) components remain the same since $\tilde{X}^i = X^i$.

(a) The first term in (A9) becomes

$$\frac{p^\alpha\partial_\alpha X^N G_{1N}}{G_{11}} = p^\alpha\partial_\alpha X^1 + p^\alpha\partial_\alpha\tilde{X}^i\frac{G_{1i}}{G_{11}} = p^\alpha\partial_\alpha X^1 - \epsilon^{\alpha\beta}\partial_\alpha\tilde{X}^1\partial_\beta\tilde{X}^i\frac{G_{1i}}{G_{11}}, \tag{A13}$$

where we substitute p^α in the second term with $\epsilon^{\alpha\beta}\partial_\beta\tilde{X}^1$.

We need also expression for $\partial_\alpha X^1$ in terms of $\tilde{X}^M$. From (A12) we have

$$\epsilon^{\alpha\beta}\partial_\beta\tilde{X}^1 = \gamma^{\alpha\beta}\partial_\beta X^1 G_{11} + \gamma^{\alpha\beta}\partial_\beta\tilde{X}^i G_{1i} - \epsilon^{\alpha\beta}\partial_\beta\tilde{X}^i B_{1i}, \tag{A14}$$

and therefore

$$\partial_\alpha X^1 = \gamma_{\alpha\rho}\epsilon^{\rho\beta}\partial_\beta\tilde{X}^1\frac{1}{G_{11}} + \gamma_{\alpha\rho}\epsilon^{\rho\beta}\partial_\beta\tilde{X}^i\frac{B_{1i}}{G_{11}} - \partial_\alpha\tilde{X}^i\frac{G_{1i}}{G_{11}}. \tag{A15}$$

Substituting (A15) into (A13) we get

$$p^\alpha\partial_\alpha X^1 = \gamma^{\sigma\beta}\partial_\sigma\tilde{X}^1\partial_\beta\tilde{X}^1\frac{1}{G_{11}} + \gamma^{\sigma\beta}\partial_\sigma\tilde{X}^1\partial_\beta\tilde{X}^{i1}\frac{B_{1i}}{G_{11}} - \epsilon^{\sigma\alpha}\partial_\alpha\tilde{X}^1\partial_\alpha\tilde{X}^i\frac{G_{1i}}{G_{11}}. \tag{A16}$$

(b) The second term in (A9) becomes

$$-p^\alpha\gamma_{\alpha\beta}\epsilon^{\beta\rho}\partial_\rho\tilde{X}^i\frac{B_{1i}}{G_{11}} = -\epsilon^{\alpha\sigma}\gamma_{\alpha\beta}\epsilon^{\beta\rho}\partial_\rho\tilde{X}^i\frac{B_{1i}}{G_{11}} \times \partial_\sigma\tilde{X}^1 = \gamma^{\sigma\rho}\partial_\sigma\tilde{X}^1\partial_\rho\tilde{X}^i\frac{B_{1i}}{G_{11}}. \tag{A17}$$

(c) The third term in (A9) can be written as

$$-\frac{1}{2}\frac{p^\alpha\gamma_{\alpha\beta}p^\beta}{G_{11}} = -\frac{1}{2}\epsilon^{\alpha\sigma}\frac{\gamma_{\alpha\beta}}{G_{11}}\epsilon^{\beta\rho}\partial_\sigma\tilde{X}^1\partial_\rho\tilde{X} = \frac{1}{2}\gamma^{\sigma\rho}\partial_\sigma\tilde{X}^1\partial_\rho\tilde{X}^i\frac{1}{G_{11}}. \tag{A18}$$

Summing up all the terms we derived above, we find

$$\frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha\tilde{X}^1\partial_\beta\tilde{X}^1\frac{1}{G_{11}} - \frac{\epsilon^{\alpha\beta}}{2}\partial_\alpha\tilde{X}^1\partial_\beta\tilde{X}^i\frac{G_{1i}}{G_{11}} + \frac{1}{2}\gamma^{\alpha\beta}\partial_\alpha\tilde{X}^1\partial_\beta\tilde{X}^i\frac{B_{1i}}{G_{11}}. \tag{A19}$$

All the other terms in the action remain unchanged. The final action has the same form as (A1) but with new background fields

$$S = -\frac{\sqrt{\lambda}}{2}\int d\tau\frac{d\sigma}{2\pi}[\gamma^{\alpha\beta}\partial_\alpha\tilde{X}^M\partial_\beta\tilde{X}^N\tilde{G}_{MN} - \epsilon^{\alpha\beta}\partial_\alpha\tilde{X}^M\partial_\beta\tilde{X}^N\tilde{B}_{MN}], \tag{A20}$$

with the following transformation laws for the background fields:

$$\tilde{G}_{11} = \frac{1}{G_{11}}, \qquad \tilde{G}_{ij} = G_{ij} - \frac{G_{1i}G_{1j} - B_{1i}B_{1j}}{G_{11}},$$
$$\tilde{G}_{1i} = \frac{B_{1i}}{G_{11}}, \qquad \tilde{B}_{ij} = B_{ij} - \frac{G_{1i}B_{1j} - B_{1i}G_{1j}}{G_{11}}, \tag{A21}$$
$$\tilde{B}_{1i} = \frac{G_{1i}}{G_{11}},$$

and the following relations between the variables:

$$\tilde{X}^i = X^i,$$
$$\epsilon^{\alpha\beta}\partial_\beta\tilde{X}^1 = \gamma^{\alpha\beta}\partial_\beta X^M G_{1M} - \epsilon^{\alpha\beta}\partial_\beta X^M B_{1M}, \tag{A22}$$

or, equivalently,

$$\partial_\alpha X^1 = \gamma_{\alpha\rho}\epsilon^{\rho\beta}\partial_\beta\tilde{X}^1\frac{1}{G_{11}} + \gamma_{\alpha\rho}\epsilon^{\rho\beta}\partial_\beta\tilde{X}^i\frac{B_{1i}}{G_{11}} - \partial_\alpha\tilde{X}^i\frac{G_{1i}}{G_{11}} = \gamma_{\alpha\rho}\epsilon^{\rho\beta}\partial_\beta\tilde{X}^M\tilde{G}_{1M} - \partial_\alpha\tilde{X}^M\tilde{B}_{1M}. \tag{A23}$$

This completes the derivation of the T-duality transformations.

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