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Chaos and control of discrete dynamic traffic model

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Abstract

This study discusses chaotic traffic flow. The discrete dynamic model proposed herein is derived from both the flow–density–speed fundamental diagram and Greenshield's model. The model employs occupancy as its variable and the ratio of free flow and average speed as its control parameter. The function form of the model is equal to logistic map that bifurcates when the value of the control parameter increases. Chaotic traffic means that traffic becomes unstable and unpredictable, which is dangerous for driving. Therefore, this study considers the implementation of chaotic control in signal or ramp metering design so as to stabilize the chaotic traffic phenomena. These results are illustrated by numerical examples.

Keywords: Chaos; Chaotic control; Discrete dynamic model; Traffic flow; Logistic map

1. Introduction

Traffic on a network or on a single link of a network can be considered as a dynamic system. A system can be described in terms of variables such as the position and speed of each vehicle from a microscopic viewpoint. On the other hand, a system may be described in terms of variables such as the total number of trips between two

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zones, the rate of traffic flow, density and speed from a macroscopic viewpoint. Dynamic traffic flow models consider the temporal evolution of these variables. Dynamic traffic flow, equilibrium and steady state operations are essential for analyzing, managing and optimizing traffic. Equilibrium and steady state operations are ideal but not assured as traffic is inherently unstable. However, many theoretical traffic flow models are based on these conditions. As a result, this type of model is actually not based on facts. The chaotic traffic flow model, however, considers more complicated traffic. Glass [1] examines the chaos of day-to-day management, and acknowledges the difficulty associated with this type of model. The inaccuracies are due to the following invalid assumptions. First, the environment is almost an elementary 'closed system'. Generally, external stimuli do not affect a closed system. Second, the operating environment is sufficiently stable to facilitate an understanding on how to develop and implement strategies. Third, the environment is a series of apparent levels. Analysis of these levels facilitates response prediction of the system. Furthermore, Glass suggested that three new realities exist. First, a system should alternately be treated as a complex 'open system', which is constantly and notably influenced by external stimuli. Intended actions are often diverted by external events. Second, a system is rapidly changing, which prohibits the formation of detailed strategies. Third, simple cause and effect linear models have failed, as situations lead to unexpected results.

Thus, investigating these models and observing chaotic traffic flow are worthwhile tasks. Research of chaotic dynamics can be separated into segments in which different interests dominate. Stoop and Radons [2] reviewed the development and extended chaotic systems. They mentioned that the first period of work concentrated on bifurcation research. A bifurcation is a change in an external parameter, which produces a crossroad of a given solution within a system. This parameter is referred to as a control parameter, since it can change the qualitative behavior of a system. Experimental observation reveals that increasing the control parameter induces period doubling, which alters systems from simple to complex. Eventually, systems will become chaotic. However, period doubling is not the only reason inducing chaotic; intermittency and quasiperiodicity also induce chaos. The mathematical description of the bifurcation sequences states that simple mechanisms are sufficient to generate complex behavior. Chaos of large number of simulated ODE systems and iterated discrete dynamic systems are observed in which the described phenomena could be traced with much finer precision.

Jarrett and Zhang [3,4] examined the chaotic traffic by adopting car-following model. Furthermore, the relationship between sensitivity and position is investigated. Disbro and Frame [5] also supported this finding. According to their results, the Lyapunov exponent is positive for certain values within the parameters, but they do not have plots or other evidence of a strange attractor. Conversely, Kirby and Smith [6] fail to detect chaos in car-following models.

The chaotic traffic flow model examined herein is a macroscopic model. The proposed model is a discrete and dynamic one derived from both the flow–density–speed fundamental diagram and the Greenshield's model. The model uses occupancy as its variable and uses the ratio of free flow and average speed as the control parameter. The rest of this paper is organized as follows. Section 2 presents the derivation of the model. Section 3 investigates the bifurcation of the model by substantial data. Section 4 introduces how to apply chaotic control to stabilize this type of model. The control method developed by Pan and Yin [7] is introduced herein. From the control method analysis, strategies are developed to stabilize traffic flow, as examined in the conclusion.

2. Discrete dynamical traffic flow model

The model developed in this section is a macroscopic traffic flow model, which is in the logistics map form. Before deriving the traffic flow model, logistic map should be introduced in brief first. Logistic map is a noted population model with discrete generations

$$x_{n+1} = \lambda x_n(1 - x_n), \tag{1}$$

where x_n, and x_{n+1} denote population density x at time span n, and $n+1$. λ is the control parameter of Eq. (1). According to computations with λ varying from 1 to 4, there is a fixed point in $1 \leqslant \lambda < 3$. $\lambda = 3$ is a bifurcation point. When $3 < \lambda \leqslant 4$ the behavior becomes increasingly complicated.

The derivation begins with the flow–density–speed fundamental diagram

$$q = ku, \tag{2}$$

where q is flow, k is density and u is speed. It is assumed that speed depends solely on density. Evidently in Eq. (2), flow is also a function of density, which is represented as

$$q(k) = ku(k). \tag{3}$$

If the road traffic flow satisfies the Greenshield's model, the relation of speed and density is shown as follows:

$$u(k) = u_{\rm f}\left(1 - \frac{k}{k_{\rm j}}\right), \tag{4}$$

where $u_{\rm f}$ is free flow speed, and $k_{\rm j}$ is jam density. From Eqs. (3) and (4), we have

$$q(k) = u_{\rm f}k\left(1 - \frac{k}{k_{\rm j}}\right). \tag{5}$$

It is assumed that the traffic flow in the next time span is decided by the current traffic conditions. Based on this, Eq. (5) is as follows:

$$q_{n+1} = u_{\rm f}k_n\left(1 - \frac{k_n}{k_{\rm j}}\right). \tag{6}$$

For computing convenience, Eq. (6) can be converted into a function, which only depends on a single variable as a result of the scaling method. Occupancy is the variable that converts Eq. (6) into a single variable function. The definition of occupancy ρ is as follows:

$$\rho = \frac{q\bar{L}}{\bar{u}} = k\bar{L}, \tag{7}$$

where $\bar{u}$ is average speed and $\bar{L}$ is average vehicle length. Occupancy is the ratio of actual occupied time and available time of a certain space, so the value of occupancy is between [0,1]. By the definition of occupancy, Eqs. (8) and (9) are

$$q = \frac{\rho\bar{u}}{\bar{L}}, \tag{8}$$

$$k = \frac{\rho}{\bar{L}}. \tag{9}$$

Superimpose Eqs. (8) and (9) into Eq. (6), and it becomes

$$\frac{\rho_{n+1}\bar{u}}{\bar{L}} = u_{\mathrm{f}}\frac{\rho_n}{\bar{L}}\left(1 - \frac{\rho_n}{k_{\mathrm{j}}\bar{L}}\right). \tag{10}$$

To simplify Eq. (10), let the modified occupancy be $\hat{\rho} = \frac{\rho}{k_{\mathrm{j}}\bar{L}}$, which concludes

$$\hat{\rho}_{n+1} = \frac{u_{\mathrm{f}}}{\bar{u}}\hat{\rho}_n(1 - \hat{\rho}_n). \tag{11}$$

Eq. (11), the discrete dynamic traffic flow model developed herein, has a form similar to Eq. (1). In Eq. (11), the ratio of free flow speed and average speed is equal to the control parameter λ represented in Eq. (1). The subsequent section demonstrates that the control parameter $u_{\mathrm{f}}/\bar{u}$ may raise chaotic behavior of the equation by using historical data.

3. Numerical example

This section considers Cho [8] and Chung's [9] empirical data and chaotic traffic flow with an increasing control parameter. Cho [8] surveyed traffic flow on the four-lane Sun Yat-sen National Freeway in Taiwan. During the study, the average density of every 30 s varied from 16.18 to 358.94 pcu/mi/two-lane and during the peak traffic the average speed was 14.82 mph. The traffic flow model supported by data is $u = 54.88 \times (1 - k/346.65)$. The free flow speed was calculated at 54.88 mph and 346.65 pcu/mi/2-lane represents the jam density. Chung [9] surveyed four-lane surface road where a mean density of every 20 s varied from 14.68 to 323.68 pcu/mi/two-lane and an average speed of 12.23 mph during the peak period. The traffic flow model supported by data was $u = 44.27 \times (1 - k/351.61)$. The free flow speed studied was 44.27 mph with a jam density of 351.61 pcu/mi/two-lane. Cho and Chung examined two short time intervals, therefore the fluctuation of the average density can be clearly detected.

Since ρ is occupancy, the range of its value is between 0 and 1. Thus $\hat{\rho}_{\max} = \rho_{\max}/k_{\mathrm{j}}\bar{L} \cong 1$ implying that $\hat{\rho} = \frac{\rho}{k_{\mathrm{j}}\bar{L}}$ is also between 0 and 1. After a review of the logistic map, with a control parameter $(u_{\mathrm{f}}/\bar{u})$ of less than 1, the fixed point is 0. This finding implies that the flow on the road is minimal and requires no control.

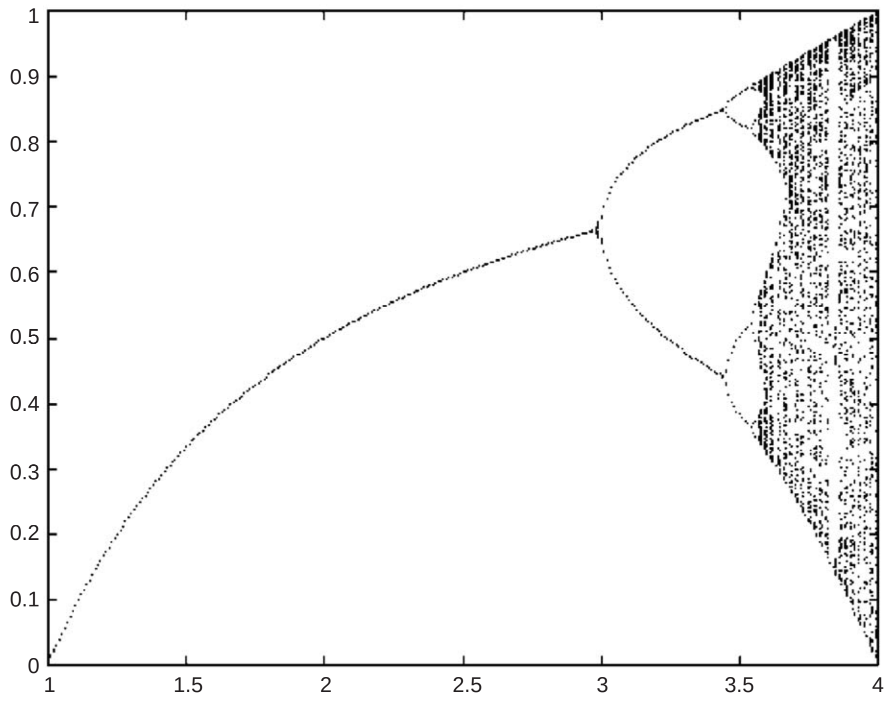

Fig. 1. Bifurcation diagram of $\hat{\rho}_{n+1} = (u_f/\bar{u})\hat{\rho}_n(1 - \hat{\rho}_n)$, where $(u_f/\bar{u}) \in [1, 4]$.

However, with a control parameter greater than 4, the traffic behavior becomes more complicated. Based on the empirical data during the peak period, Cho and Chung found a control parameter of 3.7 and 3.62. (By the way, if the observed peak period was reduced to 15 min or 30 min, the control parameter becomes larger.) Fig. 1 illustrates that traffic is unstable when the control parameter is increased from 1 to 4. The bifurcation point is 3, which can also be observed in the figure.

In empirical data, when $u_f/\bar{u} = 3$ represents the average speed, which is one-third of the free flow speed of approximately 18.64 mph on freeway in Cho [8] or 14.91 mph on surface highway in Chung [8]. Fig. 2 illustrates iteration movement when $u_f/\bar{u} = 2, 3, 3.5, 4$. A situation in which $u_f/\bar{u} = 2$, the average speed, which is half of the free flow speed, traffic can reach a stable situation. When $u_f/\bar{u} = 3$, the model is composed of a second period orbit. If the two states are in a close proximity to one another, near stable traffic flow ensues. However, for $u_f/\bar{u} = 3.5$ and 4, the traffic flow is chaotic. The following section introduces chaotic control methods.

4. Chaotic control

In dense traffic, drivers follow one another very closely, small disturbances such as the acceleration or deceleration of one vehicle might be passed over or amplified

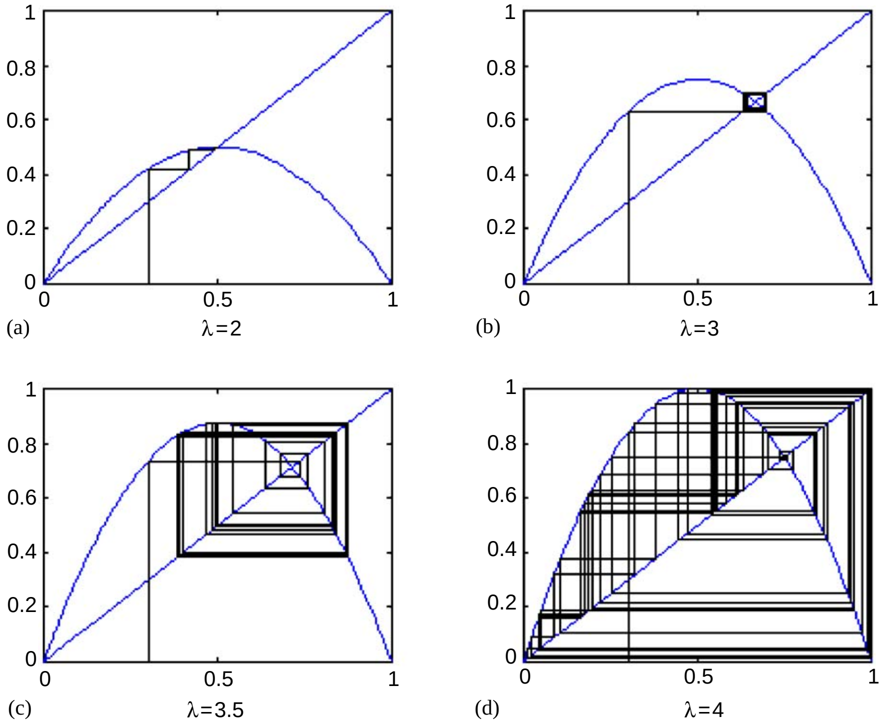

Fig. 2. Cobweb plot for $\hat{\rho}_{n+1} = (u_f/\bar{u})\hat{\rho}_n(1 - \hat{\rho}_n)$, where $\lambda = (u_f/\bar{u})$, initial value = 0.3.

along the line of vehicles. These phenomena are sensitive to the initial conditions. Chaotic flow raises, results in problems with traffic management, and can lead to accidents. Since it is the duty of the traffic operator to provide stable conditions and to improve the level of service of road system and safety, the chaos of traffic flow must be controlled. The following paragraphs consider chaotic control method and application, which influences the control model proposed herein.

Prevention or control of chaos was impossible prior to the development of a chaos control method by Ott et al. [10]. This method is the Ott–Grebogi–Yorke (OGY) control, which can be briefly described as follows:

(1) Determine some of the unstable low-period orbits that are embedded in the chaotic set.
(2) Examine the location and stability of these orbits and choose one, which yields the desired system performance.
(3) Apply limited control to stabilize the desired periodic orbit. This is possible through the use of nonlinear time-series analysis for the observation, understanding and control of the system.

This is of great importance as chaotic systems are complicated and a detailed knowledge of the equations is often unknown.

Although OGY control does not apply to chaos of functions, the concept induces the subsequent studies of chaotic control. There is a conventional assumption among developed control methods: chaotic control always corresponds with the conversion of the positive Lyapunov exponent to the negative one after application of the control force. The future behavior is predictable and controllable only when the system is insensitive to the initial conditions (i.e. it has a negative Lyapunov exponent). Variable structure control (VSC) is another recognized nonlinear feedback control method developed by DeCario et al. [11]. By using a discontinuous feedback controller, it sustains the output in a designed sliding surface.

Pan and Yin [7] used a piecewise linear controller, which differs from VSC, and kept the output in a designed chaotic attractor. Furthermore, they illustrated that the negative Lyapunov exponent is not required to obtain a practicable control. They regarded the error between the real output and the target as a stochastic process. The system is controlled when the mean value of the error is zero and the absolute value of the maximal error is less than a given threshold. For convenience, they redefine control of chaos, which is as follows:

A chaotic system is controlled when the output sequence (a stochastic process) $\Gamma(n)$, $n = 0,1,2,\ldots$, can satisfy the following two conditions:

(1) For a given $\varepsilon \geqslant 0$, $\exists N > 0$, and a closed neighborhood X_t, $S = \{X \mid \|X - X_t\| \leqslant \varepsilon\}$ such that $n > N$, $P(S) = 1$, i.e. the controlled output is restricted in S.
(2) $\forall X \in \Gamma(n)$, $n = N + 1, N + 2, \ldots$, the mean value $E(X) = X_t$

When both conditions are satisfied, an unbiased control is attained. However, if only one condition is satisfied, it is biased. If $\varepsilon = 0$, it is equivalent to classical linear control, which positively satisfies the two conditions cited above. Pan and Yin's concept of control can be described as follows:

(1) Linearize the local space near the fixed point.
(2) Apply the small feedback control force, which occurs only when the trajectory is near the target. If the controlled domain is sufficiently small, the original system can be approximated by a linear system so as to convert an unstable manifold to the stable one in this local linear system.

However, constructing an available controller, which can convert an unstable manifold to a stable one, is problematic. Pan and Yin [7] proposed linking an unstable manifold to a stable manifold to form a homoclinic orbit rather than a direct conversion. This finding implies that a small chaotic attractor can be constructed in the controlled domain, subsequently forcing all trajectories to enter the region of attraction of this new chaotic attractor. Trajectories are then captured and cannot escape without noise.

The proposed chaotic behavior of dynamic traffic model is similar to the model discussed by Pan and Yin [7]. Thus, the control force they suggested is applied to our model, so as to control chaotic behavior. Therefore, Eq. (11) becomes Eq. (12). It is assumed that Eq. (12) has an unstable fixed point $\hat{\rho}^*$.

$$\hat{\rho}_{n+1} = \frac{u_{\mathrm{f}}}{\bar{u}} \hat{\rho}_n (1 - \hat{\rho}_n) + c(\hat{\rho}_n), \tag{12}$$

where

$$\begin{aligned} c(\hat{\rho}) &= -a, & \hat{\rho} \in [\hat{\rho}^* - \varepsilon, \hat{\rho}^*) \\ &= -a + b\left(1 - \mathrm{e}^{-(\hat{\rho}-\hat{\rho}^*)}\right), & \hat{\rho} \in [\hat{\rho}^*, \hat{\rho}^* + \varepsilon] \end{aligned} \tag{13}$$

is a controller, e.g. signal or ramp-metering strategies, a and b are the parameters to be determined. With simplification of the form of Eq. (11) as $\hat{\rho}_{n+1} = f(\hat{\rho}_n)$, let $L = \mathrm{d}f(\hat{\rho})/\mathrm{d}\hat{\rho}\big|_{\hat{\rho}=\hat{\rho}^*}$, $|L| > 1$ is revealed. Careful selection of the parameters a and b is required to control chaos. Let

$$a = \varepsilon, \quad b = -2L. \tag{14}$$

For convenience, let the fixed point $\hat{\rho}^* = 0$ and $|L| = 2$. The controlled system $\hat{\rho}_{n+1} = f(\hat{\rho}_n) + c(\hat{\rho}_n)$ can be approximated by the piece-wise linear model $\hat{\rho}_{n+1} = p(\hat{\rho}_n)$ in the controlled domain $[-\varepsilon, \varepsilon]$:

$$\begin{aligned} p(\hat{\rho}) &= L\hat{\rho} - a, & \hat{\rho} \in [-\varepsilon, 0) \\ &= L\hat{\rho} - a + b\hat{\rho}, & \hat{\rho} \in [0, \varepsilon]. \end{aligned} \tag{15}$$

Applying the control force only in $S = \{\hat{\rho} \,|\, |\hat{\rho} - \hat{\rho}^*| \leqslant \varepsilon\}$, the local piecewise linear model of Eq. (15) is illustrated in Fig. 3. Fig. 3 shows that by applying the control function, the original fixed point $\hat{\rho}^* = 0$ is replaced by two new unstable fixed points $B = (-\varepsilon/3, \varepsilon/3)$ and $C = (\varepsilon, \varepsilon)$. Obviously, $O \to A \to C$ is a route from O to the unstable fixed point C. Another route from C to O can also be found. Define the left one as $p_{\mathrm{l}}^{-1}(y)$ and the right as $p_{\mathrm{r}}^{-1}(y)$. For any point $y \in (-\varepsilon, \varepsilon]$, $p^{-1}(y)$ is comprised of two symmetric points in $[-\varepsilon, 0)$ and $(0, \varepsilon]$. Also, according to Fig. 3, it is true that $p_{\mathrm{l}}^{-1}(O) \to p_{\mathrm{r}}^{-2}(O) \to p_{\mathrm{r}}^{-3}(O) \to \ldots \to p_{\mathrm{r}}^{-n}(O) \ldots$ is a convergent sequence and $\lim_{n\to\infty} p_{\mathrm{r}}^{-n}(O) = C$, where $p_{\mathrm{r}}^{-n}(y) \equiv p_{\mathrm{r}}^{-1}\left(p_{\mathrm{r}}^{-(n-1)}(y)\right)$. Thus, the sequence $C, \ldots, p_{\mathrm{r}}^{-n}(O), \ldots, p_{\mathrm{r}}^{-3}(O), p_{\mathrm{r}}^{-2}(O), p_{\mathrm{r}}^{-1}(O), O, A$ can form a homoclinic orbit based at C. The controlled model is a tent map in $[-\varepsilon, \varepsilon]$. There are some properties that can be observed:

(1) When $L = -2$, the tent map is chaotic in this small interval $[-\varepsilon, \varepsilon]$ and its Lyapunov exponent is $\ln|L|$. The chaotic attractor in $[-\varepsilon, \varepsilon]$ results in an output of uniform distribution in $[-\varepsilon, \varepsilon]$. Thus, the mean value of the controlled output is $\hat{\rho}^* = 0$ so as to satisfy the second condition of the newly formed definition of controlling chaos, i.e. it is an unbiased control.
(2) When $L \in (-2, -1)$, it can be found from Fig. 3 that the output is still restricted in $[-\varepsilon, \varepsilon]$. However, the mean value of the output is not equal to $\hat{\rho}^*$, i.e. the control is biased.

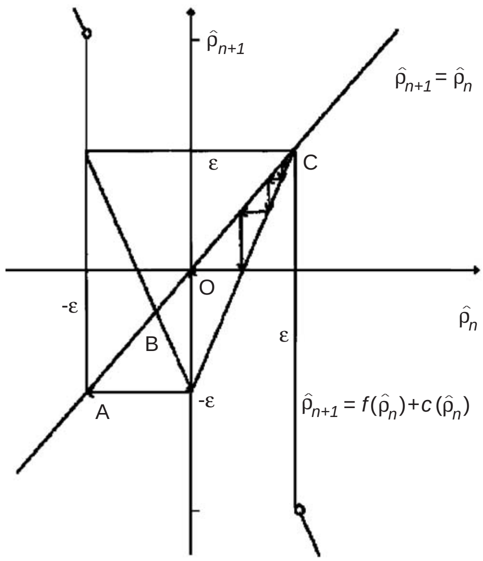

Fig. 3. The local piecewise linear model of the controlled chaotic system and its homoclinic orbit (Pan and Yin, 1997).

If $|L| \neq 2$, Pan and Yin recommended the use of an additional linear feedback controller in the controlled domain, as to satisfy the subsequent condition.

$$\begin{aligned} v(\hat{\rho}) &= b\left(1 - e^{-\hat{\rho}}\right), \quad && \hat{\rho} \geqslant 0, \\ &= 0, && \hat{\rho} < 0, \end{aligned} \tag{16}$$

where $v(\hat{\rho})$ is the occupancy. By using this controller, a feedback control function becomes

$$w(\hat{\rho}) = v(\hat{\rho}) - a. \tag{17}$$

Nevertheless, the control force $w(\hat{\rho})$ is applied only when $\hat{\rho}$ is close enough to $\hat{\rho}^*$, i.e. $\left|\hat{\rho} - \hat{\rho}^*\right| \leqslant \varepsilon$.

Within the traffic flow model, where $u_f/\bar{u} = 4$ is an example, the original fixed point is $\hat{\rho}^* = 0.75$. The original derivative is $L = \mathrm{d}f(\hat{\rho})/\mathrm{d}\hat{\rho}\big|_{\hat{\rho}=\hat{\rho}^*} = -2$. The first goal of control is to restrict the output in the closed neighborhood of $\hat{\rho}^*$, let $S = \left\{\hat{\rho}\middle|\left|\hat{\rho} - \hat{\rho}^*\right| \leqslant 0.01\right\}$; the second is to make $E(\hat{\rho}) = \hat{\rho}^*$ for any point of the controlled output after the transient time.

Since ε and L are known, from Eq. (15), $a = 0.01$, $b = 4$ are determined. If $\left|\hat{\rho} - \hat{\rho}^*\right|$ is greater than ε, the control force is zero. In this way, the control force can be greatly reduced to only a small perturbation. Most of all, when the controlled domain is very small, the local piecewise linearization can be adopted. Since the controlled output can be viewed as a random process of uniform distribution in $[\hat{\rho}^* - \varepsilon, \hat{\rho}^* + \varepsilon]$, the standard variance σ can be calculated analytically by $\sigma^2 = \frac{1}{3}\varepsilon^2$, when $\varepsilon = 0.01$,

$\sigma = 0.0058$. The uncontrolled output is illustrated in Fig. 4(a) and the controlled output is shown in Fig. 4(b).

From the experimental data, $E(\hat{\rho})$ is estimated at 0.7492. Only a small difference exists between the experimental and theoretical values. Note that the system remains chaotic even when controlled. The sole difference between the controlled and uncontrolled system is that within the controlled system, a relatively small chaotic attractoris used to replace a large one.

A practical problem must be considered, a valid controller is unavailable without a precise model. For example, the controller designed on the model $\hat{\rho}_{n+1} = 4\hat{\rho}_n(1 - \hat{\rho}_n)$ cannot control the slightly modified model $\hat{\rho}_{n+1} = 3.99\hat{\rho}_n(1 - \hat{\rho}_n)$. Furthermore, should noise exist, the control becomes invalid. This is explained by the property of the tent map. When a point is forced out of the small chaotic attractor, it will run away quickly until the chaotic trajectory reenters the small controlled domain, which can be illustrated in Fig. 3.

Most of the time, operators focus mainly on maintaining the traffic flow at a given value or within a desired range. Pan and Yin [7] regarded this condition as a solution of the problem mentioned above. First, the controlled domain should be broadened. Next, within the controlled domain, keeping the output of the controller $c(\hat{\rho} - \hat{\rho}^*)$ to be a constant if the input $\hat{\rho} - \hat{\rho}^*$ is greater than a new threshold. The modified control function is

$$\begin{aligned} c(\hat{\rho}) &= -a & &, \hat{\rho} \in [\hat{\rho}^* - d, \hat{\rho}^*) \\ &= -a + b\left(1 - e^{-(\hat{\rho}_n - \hat{\rho}^*)}\right) & &, \hat{\rho} \in [\hat{\rho}^*, \hat{\rho}^* + d_1] \\ &= -a + b\left(1 - e^{-d_1}\right) & &, \hat{\rho} \in (\hat{\rho}^* + d_1, \hat{\rho}^* + d], \end{aligned} \tag{18}$$

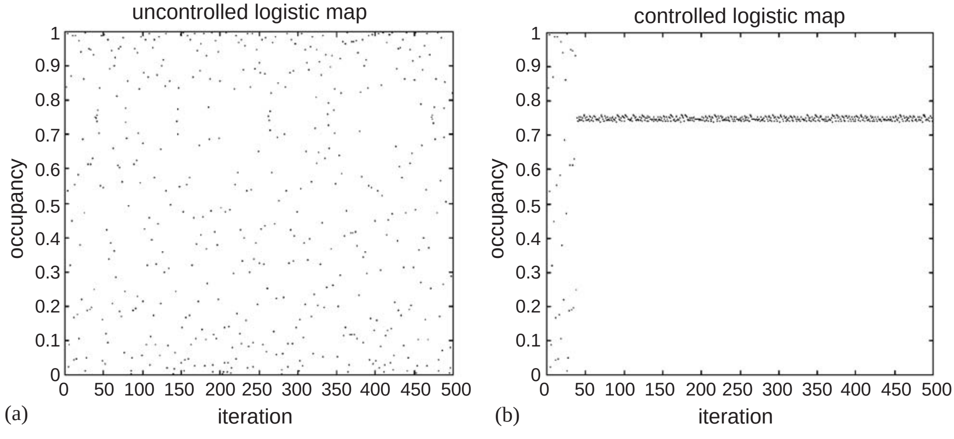

Fig. 4. The output of (a) uncontrolled and (b) controlled logistic map with piecewise linear controller (initial value = 0.3).

where d is the enlarged threshold of the controlled domain, d_1 is the new threshold. This therefore satisfies the two conditions of the new definition

$$a = \frac{\varepsilon}{2}, \quad b = -2L, \quad d_1 = \varepsilon, \quad d = 2\varepsilon \tag{19}$$

that are chosen. For convenience, assume that $\hat{\rho}^* = 0$ and $L = -2$ again. The piecewise linearized controlled model $\hat{\rho}_{n+1} = p(\hat{\rho}_n) = f(\hat{\rho}_n) + c(\hat{\rho}_n)$ is illustrated in Fig. 5. Two symmetric chaotic attractors are generated by the two adjacent tent maps. Each of the chaotic attractors can be easily destroyed by noise, but the symmetric chaotic attractors can still remain in a low noisy background. A small perturbation may result in the desired output out of the first attractor in $[-\varepsilon/2, \varepsilon/2]$, but if the noise is relatively low, the escaping point will either directly fall into the adjacent attractor, or fall into it in two steps. A similar situation occurs in the other chaotic attractor in $[\varepsilon/2, 3\varepsilon/2]$. Hence, even with a low noise the output is captured in the adjacent attractors. The chaotic trajectory switches between the two attractors. By defining the two attractors as states S_0 and S_1, a trajectory can be regarded as a Markov chain. Since the attractors are symmetric and the probability distribution of the Gaussian white noise is symmetric, the one-step transient probability p_{01} is equal to p_{10}. According to the theory of Markov chain, the probability of S_0 is equal to that of S_1. Due to the symmetry of the attractors, the conditional probability distribution $F(\hat{\rho}|S_0)$ is equal to $F(\hat{\rho}|S_1)$. Thus, the mean value of the output is located at the symmetric center of the two attractors, i.e., ($\varepsilon/2$, $\varepsilon/2$). Defined as $\hat{\rho}_t = \hat{\rho}^* + \varepsilon/2$, the controlled output is restricted in $[\hat{\rho}_t - \varepsilon, \hat{\rho}_t + \varepsilon]$ and $\hat{\rho}_t$ exists just

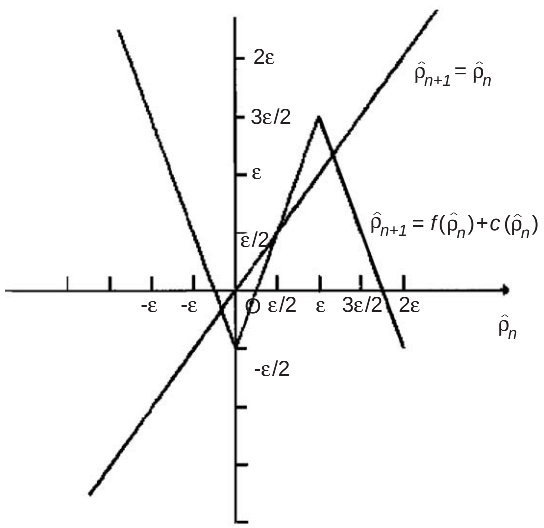

Fig. 5. Local piecewise linear model of the controlled chaotic system with modified controller (Pan and Yin, 1997).

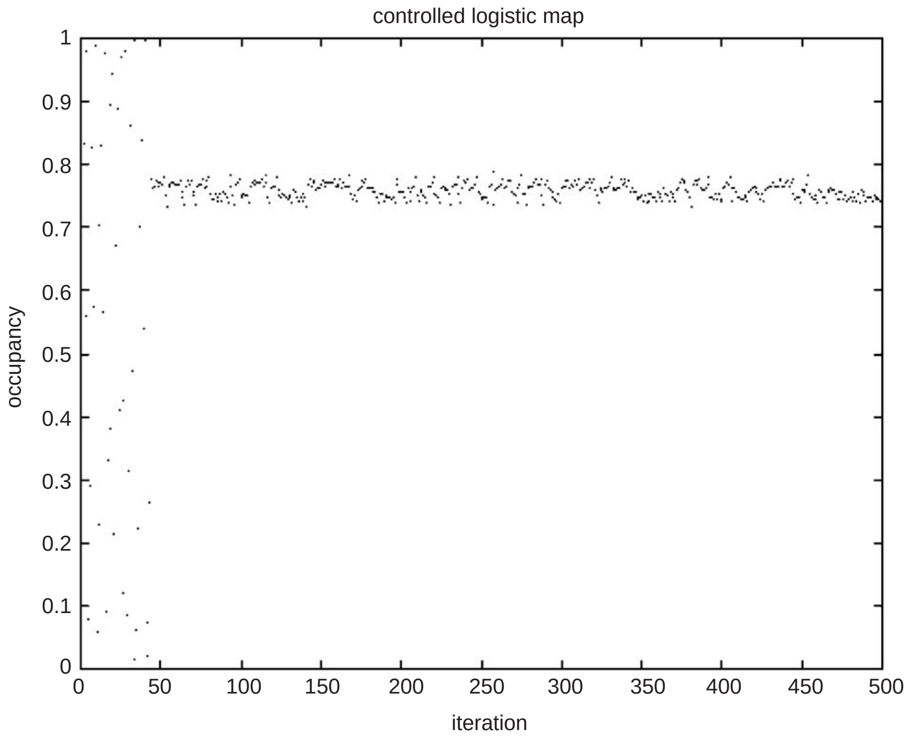

Fig. 6. The output of controlled logistic map with modified controller (initial value = 0.3).

only at the symmetric center. The target $\hat{\rho}_t$ differs slightly from the original unstable fixed point $\hat{\rho}^*$.

In the traffic flow model for example, the controlled model is Eq. (12). From the previous discussion, $L = -2$, $\hat{\rho}^* = 0.75$. Gaussian white noise N (0,0.005) is added. When $\varepsilon = 0.02$, $a = 0.01$, $b = 4$, $d = 0.04$, $d_1 = 0.02$ are determined from Eq. (18). The target is $\hat{\rho}_t = 0.76$. The controlled output is illustrated in Fig. 6. From this experiment, the mean value of the controlled output is estimated at 0.758, which is close to the target of $\hat{\rho}_t = 0.76$. In other experiments, using the same control parameters a, b, d, d_1, the traffic flow model $\hat{\rho}_{n+1} = (u_f/\bar{u})\hat{\rho}_n(1 - \hat{\rho}_n)$ permits control for all $u_f/\bar{u} \in [3.84, 4.0)$. As a result, a precise model is not required. Therefore, this kind of controller can be also designed with an experimentally determined model just as in the OGY method does. In this case, however, the control is biased.

5. Conclusion

This study presents a macroscopic chaotic traffic flow model, which is sensitive to the control parameter. As long as the bifurcation point is observed, the chaotic behavior becomes extremely complicated. Therefore, forecasting the evolution of a chaotic system is very difficult. If the road traffic flow is chaotic, the traffic flow is unstable subsequently forcing drivers to accelerate or decelerate more often. These phenomena not only make drivers uncomfortable and pressurized, but they also decrease total traffic volume. Additionally, unstable traffic typically results in accidents. Since traffic operators should provide stable traffic condition to improve the level of service of road system and safety, the chaos of traffic flow should be controlled.

The proposed control method of chaotic traffic flow follows Pan and Yin's suggestion. The controller traps the chaotic behavior in a designed attractor. In practice, traffic operators and planners design a road or a freeway based on the desired level of service. Therefore controlling chaotic traffic becomes more realizable. A control model can be constructed with a noise background and without exact model. This advantage makes the control strategy applicable to an actual highway. The proposed control method controls traffic flow by designing signal or ramp metering.

Since the control parameter of the developed model is determined by the ratio of the free flow speed and the average speed, managing or planning traffic flow on a highway speed limit control is also a workable strategy. As stated, note that the solution bifurcates when the ratio is greater than 3. Therefore, if the average speed on the highway can be maintained higher than one third of the free flow speed, traffic flow will tend to equilibrium. This may be another operational strategy to control chaotic traffic.

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