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Supermodularity in Mean-Partition Problems*

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Abstract. Supermodularity of the λ function which defines a permutation polytope has proved to be crucial for the polytope to have some nice fundamental properties. Supermodularity has been established for the λ function for the sum-partition problem under various models. On the other hand, supermodularity has not been established for the mean-partition problem even for the most basic labeled single-shape model. In this paper, we fill this gap and also settle for all other models except one. We further extend our results to other types of supermodularity.

Key words: mean-partition, supermodular

1. Introduction

Given a real-value function λ on the subsets of $\{1, \ldots, p\}$ with $\lambda(\phi) = 0$, each permutation $\sigma = (\sigma_1, \ldots, \sigma_p)$ of $\{1, \ldots, p\}$ defines a vector $\lambda_\sigma = ((\lambda_\sigma)_1, \ldots, (\lambda_\sigma)_p)$ such that

$$(\lambda_\sigma)_k = \lambda\left(\cup_{i=1}^{k}\sigma_i\right) - \lambda\left(\cup_{i=1}^{k-1}\sigma_i\right) \quad \text{for } 1 \leqslant k \leqslant p.$$

λ is called *supermodular* if for all subsets I, J of $\{1, \ldots, p\}$,

$$\lambda(I \cup J) + \lambda(I \cap J) \geqslant \lambda(I) + \lambda(J),$$

and *strictly supermodular* if the inequality is strict for all I, J not satisfying $I \subseteq J$ or $J \subseteq I$.

The *permutation polytope* induced by λ, denoted H^λ, is the convex hull of $\{\lambda_\sigma : \text{ all } \sigma\}$. These polytopes have been studied in the literature with different motivations. For example, Shapley [6] studied the case of convex p-person game. For a subset $I \subseteq \{1, \ldots, p,\}$ let $\lambda(I)$ denote the payoff to I if the members of I form an alliance. Then stability of an alliance $I \cup J$ requires λ to be supermodular. If not, say, there exist I and J with

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$$\lambda(I \cup J) + \lambda(I \cap J) < \lambda(I) + \lambda(J).$$

Let y_i be the payoff of player i for each i in $I \cup J$ under the alliance $I \cup J$. Then it is easily verified that either

$$\sum_{i \in I} y_i < \lambda(I), \qquad \text{or } \sum_{i \in J} y_i < \lambda(J).$$

In the first(second) case, $I(J)$ will form its own alliance to obtain a larger payoff.

The *core* of a convex p-person game is the solution set of the linear inequality system

$$\sum_{i \in I} x_i \geqslant \lambda(I) \quad \text{for all } I \subseteq \{1, \ldots, p\} \quad \text{and} \quad \sum_{i=1}^{p} x_i = \lambda(\{1, \ldots, p\}). \quad (1.1)$$

Let C^λ denote the polytope defined by (1.1). Among other things (See Theorem 3.2 of [2] for more details), Shapley proved

THEOREM 1. *Suppose* λ *is supermodular. Then*

(1) $H^\lambda = C^\lambda$,
(2) *the vectors of* H^λ *are precisely the* λ_σ*'s where* σ *ranges over all permutations of* $\{1, \ldots, p\}$.

The importance of Theorem 1 (1) is that if an optimization problem is to maximize a linear function of $\{x_i\}$, then C^λ provides a suitable setting for a linear programming solution. The importance of Theorem 1 (2) is that if the objective function is convex (in fact, quasi-convex suffices, see [5]), then an optimal solution can be found among the set of vertices of H^λ. Gao et al. [2] studied the single-shape sum-partition problem in which the indices of a set N of n real numbers $\theta_1 \leqslant \theta_2 \leqslant \cdots \leqslant \theta_n$ is to be partitioned into p parts $\pi_1, \ldots, \pi_p$, where the size of π_i is given to be $n_i(\{(n_1, \ldots, n_p) : \sum_{i=1}^{p} n_i = n\}$ is called a *shape*), to maximize an objective function $f(\sum_{j \in \pi_1} \theta_j, \ldots, \sum_{j \in \pi_p} \theta_j)$. For I a subset of $\{1, \ldots, p\}$, define $n(I) = \sum_{I \in I} n_i$. They defined $\lambda(I) = \sum_{j=1}^{n(I)} \theta_j$ and proved λ is supermodular. Therefore Theorem 1 is applicable. Here, H^λ is the convex hull of all $(n_i, \ldots, n_p)$-partitions (each partition is a point), and C^λ is the polytope defined by

$$\sum_{i \in I} \sum_{j \in \pi_i} \theta_j \geqslant \lambda(I) \text{ for all } I \subseteq \{1, \ldots, p\} \quad \text{and} \quad \sum_{j=1}^{n} \theta_j = \lambda(\{1, \ldots, p\}).$$

While the sum partition problem has been dominating in optimal partition problems, other partition problems have also been considered. Define $\bar{\theta}_{\pi_i} = \sum_{j\in\pi_i} \theta_j/n_i$, namely, the mean of θ_j's in π_i. Anily and Federgruen [1] first studied the single-shape mean-partition problem where the objective function is $f(\bar{\theta}_{\pi_1}, \dots, \bar{\theta}_{\pi_p})$. However, the function λ as defined in (1.1) has not been proved to be supermodular and Theorem 1 is thus not applicable. In this paper, we prove the supermodularity.

2. Supermodularity

For the single-shape partition case, without loss of generality, we can assume that $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_p$.

For $I = \{i_1, i_2, \dots, i_k\} \subseteq \{1, \dots, p\}$, we suppose that $i_i < i_2 < \cdots < i_k$. Define $N_{i_k} = \sum_{x=1}^{k} n_{i_x}$ for $1 \leqslant k \leqslant |I|$. Set

$$\lambda(I) = \sum_{k=1}^{|I|} \left(\sum_{j=N_{i_{k-1}}+1}^{N_{i_k}} \theta_j/n_{i_k} \right). \tag{2.1}$$

We first prove

LEMMA 2. *For any shape partition* $\pi = (\pi_1, \dots, \pi_p)$, $\sum_{i\in I} \bar{\theta}_{\pi_i} \geqslant \lambda(I)$.

Proof. Define $A = \{\theta_j : j \in \pi_i, i \in I\}$ and $B = \{\theta_1, \dots, \theta_{N_{i_{|I|}}}\}$ Suppose $\lambda(I)$ is defined on A but $A \neq B$. Then we can reduce $\sum_{i\in I} \bar{\theta}_{\pi_i}$ by replacing any $\theta_j \in A\backslash B$ with a $\theta_k \in B\backslash A$. Therefore we assume $A = B$. Note that

$$\bar{\theta}_{\pi_i} = \sum_{j\in\pi_i} \theta_j (1/n_i), \tag{2.2}$$

and $\theta_1, \dots, \theta_{N_{i_{|I|}}}$ are ordered from small to large. In $\lambda(I)$, the sequence of the multipliers for the θ_j's is

$$\underbrace{\frac{1}{n_{i_1}}, \dots, \frac{1}{n_{i_1}}}_{n_{i_1}}, \underbrace{\frac{1}{n_{i_2}}, \dots, \frac{1}{n_{i_2}}}_{n_{i_2}}, \dots, \underbrace{\frac{1}{n_{i_{|I|}}}, \dots, \frac{1}{n_{i_{|I|}}}}_{n_{i_{|I|}}},$$

which are ordered from large to small. Since for any π, $\sum_{i\in I} \bar{\theta}_{\pi_i}$ is computed by multiplying the same set of θ_j's with the same set of multipliers, except in different parings, $\lambda(I)$ achieves the minimum by pairing reversely. □

Define $\Delta_I(\pi) = \lambda(I) - \lambda(I\backslash\{i_1\})$.

LEMMA 3. *Suppose* $I \subset J$ *and* $i_1 = j_1$. *Then* $\Delta_I(\pi) \leqslant \Delta_J(\pi)$.

Proof. First assume $n_{j_1} = 1$

$$J:\ \overbrace{\theta_1}^{\pi_{j_1}},\ \overbrace{\theta_2,\dots,\theta_{n_{j_2}},\theta_{n_{j_2}+1}}^{\pi_{j_2}},\ \overbrace{\theta_{n_{j_2}+2},\dots,\theta_{n_{j_2}+n_{j_3}},\theta_{n_{j_2}+n_{j_3}+1},\dots}^{\pi_{j_3}}$$

$$J':\ \underbrace{\theta_1,\theta_2,\dots,\theta_{n_{j_2}}}_{\pi'_{j_2}},\ \underbrace{\theta_{n_{j_2}+1},\theta_{n_{j_2}+2},\dots,\theta_{n_{j_2}+n_{j_3}}}_{\pi'_{j_3}},\ \theta_{n_{j_2}+n_{j_3}+1},\dots$$

Figure 2.1. π'_{j_2} *and* π'_{j_3}.

Let π' represent the corresponding partition on $J' = J\backslash\{j_1\}$. We use the same subscript j_k to remind the reader that $n_{j_k} = n'_{j_k}$ for all $2 \leqslant k \leqslant |J|$.

Figure 2.1 illustrates $\pi(J)$ and $\pi'(J')$. Note that the components of $\bar{\theta}_{\pi_{j_k}}$ (as in the representation (2.2)) cancels with the components in $\bar{\theta}_{\pi'_{j_k}}$ except the first one in $\bar{\theta}_{\pi_{j_k}}$ and the last one in $\bar{\theta}_{\pi'_{j_k}}$. Hence

$$\bar{\theta}_{\pi_{j_k}} - \bar{\theta}_{\pi'_{j_k}} = \frac{(\theta_{N_{j_k}} - \theta_{N_{j_{k-1}}})}{n_{j_k}} \qquad \text{for } 1 \leqslant k \leqslant |J|.$$

Consequently,

$$\Delta_J(\pi) = \sum_{k=1}^{|J|} \frac{\theta_{N_{j_k}} - \theta_{N_{j_{k-1}}}}{n_{j_k}}.$$

Similarly,

$$\Delta_I(\pi) = \sum_{k=1}^{|I|} \frac{\theta_{N_{i_k}} - \theta_{N_{i_{k-1}}}}{n_{i_k}}.$$

Suppose $i_k = j_{g(k)}$ with $k \leqslant g(k), 2 \leqslant k \leqslant |I|$. Then

$$\begin{aligned} G_k(J) &\equiv \sum_{h=g(k-1)+1}^{g(k)} \frac{\theta_{N_{j_h}} - \theta_{N_{j_{h-1}}}}{n_{j_h}} \\ &\geqslant \sum_{h=g(k-1)+1}^{g(k)} \frac{\theta_{N_{j_h}} - \theta_{N_{j_{h-1}}}}{n_{j_{g(k)}}} = \frac{\theta_{N_{j_{g(k)}}} - \theta_{N_{j_{g(k-1)}}}}{n_{j_{g(k)}}}. \end{aligned} \tag{2.3}$$

Note that

$$\Delta_J(\pi) - \Delta_I(\pi) \geqslant \sum_{x=1}^{|I|} \left[G_x(J) - \frac{(\theta_{N_{i_x}} - \theta_{N_{i_{x-1}}})}{n_{i_x}} \right].$$

We prove for all $1 \leqslant k \leqslant |I|$,

$$\sum_{x=1}^{k} \left[G_x(J) - \frac{(\theta_{N_{i_x}} - \theta_{N_{i_{x-1}}})}{n_{i_x}} \right] \geqslant \frac{(\theta_{N_{j_{g(k)}}} - \theta_{N_{i_k}})}{n_{i_k}},$$

by induction on k. For $k=1$

$$G_1(J) - \frac{(\theta_{N_{i_1}} - \theta_{N_{i_0}})}{n_{i_1}} = \frac{(\theta_{N_{j_1}} - \theta_{N_{j_0}})}{n_{j_1}} - \frac{(\theta_{N_{i_1}} - \theta_{N_{i_0}})}{n_{i_1}} = 0$$

since $j_1 = i_1, N_{i_1} = n_{i_1} = n_{j_1} = N_{j_1} = 1, \theta_{N_{j_0}} = \theta_{N_{j_0}} = \theta_{N_{i_0}} = 0$. For general $k > 1$,

$$\begin{aligned}
\sum_{x=1}^{k} \left[G_x(J) - \frac{(\theta_{N_{i_x}} - \theta_{N_{i_x-1}})}{n_{i_x}} \right] &\geqslant G_k(J) - \frac{(\theta_{N_{i_k}} - \theta_{N_{i_k-1}})}{n_{i_k}} + \frac{(\theta_{N_{j_{g(k-1)}}} - \theta_{N_{i_k-1}})}{n_{i_k-1}} \\
&\geqslant \frac{(\theta_{N_{j_{g(k)}}} - \theta_{N_{j_{g(k-1)}}})}{n_{j_g(k)}} - \frac{(\theta_{N_{i_k}} - \theta_{N_{i_k-1}})}{n_{i_k}} \\
&\quad + \frac{(\theta_{N_{j_{g(k-1)}}} - \theta_{N_{i_{k-1}}})}{n_{i_k}} \\
&= \frac{(\theta_{N_{j_{g(k)}}} - \theta_{N_{i_k}})}{n_{i_k}},
\end{aligned}$$

since $n_{j_{g(k)}} = n_{i_k} \geqslant n_{i_{k-1}}$. Lemma 3 is proved.

For $n_j > 1$, we can handle in two ways. The first way is to notice that the only difference from the $n_{j_1} = 1$ case is that π_{j_k} and π'_{j_k} would miss each other out in n_{j_1} elements instead of 1 in Figure 2.1. So the numerator of (2.3) would be a difference between two n_{j_k} -sums; but the same logic applies. The second way is to notice that $\bar{\theta}_{n_{j_1}}$ gets canceled out in $\Delta_J(\pi) - \Delta_I(\pi)$. So the scenario is to compare the impact on I and J when both moves back n_{j_1} elements. But this is equivalent to moving one element back n_{j_1} times. □

Finally, we are ready to prove the main result of this section.

THEOREM 4. *λ as defined in* (2.1) *is supermodular.*

Proof. Let I and J, be two subsets of $\{1, \ldots, p\}$. Without loss of generality, assume $I \cup J = \{1, 2, \ldots, m\}$. We prove Theorem 4 by induction on m. Theorem 4 is trivially true for $m = 1$. We prove the general $m \geqslant 2$ case.

Case (1) $1 \in I \cap J$, i.e. both I and J contain 1. Delete π_1 and the θ_j's in it. Suppose $n_1 = k$. Then the reduced partition problem is to partition the set $\{\theta_{k+1}, \ldots, \theta_n\}$ into $p-1$ parts. Theorem 4 follows by induction.

Case (2) $1 \notin I \cap J$. Without loss of generality, assume $1 \in I$. Let $J^* = J \cup \{1\}$. By case (1),

$$\begin{aligned} 0 &\leqslant \lambda(I \cup J^*) + \lambda(I \cap J^*) - \lambda(I) - \lambda(J^*) \\ &= [\lambda(I \cup J^*) - \lambda(I)] + [\lambda(I \cap J^*) - \lambda(J^*)] \\ &\leqslant [\lambda(I \cup J) - \lambda(I)] + [\lambda(I \cap J) - \lambda(J)]. \end{aligned}$$

Since the first difference is unchanged, and the second becomes larger by Lemma 3, i.e., $\lambda(I \cap J^*) - \lambda(I \cap J) = \Delta_{I \cap J^*}(\pi) \leqslant \Delta_{J^*}(\pi) = \lambda(J^*) - \lambda(J)$. □

3. Other Mean-Partition Models

In the last two sections we studied the labeled single-shape partition problem where the λ function is defined on a single shape. In this section we study some other partition models which have been studied before [4] for the sum-partition problem. One common feature of these models is that λ is defined on a given set S of shapes. For example, in the unlabeled single-shape model, let $\{n_1, n_2, \ldots, n_p\}$ denote the given single shape. Then S consists of all permutations of $\{n_1, n_2, \ldots, n_p\}$. In the labeled bounded-shape model, a set of lower and upper bounds $L_i \leqslant n_i \leqslant U_i, i = 1, \ldots p$, is given, and S consists of all shapes $\{n_l, n_2, \ldots, n_p\}$ satisfying the bounds with $\sum_{i=1}^{p} n_i = n$. In the labeled constraint-shape model, S is a given set of shapes with each summing to n. In the unlabeled version for either the bounded-shape of the constraint-shape model, S consists of all permutations of a shape in the labeled version.

Let $\lambda_s(I)$ denote the $\lambda(I)$ in (2.1) where I is taken from the shape $s \in S$. Define $\lambda(I) = \min_{s \in S} \lambda_s(I)$. Then clearly

LEMMA 5. *For any partition* $\pi = (\pi_1, \pi_2, \ldots, \pi_p)$ *with shape* $s = (n_1, n_2, \ldots, n_p), s \in S$, $\sum\limits_{i \in I} \bar{\theta}_{\pi_i} \geqslant \lambda(I)$.

Let $X \Rightarrow Y$ mean supermodularity for model X implies for model Y. Then for both the labeled and unlabeled case, clearly,

constraint-shape ⇒ bounded-shape ⇒ single-shape.

For the sum-partition problem, the following results have been obtained [4]:

Labeled	Shape	θ	Supermodularity
Yes	Single	General	Yes
Yes	Bounded	General	Yes
Yes	Constrained	1-side	No
No	Single	1-sided	Yes
No	Single	General	No
No	Bounded	1-sided	No
No	Constrained	1-sided	No

Here, 1-sided means that θ's are either all nonnegative or all nonpositive. In this section, we also consider the supermodularity properties of λ for various mean-partition models. Note that only the ordering of θ's, but not their signs, matters for the mean-partition problem. Therefore there is no need to study the 1-sided case.

LEMMA 6. *Let $S=\{s\}$ denote the set of all permutations of the shape s. Consider $s=\{n_1,n_2,\dots,n_p\}$ and $I=\{1,2,\dots,k\}$, then for all $I'=\{i_1,i_2,\dots,i_k\}$ with $i_1<i_2<\dots<i_k$, and $\{i_1,i_2,\dots,i_k\}\subseteq\{1,2,\dots,p\}$, $\lambda(I')\geqslant\lambda(I)$.*

Proof. Since θ_j is increasing and $i_h\geqslant h$, $1\leqslant h\leqslant k$. We have $N_{i_h}\geqslant N_h$ and $\theta_{N_{i_h}+x}\geqslant\theta_{N_h+x}$ for all $x>0$.

$$\sum_{j=N_{h-1}+1}^{N_h}\frac{\theta_j}{n_h}\leqslant\sum_{j=N_{h-1}+1}^{N_{h-1}+n_{i_h}}\frac{\theta_j}{n_{i_h}}\leqslant\sum_{j=N_{i_{h-1}}+1}^{N_{i_h}}\frac{\theta_j}{n_{i_h}}.$$

Then,

$$\lambda(I)=\sum_{h=1}^{|I|}\left(\sum_{j=N_{h-1}+1}^{N_h}\frac{\theta_j}{n_h}\right)\leqslant\sum_{h=1}^{|I'|}\sum_{j=N_{i_{h-1}}+1}^{N_{i_h}}\frac{\theta_j}{n_{i_h}}=\lambda(I').$$

□

THEOREM 7. Consider $S=\{(n_1,n_2,\dots,n_p)\}$. Then λ is supermodular.

Proof.

$$\lambda(I)=\sum_{h=1}^{|I|}\left(\sum_{j=N_{h-1}+1}^{N_h}\frac{\theta j}{n_h}\right)\quad\text{for all}\quad I\subseteq\{1,2,\dots,p\}.$$

We may assume that $I\cap J=\emptyset$. Suppose to the contrary that $I\cap J\neq\emptyset$. We can delete n_i's, for all $i\leqslant|I\cap J|$ and θ_j, for all $j\leqslant N_{|I\cap J|}$. Then the reduced partition problem is to partition the set $\{\theta_{N_{I\cap J}+1},\dots,\theta_n\}$ into $p-|I\cap J|$ parts with $I' \cap J' = \emptyset$. Without loss of generality, let $I \cup J = \{1, 2, \ldots, |I| + |J|\}$, $I = \{1, 2, \ldots, |I|\}$ and $J = \{|I|+1, |I|+2, \ldots, |I|+|J|\}$. Then

$$\begin{aligned}\lambda(I)+\lambda(J) &= \sum_{h=1}^{|I|}\left(\sum_{j=N_{h-1}+1}^{N_h}\frac{\theta_j}{n_h}\right)+\sum_{h=1}^{|J|}\left(\sum_{j=N_{h-1}+1}^{N_h}\frac{\theta_j}{n_h}\right)\\ &\leqslant \sum_{h=1}^{|I|}\left(\sum_{j=N_{h-1}+1}^{N_h}\frac{\theta_j}{n_h}\right)+\sum_{h=1}^{|J|}\left(\sum_{j=N_{|I|+h-1}+1}^{N_{|I|+h}}\frac{\theta_j}{n_{|I|+h}}\right)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Lemma 6)}\\ &= \lambda(I\cup J).\end{aligned}$$

□

For a given p-vector $(a_1, a_2, \ldots, a_p)$, let $a_{[i]}$ denote the i-th smallest a_j. A p-vector $A = (a_1, a_2, \ldots, a_p)$ *majorizes* another p-vector $B = (b_1, b_2, \ldots, b_p)$ if for all $1 \leqslant k \leqslant p-1$

$$\sum_{i=1}^{k} a_{[i]} \leqslant \sum_{i=1}^{k} b_{[i]}, \quad \text{for all } 1 \leqslant k \leqslant p-1, \quad \text{and} \quad \sum_{i=1}^{p} a_i = \sum_{i=1}^{p} b_i.$$

For a set S of shapes, $A \in S$ is a *nonmajorized shape* if there does not exist a shape $B \in S$ such that B majorizes A.

Next we show by a counterexample that for the unlabled bounded-shape model, λ is not supermodular. Note that we only need to consider that λ takes values from the set of nonmajorized shapes since if a shape B is majorized by another shape A, then $\lambda_B(I) \geqslant \lambda_A(I)$ and B would not be chosen in defining $\lambda(I)$.

Let $p = 4, n = 19, l_1 = 1, l_2 = l_3 = l_4 = 2, u_1 = 13, u_2 = u_3 = u_4 = 6, \theta_1 = 1, \theta_2 = \cdots = \theta_6 = 2, \theta_i = 5, 7 \leqslant i \leqslant 19, I = \{1, 2\}, J = \{1, 3\}$. The nonmajorized shapes are {(1, 6, 6, 6), (13,2,2,2) and their permutations}

$$\begin{aligned}\lambda(I) &= \left(\frac{1+2}{2}+\frac{2+2}{2}\right)\text{or}\left(\frac{1}{1}+\frac{2+2+2+2+2+5}{6}\right)=\frac{7}{2}=\lambda(J),\\ \lambda(I\cap J) &= \frac{1}{1}=1,\\ \lambda(I\cup J) &= \frac{1+2}{2}+\frac{2+2}{2}+\frac{2+2}{2}=\frac{11}{2},\\ \lambda(I)+\lambda(J) &= 7>\frac{13}{2}=\lambda(I\cap J)+\lambda(I\cup J).\end{aligned}$$

Next we show by a counterexample that for the labeled constrained shape model, λ is not supermodular. Let $p = 4, n = 19, S = \{(2, 2, 2, 13), (1, 6, 6, 6)\}$,

$\theta_1 = 1, \theta_2 = \cdots = \theta_6 = 2, \theta_i = 5, 7 \leqslant i \leqslant 19, I = \{1, 2\}, J\{1, 3\}$.

$$\lambda(I) = \left(\frac{1+2}{2} + \frac{2+2}{2}\right) \text{or} \left(\frac{1}{1} + \frac{2+2+2+2+2+5}{6}\right) = \frac{7}{2} = \lambda(J),$$

$$\lambda(I \cap J) = \frac{1}{1} = 1,$$

$$\lambda(I \cup J) = \frac{1+2}{2} + \frac{2+2}{2} + \frac{2+2}{2} = \frac{11}{2},$$

$$\lambda(I) + \lambda(J) = 7 > \frac{13}{2} = \lambda(I \cap J) + \lambda(I \cup J).$$

The following table summarizes our results.

Labeled	Shape	Supermodularity
Yes	Single	Yes
Yes	Bounded	?
Yes	Constrained	No
No	Single	Yes
No	Bounded	No
No	Constrained	No

Finally, we give a sufficient condition for establishing supermodularity.

THEOREM 8. *Let Π be an unlabeled (labeled) bounded-shape set. If $\lambda(I \cap J)$ and $\lambda(I \cup J)$ can take values from the same shape A, then $\lambda(I) + \lambda(J) \leqslant \lambda(I \cap J) + \lambda(I \cup J)$.*

Proof. $\lambda(I) \leqslant \lambda_A(I), \lambda(J) \leqslant \lambda_A(J)$. Since supermodularity holds for the single shape A. $\lambda(I) + \lambda(J) \leqslant \lambda_A(I) + \lambda_A(J) \leqslant \lambda_A(I \cap J) + \lambda_A(I \cup J) = \lambda(I \cap J) + \lambda(I \cup J)$. □

4. Stronger Supermodularites

Hwang et al. [3] defined the notion of strong supermodularity which lies between supermodularity and strict supermodularity. Let I, J, K be subsets of $\{1, 2, \ldots, p\}$ such that $I \subset K \subset J$. Define $L = I \cup (J \backslash K)$. A triplet (I, J, K) is called *λ-flat* if $\lambda(I) + \lambda(J) = \lambda(K) + \lambda(L)$. λ is *strongly-modular* if λ is supermodular and for every pair $I \subset J$ if there exists a $K, I \subset K \subset J$ such that (I, J, K) is λ-flat, then for every $K', I \subset K' \subset J, (I, J, K')$ is λ-flat. Note that strict supermodularity implies there is no λ-flat triplet, hence strict supermodularity implies strong supermodularity.

It was shown in [3] that if the λ function defining a permutation polytope is strongly supermodular, then the polytope has many extra nice properties. They also proved that the λ function for the single-shape sum-partition problem is strongly supermodular; if the θ's are distinct, then it is strictly supermodular. In this section we study the stronger supermodularities for the mean-partition problem.

We first settle the easy strict supermodularity issue. If θ's are not all distinct, then clearly, λ is not strictly supermodular even for the single-shape model, labeled or unlabeled. On the other hand, if θ's are all distinct, then the inequalities in Theorems 4 and 7 are all strict and strict supermodularity holds.

Next we deal with the strong supermodularity case. We first show by a counterexample that for the labeled single-shape mean-partition problem, the λ function as studied in Section 2 is not strongly supermodular.

Let $(n_1, n_2, n_3, n_4) = (1, 2, 3, 4)$, $I = \{3\}$, $J = \{1, 2, 3, 4\}$ and $\theta = \{\theta_1, \theta_2, \ldots, \theta_{10}\}$. It is easily verified:

(1) $K = \{1, 3, \}, L = \{2, 3, 4\}$.
Then $\lambda(I) + \lambda(J) = \lambda(K) + \lambda(L) \Leftrightarrow \theta_1 = \theta_3, \theta_4 = \theta_{10}$,
(2) $K' = \{1, 2, 3\}, L' = \{3, 4\}$.
Then $\lambda(I) + \lambda(J) = \lambda(K') + \lambda(L') \Leftrightarrow \theta_4 = \theta_{10}$.

Since the two sets of conditions are different, we can easily construct a set θ such that the condition in (2) is satisfied but not the condition in (1), for example, $\theta = \{1, 2, 2, 2, 2, 2, 2, 2, 2, 2\}$.

We next prove

THEOREM 9. *For the unlabeled single-shape model, λ is strongly supermodular.*

Proof. Let $I \subset J$. If $\theta_{N_{|I|}+1} = \theta_{N_{|J|}}$, then every triplet (I, J, K) is λ-flat. On the other hand if there is a triplet (I, J, K) which is λ-fiat, without loss of generality, let $|K| \leqslant |L|$, then it is easily verified that

$$\sum_{i=0}^{|K|-|I|} \sum_{j=N_{|I|+i}+1}^{N_{|I|+i+1}} \frac{\theta_j}{n_{|I|+i}} = \sum_{i=0}^{|J|-|L|} \sum_{j=N_{|L|+i}+1}^{N_{|L|+i+1}} \frac{\theta_j}{n_{|L|+i}},$$

and $N_{|K|} < N_{|L|}$, then $\theta_{N_{|I|}+1} = \theta_{N_{|J|}}$. □

We summarize our results in the following table for those mean-partition models considered in Section 3 for which supermodularity holds:

Labeled	Shape	Distinctθ	Supermodularity
Yes	Single	No	Not strong
Yes	Single	Yes	Strict
No	Single	No	Strong, but not strict
No	Single	Yes	Strict

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