Optimal Static Output Feedback Simultaneous Regional Pole Placement

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***Abstract*—The problem of optimal simultaneous regional pole placement for a collection of linear time-invariant systems via a single static output feedback controller is considered. The cost function to be minimized is a weighted sum of quadratic performance indices of the systems. The constrained region for each system can be the intersection of several open half-planes and/or open disks. This problem cannot be solved by the linear matrix inequality (LMI) approach since it is a nonconvex optimization problem. Based on the barrier method, we instead solve an auxiliary minimization problem to obtain an approximate solution to the original constrained optimization problem. Moreover, solution algorithms are provided for finding the optimal solution. Furthermore, a necessary and sufficient condition for the existence of admissible solutions to the simultaneous regional pole placement problem is derived. Finally, two examples are given for illustration.**

***Index Terms*—Barrier method, constrained optimization, regional pole placement, simultaneous stabilization.**

I. Introduction

THE problem of simultaneous stabilization for a collection of linear systems via a single controller is an important issue in robust control theory (see [1], [2], [4], [13], and [25]). This problem concerned with the determination of a single controller which will simultaneously stabilize a finite collection of systems. The simultaneous stabilization problem arises frequently in practice, due to plant uncertainty, plant variation, failure modes, plants with several modes of operation, or nonlinear plants linearized at several different equilibria. In [24], a nonlinear state feedback controller which simultaneously stabilizes a collection of single input systems is presented. In [10], a necessary and sufficient condition, embedded in the solvability of a constrained optimization problem, for the existence of controllers to simultaneously stabilize a collection of single input systems is obtained. In [11], [18], and [23], the optimal simultaneous stabilizing state feedback controllers are found via numerically solving a minimization problem. The cost function to be minimized is a weighted sum of the quadratic performance indices of the systems. In [6] and [7], necessary and sufficient conditions for simultaneous stabilizability of a collection of multi-input multi-output (MIMO) systems via static output feedback and state feedback are obtained in the form of coupled algebraic Riccati inequalities. Moreover, in [20] and [21], linear periodically time-varying controllers are used for simultaneous stabilization and performance or disturbance rejection.

Although many researches have focused on the simultaneous stabilization problem in recent years, the optimal simultaneous regional pole placement problem has not been considered yet. The minimization of quadratic cost functions can indeed improve the systems' static responses (see [5] and [17]). However, it cannot guarantee that the closed-loop systems have good transient responses. The systems' transient responses are determined mainly by the locations of the systems' poles. If we can assign the systems' poles to some specified regions, then good transient responses can be guaranteed. For the single system case, in [8], [14]–[16], and [26], the authors determined a feedback controller for a system such that the closed-loop poles lie within a specified region. Moreover, a quadratic cost function being minimized by the resultant controller is found. Nevertheless, for a given cost function, how to find the optimal controller subject to the regional pole's constraint has not been discussed. In [9], the authors solved a modified Lyapunov equation to obtain a controller which minimizes an auxiliary cost and guarantees that the resultant closed-loop poles lie in a desired region. This auxiliary cost provides a guaranteed upper bound on the original quadratic cost function. However, how to find the optimal controller to minimize the actual cost subject to the regional pole's constraint is still unsolved. Up until now, the existing results about the (optimal) regional pole placement problem are focused on single system case. The optimal simultaneous regional pole placement problem for a collection of systems has yet to be addressed.

In this paper, we provide a new method to solve output feedback optimal simultaneous regional pole placement problem for a collection of systems. The considered cost function is a weighted sum of quadratic performance indices of the systems; and the constrained region for each system can be the intersection of several open half-planes and/or open disks. This is a constrained optimization problem and its minimum point may not exist. It often happens that its infimum point lies on the boundary of the admissible solution set, and it is not a stationary point. Therefore, the Lagrange multiplier method cannot be employed to derive the necessary conditions for optimum for this problem. To solve this problem analytically is quite difficult. Moreover, this problem cannot be solved via the linear matrix inequality (LMI) approach since the admissible solution set may be nonconvex. In general, static output feedback control problems are very difficult to solve [28]. It has been shown in [3] that simultaneous stabilization by static output feedback is NP-hard. In

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this paper, based on the barrier method (see [22]), we instead solve an auxiliary minimization problem to obtain an approximate solution of the original problem. The new cost function is the sum of the actual cost function of the original problem and a weighted "barrier function." Necessary and sufficient conditions for the existence of admissible solutions are given. We prove that the minimal solution of the auxiliary minimization problem exists if the admissible solution set is nonempty. Moreover, it is a stationary point. Then the Lagrange multiplier method can be used to derive the necessary conditions for optimum of the auxiliary minimization problem. In fact, the minimal solution of the auxiliary minimization problem converges to the infimal solution of the original problem if the weighting factor of the barrier function approaches zero. Unlike the approaches presented in [8], [14]–[16], and [26] for the single-system case, in our approach, we can get a solution very close to a local infimal solution of the considered problem. When the poles' constraint is relaxed, a necessary and sufficient condition for the existence of the simultaneous stabilizing static output feedback controller is found in form of coupled matrix equalities. Finally, two numerical examples are provided for illustration. Based on the gradient method, numerical algorithms are provided in Example 1 to demonstrate how to solve the auxiliary minimization problem.

A. Notations

$\boldsymbol{E}(.)$	expected value;
$\sigma(M)$	spectrum of the matrix M;
$\mathrm{Tr}(M)$	trace of the matrix M;
$M^T(M^*)$	(conjugate) transpose of the matrix M;
$\|M\|_s$	$\|M\|_s \equiv \sqrt{\lambda_{\max}(M^*M)}$, the spectral norm of the matrix M;
$M > 0 (\geq 0)$	matrix M is positive (semi)definite;
$\overline{\alpha}$	complex conjugate of $\alpha \in C$;
$a \rightarrow b$	a approaches b;
$o(\cdot)$	order of.

II. Problem Formulation and Preliminaries

Consider a collection of p linear time-invariant systems

$$\begin{aligned} \dot{x}_k(t) &= A_k x_k(t) + B_k u_k(t),\ k = 1, 2, \ldots, p \\ y_k(t) &= C_k x_k(t) \end{aligned} \tag{1}$$

where $x_k \in R^n$ is the state of the kth system, $u_k \in R^m$ is the control input of the kth system, and $\mathrm{y}_k \in R^r$ is the output of the kth system; A_k, B_k, and C_k are constant matrices of appropriate dimensions. Suppose that (A_k, B_k) is controllable, (A_k, C_k) is observable and C_k has full row rank for all k. Let $\alpha \in \boldsymbol{C}, \beta \in \boldsymbol{C}$, $\theta \in R$, and $\gamma > 0$. Define

$$\begin{aligned} H(\alpha, \theta) &\equiv \{ s \in \boldsymbol{C} | \mathrm{Re}[e^{-\mathbf{i}\theta}(s - \alpha)] < 0 \} \\ D(\beta, \gamma) &\equiv \{ s \in \boldsymbol{C} \, | |s - \beta| < \gamma \}. \end{aligned}$$

Note that $H(\alpha, \theta)$ denotes an open half-plane and $D(\beta, \gamma)$ is an open disk with radius γ and centered at β. The region $H(0,0)$ is the open left half-plane.

The design goal is to find a static output feedback gain F such that the controllers

$$u_k(t) = F \cdot y_k(t), \quad k = 1, 2, \ldots, p \tag{2}$$

achieve the infimum of the cost function

$$J(F) = \sum_{k=1}^{p} w_k J_k(F) \tag{3}$$

subject to the constraints that

$$\sigma(A_k + B_k F C_k) \in \Omega_k,\ k = 1, 2, \ldots, p$$

where $w_k > 0$, $k = 1, 2, \ldots, p$, are weighting factors and $J_k(F)$ is defined as

$$J_k(F) = \boldsymbol{E} \left\{ \int_0^\infty \left(x_k^T(t) Q_k x_k(t) + u_k^T(t) R_k u_k(t) \right) dt \right\}.$$

Suppose $R_k = R_k^T > 0$, $Q_k = D_k^T D_k \geq 0$ with (A_k, D_k) being observable, and the constrained region Ω_k is represented by

$$\begin{aligned} \Omega_k \equiv \Big\{ s \in \boldsymbol{C} | \, &\mathrm{Re}[e^{-\mathbf{i}\theta_{ki}}(s - \alpha_{ki})] < 0,\ i = 1, 2, \ldots, l_k \\ &\text{and } |s - \beta_{kj}| < \gamma_{kj},\ j = 1, 2, \ldots, c_k \Big\} \subset H(0,0). \end{aligned}$$

Each Ω_k can be the intersection of several open half-planes and open disks. Note that the region Ω_k must be symmetric with respect to the real axis in order to obtain a real feedback gain. The selection of weighting factors w_k, $k = 1, 2, \ldots, p$, depends on requirements of practical applications. If we want the i-*th* system has better *LQ* performance, then we can choose larger w_i. In contrast, if the *LQ* performance of the j-*th* system is less important comparing to the other systems, then we can choose smaller w_j.

Let $A_{kc} = A_k + B_k F C_k$ and let

$$\begin{aligned} \Gamma_k^s &\equiv \{ F \in R^{m \times r} \mid \sigma(A_k + B_k F C_k) \subset H(0,0) \} \\ \Gamma^s &= \Gamma_1^s \cap \Gamma_2^s \cap \cdots \cap \Gamma_p^s \\ \Gamma_k^r &\equiv \{ F \in R^{m \times r} \mid \sigma(A_k + B_k F C_k) \subset \Omega_k \} \\ \text{and } \Gamma^r &= \Gamma_1^r \cap \Gamma_2^r \cap \cdots \cap \Gamma_p^r. \end{aligned}$$

The set Γ_k^s is the collection of all matrices $F \in R^{m \times r}$ such that the kth closed-loop system is stable; the set Γ^s is the collection of all matrices $F \in R^{m \times r}$ such that all the closed-loop systems are stable; the set Γ_k^r is the collection of all matrices $F \in R^{m \times r}$ such that all the closed-loop poles of the kth system lie in the region Ω_k; and the set Γ^r is the collection of all matrices $F \in R^{m \times r}$ such that all the closed-loop poles of the systems are located in the desired regions.

It is shown in [17] that the objective function $J_k(F)$ is equivalent to

$$J_k(F) = \begin{cases} \mathrm{Tr}(P_k X_{k0}), & \text{if } F \in \Gamma_k^s \\ \infty, & \text{otherwise} \end{cases}$$

where $X_{k0} = \boldsymbol{E}\{x_k(0) x_k^T(0)\}$ and $P_k = P_k^T \geq 0$ is the unique solution of

$$A_{kc}^T P_k + P_k A_{kc} + C_k^T F^T R_k F C_k + Q_k = 0. \tag{4}$$

Therefore, the cost function $J(F)$ becomes

$$J(F) = \begin{cases} \sum_{k=1}^{p} w_k \mathrm{Tr}(P_k X_{k0}), & \text{if } F \in \Gamma^s \\ \infty, & \text{otherwise.} \end{cases} \quad (5)$$

Suppose that X_{k0}, $k = 1, 2, \ldots, p$, are positive definite. Two useful lemmas are introduced in the following.

Lemma 1 [12]: All the eigenvalues of the matrix M lie in the region $H(\alpha, \theta)$ if, and only if, for any given matrix $Q = Q^T > 0$, the equation

$$e^{\mathbf{i}\theta}(M - \alpha I)^* P + e^{-\mathbf{i}\theta} P(M - \alpha I) + Q = 0$$

has a unique solution $P = P^* > 0$. $\Delta\Delta$

Lemma 2 [8]: All the eigenvalues of the matrix M lie in the region $D(\beta, \gamma)$ if, and only if, for any given matrix $Q = Q^T > 0$, the equation

$$\frac{1}{\gamma^2}(M - \beta I)^* P(M - \beta I) - P + Q = 0$$

has a unique solution $P = P^* > 0$. $\Delta\Delta$

III. AUXILIARY MINIMIZATION PROBLEM

The considered problem is a constrained optimization problem. To solve this problem analytically is difficult since its minimal solution may not exist. In fact, its infimal solution may lie on the boundary of the set Γ^r; and furthermore, it may not be a stationary point. In this paper, motivated by the barrier method (Luenberger [22]), we instead solve an *auxiliary minimization problem* to obtain an approximate solution of the original problem. The auxiliary cost function $J_{\mathrm{aux}}(F)$ is the sum of the actual cost function $J(F)$ and an additional weighted barrier function $J_{\mathrm{pole}}(F)$. The *auxiliary minimization problem* is formulated as: Find F, over Γ^r, to minimize the *auxiliary cost function*

$$J_{\mathrm{aux}}(F) = J(F) + \rho \cdot J_{\mathrm{pole}}(F)$$

where the term $J(F)$ is defined in (3), ρ is the weighting factor

$$J_{\mathrm{pole}}(F) = \begin{cases} \sum_{k=1}^{p} \left(\sum_{i=1}^{l_k} \mathrm{Tr}(\widetilde{P}_{ki}) + \sum_{j=1}^{c_k} \mathrm{Tr}(\hat{P}_{kj}) \right), & \text{if } F \in \Gamma^r \\ \infty, & \text{otherwise} \end{cases} \quad (6)$$

and matrices $\widetilde{P}_{ki} > 0$ and $\hat{P}_{kj} > 0$ are the solutions of

$$e^{\mathbf{i}\theta_{ki}}(A_{kc} - \alpha_{ki} I)^* \widetilde{P}_{ki} + e^{-\mathbf{i}\theta_{ki}} \widetilde{P}_{ki}(A_{kc} - \alpha_{ki} I) + C_k^T F^T \widetilde{R}_{ki} F C_k + \widetilde{Q}_{ki} = 0 \quad (7)$$

and

$$\frac{1}{\gamma_{kj}^2}(A_{kc} - \beta_{kj} I)^* \hat{P}_{kj}(A_{kc} - \beta_{kj} I) - \hat{P}_{kj} + C_k^T F^T \hat{R}_{kj} F C_k + \hat{Q}_{kj} = 0 \quad (8)$$

respectively, with $\widetilde{Q}_{ki} = \widetilde{Q}_{ki}^T > 0$ and $\hat{Q}_{kj} = \hat{Q}_{kj}^T > 0$.

Let $\otimes$ denote the *Kronecker product*, vec$(\cdot)$ denote the operator of stacking the column vectors of a $n \times m$ matrix to a $1 \times nm$ column vector, and vec^{-1} be the inverse operator of vec (see [9]). As shown in [22], a barrier function must satisfy: 1) it is continuous, 2) it is non-negative over the set Γ^r, and 3) it will approach infinity as F approaches the boundary of the set Γ^r. Now we will show that the function $J_{\mathrm{pole}}(F)$ satisfies these three conditions.

Lemma 3: The function $J_{\mathrm{pole}}(F)$ defined in (6) satisfies the following.

1) $J_{\mathrm{pole}}(F)$ is continuous in the set Γ^r.
2) $J_{\mathrm{pole}}(F) > 0$ over the set Γ^r.
3) $J_{\mathrm{pole}}(F)$ approaches infinity as F approaches the boundary of the set Γ^r.

Proof:

1) We first show that Tr$(\widetilde{P}_{ki})$ is continuous in the set Γ_k^r for fixed k and i. Using vec(.) operator in (7) yields

$$\Psi(F) \cdot \mathrm{vec}(\widetilde{P}_{ki}) = -\,\mathrm{vec}\left(C_k^T F^T \widetilde{R}_{ki} F C_k + \widetilde{Q}_{ki}\right)$$

$$\text{where } \Psi(F) \equiv (e^{-\mathbf{i}\theta_{ki}}(A_{kc} - a_{ki} I))^T \otimes I + I \otimes (e^{-\mathbf{i}\theta_{ki}}(A_{kc} - a_{ki} I))^*.$$

If $F \in \Gamma_k^r$, then $\Psi(F)$ is nonsingular and

$$\mathrm{Tr}(\widetilde{P}_{ki}) = -\mathrm{Tr}\left(\mathrm{vec}^{-1}\left(\Psi(F)^{-1} \cdot \mathrm{vec}\left(C_k^T F^T \widetilde{R}_{ki} F C_k + \widetilde{Q}_{ki}\right)\right)\right). \quad (9)$$

The right-hand side of (9) is smooth in Γ_k^r.

Note that the solution of discrete Lyapunov equation (8) can be expressed as (10), shown at the bottom of the page, which is a rational function of the matrix F. So, Tr$(\hat{P}_{kj})$ is smooth in the set Γ_k^r (see [19]). From the definitions of Γ^r and $J_{\mathrm{pole}}(F)$, it follows that $J_{\mathrm{pole}}(F)$ is continuous in the set Γ^r.

2) As stated in Lemmas 1 and 2, $\widetilde{P}_{ki}$ and $\hat{P}_{kj}$ are positive definite in the set Γ_k^r. Therefore, $J_{\mathrm{pole}}(F) > 0$ in the set Γ^r.

3) Let $\{F(q)\}$ be an infinite sequence of gain approaching the boundary of Γ_k^r from the interior. Then there exists an eigenvalue $\lambda(q) \in \sigma(A_k + B_k F(q) C_k)$ approaching the boundary of Ω_k as $q \to \infty$. Suppose $\widetilde{P}_{ki}(q)$ and $\hat{P}_{kj}(q)$ are the solutions of (7) and (8), respectively, with F being replaced by $F(q)$. We first show that if the sequence $F(q)$ is such that Re$\left\{e^{-\mathbf{i}\theta_{ki}}(\lambda(q) - \alpha_{ki})\right\} \to 0^-$, then

$$\hat{P}_{kj} = \sum_{q=0}^{\infty} \frac{1}{\gamma_{kj}^2} \left((A_k + B_k F C_k - \beta_{kj} I)^*\right)^q \left(C_k^T F^T \hat{R}_{kj} F C_k + \hat{Q}_{kj}\right) (A_k + B_k F C_k - \beta_{kj} I)^q \quad (10)$$

$\mathrm{Tr}(\widetilde{P}_{ki}(q)) \rightarrow \infty$. Suppose $v(q)$ is the normalized eigenvector corresponding to $\lambda(q)$. Premultiplying and postmultiplying (7) by $v^*(q)$ and $v(q)$, respectively, and after some manipulations, we have

$$v^*(q)\widetilde{P}_{ki}(q)v(q) = -\frac{v^*(q)\cdot(C_k^T F^T(q)\widetilde{R}_{ki}F(q)C_k + \widetilde{Q}_{ki})\cdot v(q)}{2Re\{e^{-\mathbf{i}\theta_{ki}}(\lambda(q)-\alpha_{ki})\}}.$$

Since

$$v^*(q)\cdot\left(C_k^T F^T(q)\widetilde{R}_{ki}F(q)C_k + \widetilde{Q}_{ki}\right)\cdot v(q) > 0$$

we have $v^*(q)\widetilde{P}_{ki}v(q) \rightarrow \infty$ as $Re\{e^{-\mathbf{i}\theta_{ki}}(\lambda(q)-\alpha_{ki})\} \rightarrow 0^-$. Note that since $\widetilde{P}_{ki}(q)$ is positive definite, $Tr\left(\widetilde{P}_{ki}(q)\right) \rightarrow \infty$ as $Re\{e^{-\mathbf{i}\theta_{ki}}(\lambda(q)-\alpha_{ki})\} \rightarrow 0^-$. Similarly, we can prove if the sequence $F(q)$ is such that $|\lambda(q)-\beta_{kj}| - \gamma_{kj} \rightarrow 0^-$, then $\mathrm{Tr}(\hat{P}_{kj}(q)) \rightarrow \infty$. These show that $\sum_i^{l_k}\mathrm{Tr}(\widetilde{P}_{ki}(q)) + \sum_{j=1}^{c_k}\mathrm{Tr}(\hat{P}_{kj}(q)) \rightarrow \infty$ if the sequence $F(q)$ approaches the boundary of Γ_k^r.

From the definitions of Γ^r and $J_{\mathrm{pole}}(F)$, we can conclude that $J_{\mathrm{pole}}(F) \rightarrow \infty$ if F approaches the boundary of Γ^r. $\Delta\Delta$

As stated in [22], although the auxiliary minimization problem is, from a formal viewpoint, a minimization problem with inequality constraints; for a computational viewpoint it is unconstrained. The advantage of the auxiliary minimization problem is that it can be solved by unconstrained search techniques.

Remark 1: It is shown in [22] that the optimal solution of the auxiliary minimization problem converges to the solution of the original problem as the weighting factor $\rho \rightarrow 0^+$. This suggests a way to approximate the infimal solution of the original problem in our approach. It should be noted that even for the single system case, the optimal solutions obtained by the approaches presented in [9], [27], and [29] might be far away from the infimal solutions of the original constrained optimization problems.

Next, we will prove that if the set Γ^r is nonempty, then the auxiliary cost function $J_{\mathrm{aux}}(F)$ has a minimum point in the set Γ^r.

Lemma 4: If the admissible set Γ^r is nonempty, then the auxiliary cost function $J_{\mathrm{aux}}(F)$ has a minimum point in the interior of the set Γ^r.

Proof: From (4), we have

$$2\mathrm{Tr}(-P_k A_{kc}) = \mathrm{Tr}(C_k^T F^T R_k F C_k + Q_k).$$

For matrices M and $N \geq 0$, $\mathrm{Tr}(MN) \leq \|M\|_s \cdot \mathrm{Tr}(N)$ (see [29]). Therefore

$$\mathrm{Tr}(P_k) \geq \frac{\mathrm{Tr}(C_k^T F^T R_k F C_k + Q_k)}{2\|A_{kc}\|_s}. \quad (11)$$

Since C_k has full rank, $R_k > 0$, and $Q_k \geq 0$, then the right hand side of (11) is $o(\|F\|)$ as $\|F\| \rightarrow \infty$. This means $\mathrm{Tr}(P_k) \rightarrow \infty$ as $\|F\| \rightarrow \infty$. Moreover, since X_{k0} is by assumption positive definite, then $\mathrm{Tr}(P_k X_{k0}) \rightarrow \infty$ as $\|F\| \rightarrow \infty$. This implies that $J_{\mathrm{aux}}(F) \rightarrow \infty$ as $\|F\| \rightarrow \infty$. As a result, the level set $\Gamma_a(F_0) \equiv \{F \in \Gamma^r | J_{\mathrm{aux}}(F) \leq J_{\mathrm{aux}}(F_0)\}$ is bounded for any $F_0 \in \Gamma^r$. Moreover, since $J_{\mathrm{aux}}(F)$ is continuous in the set Γ^r and $J_{\mathrm{aux}}(F) \rightarrow \infty$ as F approaches the boundary of the set Γ^r from the interior, the set $\Gamma_a(F_0)$ is closed and then is compact. From the Weiestrass theorem (see [19]), there exists a $F_{\mathrm{opt}} \in \Gamma_a(F_0)$ such that

$$J_{\mathrm{aux}}(F_{\mathrm{opt}}) \leq J_{\mathrm{aux}}(F), \text{ for all } F \in \Gamma_a(F_0).$$

This implies that

$$J_{\mathrm{aux}}(F_{\mathrm{opt}}) \leq J_{\mathrm{aux}}(F), \text{ for all } F \in \Gamma^r$$

and completes the proof. $\Delta\Delta$

Since the minimum point of the auxiliary cost function $J_{\mathrm{aux}}(F)$ lies in the interior of the admissible solution set, it must be a stationary point. The Lagrange multiplier method can be employed to derive the necessary conditions for local optimum of cost function $J_{\mathrm{aux}}(F)$.

Theorem 1: Let $F \in \Gamma^r$ minimize $J_{\mathrm{aux}}(F)$. Then there exist $P_k \geq 0$, $\widetilde{P}_{ki} > 0$, $\hat{P}_{kj} > 0$, $L_k > 0$, $\widetilde{L}_{ki} > 0$, and $\hat{L}_{kj} > 0$ ($k = 1, 2, \ldots, p$, $i = 1, 2, \ldots, l_k$, and $j = 1, 2, \ldots, c_k$) satisfying

$$\widehat{A}_{kc}^T P_k + P_k\widehat{A}_{kc} + C_k^T\left(\mathrm{vec}^{-1}(\Phi^{-1}\times\mathrm{vec}(\Theta))\right)^T \cdot R_k\left(\mathrm{vec}^{-1}(\Phi^{-1}\times\mathrm{vec}(\Theta))\right)C_k + Q_k = 0 \quad (12)$$

$$\widehat{A}_{kc}L_k + L_k\widehat{A}_{kc}^T + w_k X_{k0} = 0 \quad (13)$$

$$e^{\mathbf{i}\theta_{ki}}(\widehat{A}_{kc} - \alpha_{ki}I)^*\widetilde{P}_{ki} + e^{-\mathbf{i}\theta_{ki}}\widetilde{P}_{ki}(\widehat{A}_{kc} - \alpha_{ki}I) + C_k^T(\mathrm{vec}^{-1}(\Phi^{-1}\times\mathrm{vec}(\Theta)))^T \cdot\widetilde{R}_{ki}(\mathrm{vec}^{-1}(\Phi^{-1}\times\mathrm{vec}(\Theta)))C_k + \widetilde{Q}_{ki} = 0 \quad (14)$$

$$e^{-\mathbf{i}\theta_{ki}}(\widehat{A}_{kc} - \alpha_{ki}I)\widetilde{L}_{ki} + e^{\mathbf{i}\theta_{ki}}\widetilde{L}_{ki}(\widehat{A}_{kc} - \alpha_{ki}I)^* + \rho I = 0 \quad (15)$$

$$\frac{1}{\gamma_{kj}^2}(\widehat{A}_{kc} - \beta_{kj}I)^*\hat{P}_{kj}(\widehat{A}_{kc} - \beta_{kj}I) - \hat{P}_{kj} + C_k^T\left(\mathrm{vec}^{-1}(\Phi^{-1}\times\mathrm{vec}(\Theta))\right)^T \cdot\hat{R}_{kj}\left(\mathrm{vec}^{-1}(\Phi^{-1}\times\mathrm{vec}(\Theta))\right)C_k + \hat{Q}_{kj} = 0 \quad (16)$$

and

$$\frac{1}{\gamma_{kj}^2}(\widehat{A}_{kc} - \beta_{kj}I)\hat{L}_{kj}(\widehat{A}_{kc} - \beta_{kj}I)^* - \hat{L}_{kj} + \rho I = 0 \quad (17)$$

where we have the first equation at the bottom of the next page, and

$$\widehat{A}_{kc} = A_k - B_k\cdot\mathrm{vec}^{-1}\left(\Phi^{-1}\times\mathrm{vec}(\Theta)\right)\cdot C_k$$

such that the optimal feedback gain F is given by

$$F = -\mathrm{vec}^{-1}\left(\Phi^{-1}\times\mathrm{vec}(\Theta)\right). \quad (18)$$

Proof: The Lagragian H_{am} is defined as the second equation at the bottom of the next page. The necessary conditions for local optimum are $\partial H_{am}/\partial F = 0$, $\partial H_{am}/\partial L = 0$, $\partial H_{am}/\partial P = 0$, $\partial H_{am}/\partial\widetilde{L}_{ki} = 0$, $\partial H_{am}/\partial\widetilde{P}_{ki} = 0$,

$\partial H_{am}/\partial \hat{L}_{kj} = 0$, and $\partial H_{am}/\partial \hat{P}_{kj} = 0$. After some manipulations, we have (19)–(25), shown at the bottom of the page. From (19), we can derive (18). By substituting (18) into (19)–(25), (12)–(17) can be obtained. $\Delta\Delta$

The above theorem provides not only a necessary conditions for optimum but a method to calculate the gradient direction of $J_{\text{aux}}(F)$ at a given point F as well. The gradient of $J_{\text{aux}}(F)$ at a fixed point F is shown in the equation at the bottom of the next page, where P_k, $\widetilde{P}_{ki}$, $\hat{P}_{kj}$, L_k, $\widetilde{L}_{ki}$, and $\hat{L}_{kj}$ ($k = 1, 2, \ldots, p$, $i = 1, 2, \ldots, l_k$, and $j = 1, 2, \ldots, c_k$) are the solution of (20)–(25). In the solution algorithms, this gradient direction is used as the searching direction.

Note that if $Q_k > 0$, then the solution P_k of (4) is positive definite. Based on the Theorem 1, a necessary and sufficient condition for the existence of admissible solutions to the simultaneous static output feedback regional pole placement problem is given in the following.

Corollary 1: The set Γ^r is nonempty if, and only if, for any given positive definite Hermitian matrices R_k, Q_k, $\widetilde{R}_{ki}$, $\widetilde{Q}_{ki}$, $\hat{R}_{kj}$, and $\hat{Q}_{kj}$ ($k = 1, 2, \ldots, p$, $i = 1, 2, \ldots, l_k$, and

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$$\Phi = \sum_{k=1}^{p}\left((C_kL_kC_k^T)\otimes R_k + \sum_{i=1}^{l_k}(C_k\widetilde{L}_{ki}C_k^T)\otimes \widetilde{R}_{ki} + \sum_{j=1}^{c_k}\left(\frac{1}{\gamma_{kj}^2}(C_k\hat{L}_{kj}C_k^T)\otimes B_k^T\hat{P}_{kj}B_k + (C_k\hat{L}_{kj}C_k^T)\otimes \hat{R}_{kj}\right)\right)$$

$$\Theta = \sum_{k=1}^{p}B_k^T\left(P_kL_k + \frac{1}{2}\sum_{i=1}^{l_k}\left(e^{\mathbf{i}\theta_{ki}}\widetilde{P}_{ki}\widetilde{L}_{ki} + e^{-\mathbf{i}\theta_{ki}}\widetilde{P}_{ki}^T\widetilde{L}_{ki}^T\right) + \frac{1}{2}\sum_{j=1}^{c_k}\frac{1}{\gamma_{kj}^2}\left(\hat{P}_{kj}(A_k-\beta_{kj}I)\hat{L}_{kj} + \hat{P}_{kj}^T(A_k^T-\overline{\beta}_{kj}I)\hat{L}_{kj}^T\right)\right)C_k^T$$

$$\begin{aligned}H_{am} = &\sum_{k=1}^{p} w_k\mathrm{Tr}\left(P_kX_{k0}\right) + \rho\cdot\sum_{k=1}^{p}\left(\sum_{i}^{l_k}\mathrm{Tr}\left(\widetilde{P}_{ki}\right) + \sum_{j=1}^{c_k}\mathrm{Tr}\left(\hat{P}_{kj}\right)\right)\\ &+\sum_{k=1}^{p}\mathrm{Tr}\left(L_k\left(A_{kc}^TP_k + P_kA_{kc} + C_k^TF^TR_kFC_k + Q_k\right)\right)\\ &+\sum_{k=1}^{p}\sum_{i=1}^{l_k}\mathrm{Tr}\left(\widetilde{L}_{ki}\left(e^{\mathbf{i}\theta_{ki}}(A_{kc}-\alpha_{ki}I)^*\widetilde{P}_{ki} + e^{-\mathbf{i}\theta_{ki}}\widetilde{P}_{ki}(A_{kc}-\alpha_{ki}I) + C_k^TF^T\widetilde{R}_{ki}FC_k + \widetilde{Q}_{ki}\right)\right)\\ &+\sum_{k=1}^{p}\sum_{j=1}^{c_k}\mathrm{Tr}\left(\hat{L}_{kj}\left(\frac{1}{\gamma_{kj}^2}(A_{kc}-\beta_{kj}I)^*\hat{P}_{kj}(A_{kc}-\beta_{ki}I) - \hat{P}_{kj} + C_k^TF^T\hat{R}_{kj}FC_k + \hat{Q}_{kj}\right)\right).\end{aligned}$$

$$\begin{aligned}&2\sum_{k=1}^{p}\left(R_kFC_kL_kC_k^T + \sum_{i=1}^{l_k}\widetilde{R}_{ki}FC_k\widetilde{L}_{ki}C_k^T + \sum_{j=1}^{c_k}\left(\frac{1}{\gamma_{kj}^2}B_k^T\hat{P}_{kj}B_kFC_k\hat{L}_{kj}C_k^T + \hat{R}_{kj}FC_k\hat{L}_{kj}C_k^T\right)\right)\\ &+2\sum_{k=1}^{p}B_k^T\left(P_kL_k + \frac{1}{2}\sum_{i=1}^{l_k}\left(e^{\mathbf{i}\theta_{ki}}\widetilde{P}_{ki}\widetilde{L}_{ki} + e^{-\mathbf{i}\theta_{ki}}\widetilde{P}_{ki}^T\widetilde{L}_{ki}^T\right) + \frac{1}{2}\sum_{j=1}^{c_k}\frac{1}{\gamma_{kj}^2}\left(\hat{P}_{kj}(A_k-\beta_{kj}I)\hat{L}_{kj} + \hat{P}_{kj}^T(A_k^T-\overline{\beta}_{kj}I)\hat{L}_{kj}^T\right)\right)C_k^T = 0 \quad (19)\end{aligned}$$

$$A_{kc}^TP_k + P_kA_{kc} + C_k^TF^TR_kFC_k + Q_k = 0 \quad (20)$$

$$A_{kc}L_k + L_kA_{kc}^T + w_kX_{k0} = 0 \quad (21)$$

$$e^{\mathbf{i}\theta_{ki}}(A_{kc}-\alpha_{ki}I)^*\widetilde{P}_{ki} + e^{-\mathbf{i}\theta_{ki}}\widetilde{P}_{ki}(A_{kc}-\alpha_{ki}I) + C_k^TF^T\widetilde{R}_{ki}FC_k + \widetilde{Q}_{ki} = 0 \quad (22)$$

$$e^{-\mathbf{i}\theta_{ki}}(A_{kc}-\alpha_{ki}I)\widetilde{L}_{ki} + e^{\mathbf{i}\theta_{ki}}\widetilde{L}_{ki}(A_{kc}-\alpha_{ki}I)^* + \rho\cdot I = 0 \quad (23)$$

$$\frac{1}{\gamma_{kj}^2}(A_{kc}-\beta_{kj}I)^*\hat{P}_{kj}(A_{kc}-\beta_{kj}I) - \hat{P}_{kj} + C_k^TF^T\hat{R}_{kj}FC_k + \hat{Q}_{kj} = 0 \quad (24)$$

$$\frac{1}{\gamma_{kj}^2}(A_{kc}-\beta_{kj}I)\hat{L}_{kj}(A_{kc}-\beta_{kj}I)^* - \hat{L}_{kj} + \rho I = 0. \quad (25)$$

$j = 1, 2, \ldots, c_k$), there exist positive definite Hermitian solutions P_k, L_k, $\widetilde{P}_{ki}$, $\widetilde{L}_{ki}$, $\hat{P}_{kj}$, and $\hat{L}_{kj}$ ($k = 1, 2, \ldots, p$, $i = 1, 2, \ldots, l_k$, and $j = 1, 2, \ldots, c_k$) to the cross-coupled (12) to (17).

Proof: The "Sufficiency" part is obvious and, therefore, is omitted.

Necessity: Suppose the set Γ^r is nonempty. From Lemma 4, $J(F)$ has a minimum in Γ^r. It has been shown in the Theorem 1 that there must exist some positive definite matrices P_k, L_k, $\widetilde{P}_{ki}$, $\widetilde{L}_{ki}$, $\hat{P}_{kj}$, and $\hat{L}_{kj}$ ($k = 1, 2, \ldots, p$, $i = 1, 2, \ldots, l_k$, and $j = 1, 2, \ldots, c_k$) satisfying the cross-coupled (12) to (17). This completes the proof. $\Delta\Delta$

When the poles' constraints are relaxed, the considered problem is reduced to the *optimal simultaneous static output feedback stabilization problem*. It is obvious that $J(F)$ is finite if $F \in \Gamma^s$, and $J(F)$ will approach infinity if F approaches the boundary of Γ^s. Following the same procedure provided above, we can show that the cost function $J(F)$ is continuous in the set Γ^s and the level set $\Gamma_l(F_0) \equiv \{F \in \Gamma^s | J(F) \leq J(F_0)\}$ for any $F_0 \in \Gamma^s$ is compact. Therefore, $J(F)$ has a minimum in Γ^s. Hence, the following results can be obtained.

Corollary 2: Let $F \in \Gamma^s$ minimize $J(F)$. Then there exist $P_k = P_k^T \geq 0$ and $L_k = L_k^T > 0$, $k = 1, 2, \ldots, p$, satisfy

$$\begin{aligned} A_k^T P_k + P_k A_k - C_k^T &\left(\mathrm{vec}^{-1}\left(\Pi^{-1} \times \mathrm{vec}(\Xi)\right)\right)^T \\ &\cdot B_k^T P_k - P_k B_k \left(\mathrm{vec}^{-1}\left(\Pi^{-1} \times \mathrm{vec}(\Xi)\right)\right) C_k \\ &+ C_k^T \left(\mathrm{vec}^{-1}\left(\Pi^{-1} \times \mathrm{vec}(\Xi)\right)\right)^T \\ &\cdot R_k \left(\mathrm{vec}^{-1}\left(\Pi^{-1} \times \mathrm{vec}(\Xi)\right)\right) C_k + Q_k = 0 \quad (26) \end{aligned}$$

and

$$\begin{aligned} A_k L_k + L_k A_k^T - B_k &\left(\mathrm{vec}^{-1}\left(\Pi^{-1} \times \mathrm{vec}(\Xi)\right)\right) C_k L_k \\ - L_k C_k^T &\left(\mathrm{vec}^{-1}\left(\Pi^{-1} \times \mathrm{vec}(\Xi)\right)\right)^T B_k^T + w_k X_{k0} = 0 \quad (27) \end{aligned}$$

where

$$\Pi = \sum_{k=1}^{p} (C_k L_k C_k^T) \otimes R_k, \; \Xi = \sum_{k=1}^{p} B_k^T P_k L_k C_k^T$$

such that the optimal feedback gain F is given by

$$F = -\mathrm{vec}^{-1}\left(\Pi^{-1} \times \mathrm{vec}(\Xi)\right). \quad (28)$$

Proof: The proof is similar to that of the Theorem 1 and, therefore, is omitted. $\Delta\Delta$

Consequently, a necessary and sufficient condition for the existence of admissible solutions to the *simultaneous output feedback stabilization problem* can be obtained.

Corollary 3: The set Γ^s is nonempty if, and only if, for any given positive definite symmetry matrices R_k and Q_k, $k = 1, 2, \ldots, p$, there exist positive definite symmetry solutions P_k and L_k, $k = 1, 2, \ldots, p$, to the cross-coupled (26) and (27). In this case, the output feedback gain F given in (28) will simultaneously stabilize the collection of systems (1).

Proof: The "Sufficiency" part is obvious and, therefore, is omitted.

Necessity: Suppose the set Γ^s is nonempty. We have shown that $J(F)$ has a minimum in Γ^s. It has been shown in Corollary 2 that there exists positive definite matrices P_k and L_k, $k = 1, 2, \ldots, p$, satisfying the cross-coupled (26) and (27). In this case, it is obvious that the feedback matrix F given in (28) is a solution to the simultaneous output feedback stabilization problem. $\Delta\Delta$

Remark 2: For state feedback case, we only need to let $C_k = I$ for all $k = 1, 2, \ldots, p$ in the above results.

IV. Illustrative Examples

Example 1: Consider the following collection of systems

$$\begin{aligned} \dot{x}_k(t) &= A_k x_k(t) + B_k u_k(t), \; k = 1 \text{ and } 2. \\ y_k(t) &= C_k x_k(t) \end{aligned}$$

where $A_1 = \begin{bmatrix} -4 & 5 & -4 \\ 4 & -21 & -18 \\ -32 & -4 & 34.5 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 & 1 \\ -5 & 3 \\ 0 & -4 \end{bmatrix}$

$C_1 = \begin{bmatrix} 0 & 1 & 1 \\ -2 & 0 & 3 \end{bmatrix}$

and $A_2 = \begin{bmatrix} -43 & 24 & -4 \\ -98 & 33 & 20 \\ 49 & 49 & -94 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ -3 & 1 \end{bmatrix}$

$C_2 = \begin{bmatrix} -4 & -8 & 12 \\ 8 & 8 & 4 \end{bmatrix}$.

Suppose $\boldsymbol{E}\{x_1(0)x_1^T(0)\} = X_{10} = I_{3\times 3}$ and $\boldsymbol{E}\{x_2(0)x_2^T(0)\} = X_{20} = I_{3\times 3}$.

$$\begin{aligned} F_{grad}(F) = 2\sum_{k=1}^{p} &\left(R_k F C_k L_k C_k^T + \sum_{i=1}^{l_k} \widetilde{R}_{ki} F C_k \widetilde{L}_{ki} C_k^T + \sum_{j=1}^{c_k} \left(\frac{1}{\gamma_{kj}^2} B_k^T \hat{P}_{kj} B_k F C_k \hat{L}_{kj} C_k^T + \hat{R}_{kj} F C_k \hat{L}_{kj} C_k^T \right) \right) \\ &+ 2\sum_{k=1}^{p} B_k^T \left(P_k L_k + \frac{1}{2} \sum_{i=1}^{l_k} \left(e^{\mathbf{i}\theta_{ki}} \widetilde{P}_{ki} \widetilde{L}_{ki} + e^{-\mathbf{i}\theta_{ki}} \widetilde{P}_{ki}^T \widetilde{L}_{ki}^T \right) \right. \\ &\left. + \frac{1}{2} \sum_{j=1}^{c_k} \frac{1}{\gamma_{kj}^2} \left(\hat{P}_{kj} (A_k - \beta_{kj} I) \hat{L}_{kj} + \hat{P}_{kj}^T (A_k^T - \overline{\beta}_{kj} I) \hat{L}_{kj}^T \right) \right) C_k^T \end{aligned}$$

The design goal is to find a static output feedback gain $F \in R^{2\times 2}$ such that the controllers

$$u_k(t) = F \cdot y_k(t),\ k = 1 \text{ and } 2$$

achieve the infimum of the cost function

$$J(F) = \sum_{k=1}^{2} w_k \cdot \boldsymbol{E}\left\{\int_0^\infty (x_k^T Q_k x_k + u_k^T R_k u_k)dt\right\}$$

subject to the constraints that $\sigma(A_1+B_1FC_1) \in \Omega_1$ and $\sigma(A_2+B_2FC_2) \in \Omega_2$, where

$$Q_1=\begin{bmatrix}3&0&0\\0&5&0\\0&0&1\end{bmatrix}, Q_2=\begin{bmatrix}1&0&0\\0&1&0\\0&0&5\end{bmatrix}, R_1=\begin{bmatrix}1&0\\0&2\end{bmatrix}, R_2=\begin{bmatrix}2&0\\0&1\end{bmatrix}$$

and the constrained regions Ω_1 and Ω_2 are represented by

$$\Omega_1 = \{s \in \boldsymbol{C} \,||s+15| < 10, |s+5| < 17\}$$
$$\Omega_2 = \left\{\text{s} \in \boldsymbol{C} \middle| \text{Re}(s) < -5, \text{Re}(e^{\text{i}180^\circ} s) < -20\right\}$$
$$= \{\text{s} \in \boldsymbol{C}\,|\text{Re}(s) < -5, \text{Re}(s) > -20\}.$$

Let

$$\Gamma^r \equiv \{F \in R^{2\times 2} | \sigma(A_1+B_1FC_1) \in \Omega_1, \sigma(A_2+B_2FC_2) \in \Omega_2\}.$$

Suppose the weighting factors $w_1 = 1$ and $w_2 = 1$. As shown in Section II, we have

$$J(F) = \text{Tr}(P_1X_{10}) + \text{Tr}(P_2X_{20}) = \text{Tr}(P_1) + \text{Tr}(P_2)$$

where P_1 and P_2 are the positive definite solutions of

$$(A_1 + B_1FC_1)^T P_1 + P_1(A_1 + B_1FC_1) + C_1^T F^T Q_1 FC_1 + R_1 = 0 \quad (29)$$
$$(A_2 + B_2FC_2)^T P_2 + P_2(A_2 + B_2FC_2) + C_2^T F^T Q_2 FC_2 + R_2 = 0. \quad (30)$$

Let the infimal solution of this problem be denoted by $F = F_{\text{opt}}$. Choose $\hat{Q}_{11} = \hat{Q}_{12} = \widetilde{Q}_{21} = \widetilde{Q}_{22} = I$ and $\hat{R}_{11} = \hat{R}_{12} = \widetilde{R}_{21} = \widetilde{R}_{22} = I$. From the discussions in Section III, we solve the following auxiliary minimization problem: Find $F = F^o$, over Γ^r, to minimize the *auxiliary cost function*

$$J_{\text{aux}}(F) = J(F) + \rho \cdot J_{\text{pole}}(F) = \text{Tr}(P_1 + P_2) + \rho \cdot Tr\left(\hat{P}_{11} + \hat{P}_{12} + \widetilde{P}_{21} + \widetilde{P}_{22}\right)$$

where ρ is a weighting factor to be chosen, and matrices $\hat{P}_{11}$, $\hat{P}_{12}$, $\widetilde{P}_{21}$, and $\widetilde{P}_{22}$ are the positive definite solutions of

$$\frac{1}{10^2}(A_1 + B_1FC_1 + 15I)^T \hat{P}_{11}(A_1 + B_1FC_1 + 15I) - \hat{P}_{11} + C_1^T F^T FC_1 + I = 0 \quad (31)$$
$$\frac{1}{17^2}(A_1 + B_1FC_1 + 5I)^T \hat{P}_{12}(A_1 + B_1FC_1 + 5I) - \hat{P}_{12} + C_1^T F^T FC_1 + I = 0 \quad (32)$$
$$(A_2 + B_2FC_2 + 5I)^T \widetilde{P}_{21} + \widetilde{P}_{21}(A_2 + B_2FC_2 + 5I) + C_2^T F^T FC_2 + I = 0 \quad (33)$$
$$-(A_2 + B_2FC_2 + 20I)^T \widetilde{P}_{22} - \widetilde{P}_{22}(A_2 + B_2FC_2 + 20I) + C_2^T F^T FC_2 + I = 0. \quad (34)$$

Suppose matrices L_1, L_2, $\hat{L}_{11}$, $\hat{L}_{12}$, $\widetilde{L}_{21}$, and $\widetilde{L}_{22}$ are the solutions of

$$(A_1 + B_1FC_1)L_1 + L_1(A_1 + B_1FC_1)^T + I = 0 \quad (35)$$
$$(A_2 + B_2FC_2)L_2 + L_2(A_2 + B_2FC_2)^T + I = 0 \quad (36)$$
$$\frac{1}{10^2}(A_1 + B_1FC_1 + 15I)\hat{L}_{11}(A_1 + B_1FC_1 + 15I)^T - \hat{L}_{11} + \rho \cdot I = 0 \quad (37)$$
$$\frac{1}{17^2}(A_1 + B_1FC_1 + 5I)\hat{L}_{12}(A_1 + B_1FC_1 + 5I)^T - \hat{L}_{12} + \rho \cdot I = 0 \quad (38)$$
$$(A_2 + B_2FC_2 + 5I)\widetilde{L}_{21} + \widetilde{L}_{21}(A_2 + B_2FC_2 + 5I)^T + \rho \cdot I = 0 \quad (39)$$
$$-(A_2 + B_2FC_2 + 20I)\widetilde{L}_{22} - \widetilde{L}_{22}(A_2 + B_2FC_2 + 20I)^T + \rho \cdot I = 0. \quad (40)$$

From the Theorem 1, we know that the gradient of $J_{\text{aux}}(F)$ at a fixed point F is

$$\begin{aligned}F_{grad}(F) = {} & 2R_1FC_1L_1C_1^T + 2R_2FC_2L_2C_2^T \\ & + 2F\left(C_1(\hat{L}_{11} + \hat{L}_{12})C_1^T + C_2(\widetilde{L}_{21} + \widetilde{L}_{22})C_2^T\right) \\ & + 2\left(\frac{1}{10^2}B_1^T\hat{P}_{11}B_1FC_1\hat{L}_{11}C_1^T + \frac{1}{17^2}B_1^T\hat{P}_{12}B_1FC_1\hat{L}_{12}C_1^T\right) \\ & + 2B_1^T\left(P_1L_1 + \frac{1}{2}\left(\frac{1}{10^2}\left(\hat{P}_{11}(A_1 + A_1^T + 30I)\hat{L}_{11}\right) + \frac{1}{17^2}\left(\hat{P}_{12}(A_1 + A_1^T + 10I)\hat{L}_{12}\right)\right)\right)C_1^T \\ & + 2B_2^T\left(P_2L_2 + \widetilde{P}_{21}\widetilde{L}_{21} - \widetilde{P}_{22}\widetilde{L}_{22}\right)C_2^T.\end{aligned}$$

Based on the gradient method, an algorithm is presented in the following to solve the *auxiliary minimization problem*.

Main-Algorithm: Find the optimal solution F^o of the auxiliary minimization problem.

1) Choose a $F(0) \in \Gamma^r$. Set $q = 0$.
2) Solving (29)–(40), where F is substituted by $F(q)$, yields $P_1(q)$, $P_2(q)$, $\hat{P}_{11}(q)$, $\hat{P}_{12}(q)$, $\widetilde{P}_{21}(q)$, $\widetilde{P}_{22}(q)$, $L_1(q)$, $L_2(q)$, $\hat{L}_{11}(q)$, $\hat{L}_{12}(q)$, $\widetilde{L}_{21}(q)$, and $\widetilde{L}_{22}(q)$.
3) Let $F^g(q) = F_{grad}(F(q))$.
4) If $\|F^g(q)\| < \varepsilon$, where $\varepsilon > 0$ is a small positive number, then $F^o = F(q)$, end; else find $\Delta(q) > 0$, via line search, such that $F(q+1) = F(q) - \Delta(q)F^g(q)$ shall minimize $J_{\text{aux}}(F(q+1))$. Let $q = q+1$, go to step 2).

In fact, the step 1) of the Main-Algorithm is not an easy task. In the following, we will provide a Pre-Algorithm to find a $F \in \Gamma^r$. Let

$$\Omega_1^a(\mu_{11}, \mu_{12}) = \{s \in \boldsymbol{C}\,||s+15| < 10 + \mu_{11}, |s+5| < 17 + \mu_{12}\}$$
$$\Omega_2^a(\mu_{21}, \mu_{22}) = \{\text{s} \in \boldsymbol{C}\,|\text{Re}(s) < -5 + \mu_{21}, \text{Re}(s) > -20 - \mu_{22}\}$$

for non-negative μ_{11}, μ_{12}, μ_{21}, and μ_{22}. It is clear that $\Omega_1^a(0,0)=\Omega_1$ and $\Omega_2^a(0,0)=\Omega_2$. Define

$$\Gamma^a(\mu_{11},\mu_{12},\mu_{21},\mu_{22})=\{F\in R^{2\times 2}|\sigma(A_1+B_1FC_1)\in\Omega_1^a(\mu_{11},\mu_{12}) \text{ and } \sigma(A_2+B_2FC_2)\in\Omega_2^a(\mu_{21},\mu_{22})\}.$$

Note that $\Gamma^r\subset\Gamma^a(\mu_{11},\mu_{12},\mu_{21},\mu_{22})$ and $\Gamma^a(0,0,0,0)=\Gamma^r$. Let $\hat{P}_{11}(\mu_{11})$, $\hat{P}_{12}(\mu_{12})$, $\widetilde{P}_{21}(\mu_{21})$, $\widetilde{P}_{22}(\mu_{22})$, $\hat{L}_{11}(\mu_{11})$, $\hat{L}_{12}(\mu_{12})$, $\widetilde{L}_{21}(\mu_{21})$, and $\widetilde{L}_{22}(\mu_{22})$ are the solutions of

$$\frac{1}{(10+\mu_{11})^2}(A_1+B_1FC_1+15I)^T\hat{P}_{11}(A_1+B_1FC_1+15I)-\hat{P}_{11}+C_1^TF^TFC_1+I=0 \quad (41)$$

$$\frac{1}{(17+\mu_{12})^2}(A_1+B_1FC_1+5I)^T\hat{P}_{12}(A_1+B_1FC_1+5I)-\hat{P}_{12}+C_1^TF^TFC_1+I=0 \quad (42)$$

$$(A_2+B_2FC_2+5I-\mu_{21}I)^T\widetilde{P}_{21}+\widetilde{P}_{21}(A_2+B_2FC_2+5I-\mu_{21}I)+C_2^TF^TFC_2+I=0 \quad (43)$$

$$-(A_2+B_2FC_2+20I+\mu_{22}I)^T\widetilde{P}_{22}-\widetilde{P}_{22}(A_2+B_2FC_2+20I+\mu_{22}I)+C_2^TF^TFC_2+I=0 \quad (44)$$

$$\frac{1}{(10+\mu_{11})^2}(A_1+B_1FC_1+15I)\hat{L}_{11}(A_1+B_1FC_1+15I)^T-\hat{L}_{11}+I=0 \quad (45)$$

$$\frac{1}{(17+\mu_{12})^2}(A_1+B_1FC_1+5I)\hat{L}_{12}(A_1+B_1FC_1+5I)^T-\hat{L}_{12}+I=0 \quad (46)$$

$$(A_2+B_2FC_2+5I-\mu_{21}I)\widetilde{L}_{21}+\widetilde{L}_{21}\cdot(A_2+B_2FC_2+5I-\mu_{21}I)^T+I=0 \quad (47)$$

$$-(A_2+B_2FC_2+20I+\mu_{22}I)\widetilde{L}_{22}-\widetilde{L}_{22}\cdot(A_2+B_2FC_2+20I+\mu_{22}I)^T+I=0. \quad (48)$$

Let

$$J_{\text{pole}}^a(F,\mu_{11},\mu_{12},\mu_{21},\mu_{22})=\text{Tr}\left(\hat{P}_{11}(\mu_{11})+\hat{P}_{12}(\mu_{12})+\widetilde{P}_{21}(\mu_{21})+\widetilde{P}_{22}(\mu_{22})\right).$$

From the discussions in Section III, we know that $J_{\text{pole}}^a(F,\mu_{11},\mu_{12},\mu_{21},\mu_{22})\to\infty$ if F approaches the boundary of $\Gamma^a(\mu_{11},\mu_{12},\mu_{21},\mu_{22})$ from the interior. The gradient of $J_{\text{pole}}^a(F,\mu_{11},\mu_{12},\mu_{21},\mu_{22})$ at a fixed point F is

$$\begin{aligned}F_{grad}^a&(F,\mu_{11},\mu_{12},\mu_{21},\mu_{22})\\&=2F\cdot\left(C_1(\hat{L}_{11}(\mu_{11})+\hat{L}_{12}(\mu_{12}))C_1^T+C_2(\widetilde{L}_{21}(\mu_{21})+\widetilde{L}_{22}(\mu_{22}))C_2^T\right)\\&+2\left(\frac{1}{(10+\mu_{11})^2}B_1^T\hat{P}_{11}(\mu_{11})B_1FC_1\hat{L}_{11}(\mu_{11})C_1^T+\frac{1}{(17+\mu_{12})^2}B_1^T\hat{P}_{12}(\mu_{12})B_1FC_1\hat{L}_{12}(\mu_{12})C_1^T\right)\\&+B_1^T\left(\frac{1}{(10+\mu_{11})^2}\left(\hat{P}_{11}(\mu_{11})(A_1+A_1^T+30I)\hat{L}_{11}(\mu_{11})\right)+\frac{1}{(17+\mu_{12})^2}\left(\hat{P}_{12}(\mu_{12})(A_1+A_1^T+10I)\hat{L}_{12}(\mu_{12})\right)\right)C_1^T\\&+2B_2^T\left(\widetilde{P}_{21}(\mu_{21})\widetilde{L}_{21}(\mu_{21})-\widetilde{P}_{22}(\mu_{22})\widetilde{L}_{22}(\mu_{22})\right)C_2^T.\end{aligned}$$

Now we are ready to provide the Pre-Algorithm for finding a $F\in\Gamma^r$.

Pre-Algorithm: Find a $F\in\Gamma^r$.

1) Choose arbitrary $F(0)$. Find sufficient large $\mu_{11}(0)\geq 0$, $\mu_{12}(0)\geq 0$, $\mu_{21}(0)\geq 0$, and $\mu_{22}(0)\geq 0$ such that $F(0)\in\Gamma^a(\mu_{11}(0),\mu_{12}(0),\mu_{21}(0),\mu_{22}(0))$. Set $q=0$.

2) Solving (41)-(48), where F is substituted by $F(q)$, yields $\hat{P}_{11}(\mu_{11}(q))$, $\hat{P}_{12}(\mu_{12}(q))$, $\widetilde{P}_{21}(\mu_{21}(q))$, $\widetilde{P}_{22}(\mu_{22}(q))$, $\hat{L}_{11}(\mu_{11}(q))$, $\hat{L}_{12}(\mu_{12}(q))$, $\widetilde{L}_{21}(\mu_{21}(q))$, and $\widetilde{L}_{22}(\mu_{22}(q))$.

3) Let $F^d(q)=F_{grad}^a(F(q),\mu_{11}(q),\mu_{12}(q),\mu_{21}(q),\mu_{22}(q))$.

4) Find $\delta(q)>0$, via line search, such that $F(q+1)=F(q)-\delta(q)F^d(q)$ shall minimize

$$J_{\text{pole}}^a(F(q+1),\mu_{11}(q),\mu_{12}(q),\mu_{21}(q),\mu_{22}(q)).$$

5) Let $q=q+1$. Suppose $\lambda_{1i}(q)$, $i=1,2,\ldots,n$, are the eigenvalues of matrix $A_1+B_1F(q)C_1$ and $\lambda_{2i}(q)$, $i=1,2,\ldots,n$, are the eigenvalues of matrix $A_2+B_2F(q)C_2$. Choose $0<\eta(q)<1$.

If

$$\mu_{11}^d(q)=\max\left\{\max_i\left\{\sqrt{(\text{Re}(\lambda_{1i}(q)+15)^2+\text{Im}(\lambda_{2i}(q))^2}\right\}-10,0\right\}=0$$

let $\mu_{11}(q)=0$,

else $\mu_{11}(q)=\mu_{11}(q-1)-\eta(q)\cdot(\mu_{11}(q-1)-\mu_{11}^d(q))$.

If

$$\mu_{12}^d(q)=\max\left\{\max_i\left\{\sqrt{(\text{Re}(\lambda_{1i}(q)+5)^2+\text{Im}(\lambda_{2i}(q))^2}\right\}-17,0\right\}=0$$

let $\mu_{12}(q)=0$,

else $\mu_{12}(q)=\mu_{12}(q-1)-\eta(q)\cdot(\mu_{12}(q-1)-\mu_{12}^d(q))$.

If $\mu_{21}^d(q)=\max\left\{\max_i\{\text{Re}(\lambda_{2i}(q)+5\},0\right\}=0$, let $\mu_{21}(q)=0$,

else $\mu_{21}(q)=\mu_{21}(q-1)-\eta(q)\cdot(\mu_{21}(q-1)-\mu_{21}^d(q))$.

If $\mu_{22}^d(q)=\max\left\{-\min_i\{\text{Re}(\lambda_{2i}(q)+20\},0\right\}=0$, let $\mu_{22}(q)=0$,

else $\mu_{22}(q)=\mu_{22}(q-1)-\eta(q)\cdot(\mu_{22}(q-1)-\mu_{22}^d(q))$.

Repeat 2)-5) until $\mu_{11}(q)=0$, $\mu_{12}(q)=0$, $\mu_{21}(q)=0$, and $\mu_{22}(q)=0$. Then, $F(q)\in\Gamma^r$.

From the Pre-Algorithm we know that $\mu_{11}(q)<\mu_{11}(q-1)$ if $\mu_{11}(q-1)\neq 0$, $\mu_{12}(q)<\mu_{12}(q-1)$ if $\mu_{12}(q-1)\neq 0$, $\mu_{21}(q)<\mu_{21}(q-1)$ if $\mu_{21}(q-1)\neq 0$, and $\mu_{22}(q)<\mu_{22}(q-1)$ if $\mu_{22}(q-1)\neq 0$. The values of $\mu_{11}(q)$, $\mu_{12}(q)$, $\mu_{21}(q)$, and $\mu_{22}(q)$ are monotonically decreasing if they are nonzero. Thus we can expect that if the admissible solution set Γ^r is nonempty and the set $\Gamma^a(\mu_{11}(q),\mu_{12}(q),\mu_{21}(q),\mu_{22}(q))$ is connected in the iteration, there is some finite v such that $\mu_{11}(v)=0$, $\mu_{12}(v)=0$, $\mu_{21}(v)=0$, $\mu_{22}(v)=0$, and $F(v)\in\Gamma^r$.

For the considered problem, we choose the weighting factor $\rho=0.000\,01$. The Pre-Algorithm is started with an initial guess $F(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. After some iteration, a matrix $F = \begin{bmatrix} -1.0745 & 1.3649 \\ 3.2846 & 2.3229 \end{bmatrix} \in \Gamma^r$ is obtained. Then, let $F(0) = \begin{bmatrix} -1.0745 & 1.3649 \\ 3.2846 & 2.3229 \end{bmatrix}$ and start the Main-Algorithm with stop condition $\|F_{grad}(F)\| < \varepsilon = 0.000\,01$. After some iteration, we get the following results (the solutions of (29)–(40)):

$$P_1 = \begin{bmatrix} 10.4872 & 13.0665 & -3.3851 \\ 13.0665 & 25.0656 & 3.2529 \\ -3.3851 & 3.2529 & 8.0431 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 154.7090 & -82.7574 & -93.1120 \\ -82.7574 & 74.5960 & 43.7303 \\ -93.1120 & 43.7303 & 175.1064 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 0.0711 & -0.0238 & 0.0469 \\ -0.0238 & 0.1403 & -0.1598 \\ 0.0469 & -0.1598 & 0.2503 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 0.0820 & 0.0215 & 0.0450 \\ 0.0215 & 0.1370 & -0.0027 \\ 0.0450 & -0.0027 & 0.0440 \end{bmatrix}$$

$$\hat{P}_{11} = \begin{bmatrix} 1966.72 & 3797.92 & 1005.24 \\ 3797.92 & 10\,889.76 & 5008.07 \\ 1005.24 & 5008.07 & 3177.10 \end{bmatrix}$$

$$\hat{P}_{12} = \begin{bmatrix} 107.4386 & 160.8044 & 6.7192 \\ 160.8044 & 568.9773 & 330.6886 \\ 6.7192 & 330.6886 & 318.9183 \end{bmatrix}$$

$$\widetilde{P}_{21} = \begin{bmatrix} 348.1315 & -166.1753 & -147.0505 \\ -166.1753 & 109.5204 & 61.7494 \\ -147.0505 & 61.7494 & 188.8397 \end{bmatrix}$$

$$\widetilde{P}_{22} = \begin{bmatrix} 15\,584.91 & -5976.24 & -29\,054.91 \\ -5976.24 & 9074.14 & 11\,332.01 \\ -29\,054.9 & 11\,332.01 & 54\,400.83 \end{bmatrix}$$

$$\hat{L}_{11} = \begin{bmatrix} 0.1246 & -0.0321 & 0.0416 \\ -0.0321 & 0.0220 & -0.0473 \\ 0.0416 & -0.0473 & 0.1272 \end{bmatrix} \times 10^{-2}$$

$$\hat{L}_{12} = \begin{bmatrix} 0.4540 & -0.0546 & -0.0581 \\ -0.0546 & 0.1248 & -0.0654 \\ -0.0581 & -0.0654 & 0.4647 \end{bmatrix} \times 10^{-4}$$

$$\widetilde{L}_{21} = \begin{bmatrix} 0.2300 & 0.0382 & 0.1240 \\ 0.0382 & 0.1891 & -0.0041 \\ 0.1240 & -0.0041 & 0.0939 \end{bmatrix} \times 10^{-5}$$

$$\widetilde{L}_{22} = \begin{bmatrix} 0.0954 & 0.2140 & 0.0177 \\ 0.2140 & 0.4945 & 0.0109 \\ 0.0177 & 0.0109 & 0.0751 \end{bmatrix} \times 10^{-3}.$$

Since all the above matrices are positive definite, this verifies the results of the Corollary 1 that the admissible solution set Γ^r is nonempty. The resultant optimal feedback gain for the auxiliary minimization problem is

$$F^o = \begin{bmatrix} -0.7183 & 1.4953 \\ 4.8755 & 2.4765 \end{bmatrix}.$$

We have

$$\sigma(A_1 + B_1 F^o C_1) = \{-8.2386, -10.6034 \pm \mathbf{i} \times 8.6201\} \in \Omega_1$$

and

$$\sigma(A_2 + B_2 F^o C_2) = \{-8.1571, -19.9413 \pm \mathbf{i} \times 54.5002\} \in \Omega_2$$

as desired. The resultant optimal value of cost function is

$$J(F^o) = 448.0074.$$

From the discussions in Remark 1, we can expect that it is very close to the (local) infimal value $J(F_{\rm opt})$ since $\rho = 0.000\,01$ is very small.

Note that the Pre-Algorithm is not sensitive with respect to the initial guess $F(0)$. Even for the extreme case $F(0) = \begin{bmatrix} 100\,000 & 0 \\ 0 & 100\,000 \end{bmatrix}$, which is far away from the admissible solution set Γ^r, a matrix $F = \begin{bmatrix} -0.9816 & 1.3516 \\ 3.4358 & 2.2895 \end{bmatrix} \in \Gamma^r$ is obtained after some iteration. Note also that since the considered constrained optimization problem is not a convex optimization problem, it may have several local infimal (minimal) solutions. Thus the result obtained via the Main-Algorithm may be a local optimal solution of the auxiliary minimization problem. However, for this example, we have started the algorithms with several different initial guesses; they all finally converge to the solution $F^o = \begin{bmatrix} -0.7183 & 1.4953 \\ 4.8755 & 2.4765 \end{bmatrix}$.

For comparison, we consider the same optimization problem with the following new constrained regions:

$$\Omega_1 = \{s \in \mathbf{C} \,|\, |s+15| < 10, |s+5| < 7\}$$
$$\Omega_2 = \{s \in \mathbf{C} \,|\, \mathrm{Re}(s) < -15, \mathrm{Re}(s) > -20\}.$$

We find that for several different initial guesses $F(0)$, the Sub-Algorithm never converges. Therefore, we expect that the set

$$\Gamma^r \equiv \{F \in R^{2\times 2} | \sigma(A_1 + B_1 F C_1) \in \Omega_1 \text{and} \sigma(A_2 + B_2 F C_2) \in \Omega_2\}$$

is empty and the considered problem is unsolvable. $\Delta\Delta$

Example 2: For comparison, we now consider a static state feedback simultaneous regional pole placement problem. Consider the systems described in Howitt and Luus [11], Paskota *et al.* [23], and Petersen [24]

$$\dot{x}(t) = A_k x(t) + B_k u(t), \; k = 1, 2, 3, \text{and } 4$$

where

$$A_1 = \begin{bmatrix} -0.989\,60 & 17.4100 & 96.15 \\ 0.264\,80 & -0.8512 & -11.89 \\ 0 & 0 & -30 \end{bmatrix}, \; B_1 = \begin{bmatrix} -97.78 \\ 0 \\ 30 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.660\,70 & 18.1100 & 84.34 \\ 0.082\,01 & -0.6587 & -10.81 \\ 0 & 0 & -30 \end{bmatrix}, \; B_2 = \begin{bmatrix} -272.2 \\ 0 \\ 30 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1.702\,00 & 50.7200 & 263.50 \\ 0.220\,10 & -1.4180 & -31.99 \\ 0 & 0 & -30 \end{bmatrix}, \; B_3 = \begin{bmatrix} -85.09 \\ 0 \\ 30 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -0.516\,20 & 26.9600 & 178.90 \\ -0.689\,60 & -1.2250 & -30.38 \\ 0 & 0 & -30 \end{bmatrix}, \; B_4 = \begin{bmatrix} 175.6 \\ 0 \\ 30 \end{bmatrix}.$$

Suppose $\mathbf{E}\{x_k(0) x_k^T(0)\} = X_{k0} = I_{3\times 3}$, $k = 1, 2, 3$, and 4.

The design goal is to find a static state feedback gain F such that the controllers

$$u_k(t) = F \cdot x_k(t),\ k = 1, 2, 3, \text{and } 4$$

achieve the infimum of the cost function

$$J(F) = \sum_{k=1}^{4} \boldsymbol{E}\left\{\int_0^{\infty} (x_k^T x_k + u_k^T u_k)dt\right\}$$

subject to the constraints that

$$\sigma(A_k + B_k F C_k) \in \Omega_k,\ k = 1, 2, 3, \text{and } 4$$

where the constrained regions Ω_k, $k = 1, 2, 3$, and 4, are represented by

$$\begin{aligned}\Omega_1 &= \{s \in \boldsymbol{C} \mid \text{Re}(s) < -3\}\\ \Omega_2 &= \{s \in \boldsymbol{C} \mid \text{Re}(s) < -1.5, \text{Re}(e^{-\mathbf{i}45^\circ}s) < 0\\ &\quad \text{and Re}(e^{\mathbf{i}45^\circ}s) < 0\}\\ \Omega_3 &= \{s \in \boldsymbol{C} \mid \text{Re}(s) < -10, |s + 30| < 90\}\\ \Omega_4 &= \{s \in \boldsymbol{C} \mid \text{Re}(e^{-\mathbf{i}60^\circ}s) < 0, \text{Re}(e^{\mathbf{i}60^\circ}s) < 0\\ &\quad \text{and} |s + 30| < 100\}.\end{aligned}$$

Let

$$\Gamma^r \equiv \left\{F \in R^{1\times 3} \,|\sigma(A_k + B_k F) \in \Omega_k, k = 1, 2, 3, \text{ and } 4\right\}.$$

As shown in Section II

$$J(F) = \sum_{k=1}^{4} \text{Tr}(P_k X_{k0}) = \sum_{k=1}^{4} \text{Tr}(P_k)$$

where P_1, P_2, P_3, and P_4 are the positive definite solutions of

$$(A_1 + B_1 F)^T P_1 + P_1(A_1 + B_1 F) + F^T F + I = 0 \quad (49)$$
$$(A_2 + B_2 F)^T P_2 + P_2(A_2 + B_2 F) + F^T F + I = 0 \quad (50)$$
$$(A_3 + B_3 F)^T P_3 + P_3(A_3 + B_3 F) + F^T F + I = 0 \quad (51)$$
$$(A_4 + B_4 F)^T P_4 + P_4(A_4 + B_4 F) + F^T F + I = 0. \quad (52)$$

Let the infimal solution of this problem be denoted by $F = F_{\text{opt}}$.

Note that for this problem, $C_1 = C_2 = C_3 = C_4 = I$, $w_1 = w_2 = w_3 = w_4 = 1$, $Q_1 = Q_2 = Q_3 = Q_4 = I$, and $R_1 = R_2 = R_3 = R_4 = 1$. Moreover, choose

$$\widetilde{Q}_{11} = \widetilde{Q}_{21} = \widetilde{Q}_{22} = \widetilde{Q}_{23} = \widetilde{Q}_{31} = \hat{Q}_{31} = \widetilde{Q}_{41} = \widetilde{Q}_{42} = \hat{Q}_{41} = I$$

and

$$\widetilde{R}_{11} = \widetilde{R}_{21} = \widetilde{R}_{22} = \widetilde{R}_{23} = \widetilde{R}_{31} = \hat{R}_{31} = \widetilde{R}_{41} = \widetilde{R}_{42} = \hat{R}_{41} = 1.$$

From the discussions in Section III, we solve the following auxiliary minimization problem: Find $F = F^o$, over Γ^r, to minimize the *auxiliary cost function*

$$\begin{aligned}J_{\text{aux}}(F) &= J(F) + \rho \cdot J_{\text{pole}}(F)\\ &= \sum_{k=1}^{4} \text{Tr}(P_k) + \rho \cdot \text{Tr}\left(\widetilde{P}_{11} + \widetilde{P}_{21} + \widetilde{P}_{22} + \widetilde{P}_{23}\right.\\ &\quad \left. + \widetilde{P}_{31} + \hat{P}_{31} + \widetilde{P}_{41} + \widetilde{P}_{42} + \hat{P}_{41}\right)\end{aligned}$$

where ρ is a weighting factor to be chosen, and matrices $\widetilde{P}_{11}$, $\widetilde{P}_{21}$, $\widetilde{P}_{22}$, $\widetilde{P}_{23}$, $\widetilde{P}_{31}$, $\hat{P}_{31}$, $\widetilde{P}_{41}$, $\widetilde{P}_{42}$, and $\hat{P}_{41}$ are the positive definite solutions of

$$(A_1 + B_1 F + 3I)^T \widetilde{P}_{11} + \widetilde{P}_{11}(A_1 + B_1 F + 3I) + F^T F + I = 0 \quad (53)$$
$$(A_2 + B_2 F + 1.5I)^T \widetilde{P}_{21} + \widetilde{P}_{21}(A_2 + B_2 F + 1.5I) + F^T F + I = 0 \quad (54)$$
$$e^{\mathbf{i}45^0}(A_2 + B_2 F)^T \widetilde{P}_{22} + \widetilde{P}_{22}(A_2 + B_2 F)e^{-\mathbf{i}45^0} + F^T F + I = 0 \quad (55)$$
$$e^{-\mathbf{i}45^0}(A_2 + B_2 F)^T \widetilde{P}_{23} + \widetilde{P}_{23}(A_2 + B_2 F)e^{\mathbf{i}45^0} + F^T F + I = 0 \quad (56)$$
$$(A_3 + B_3 F + 10I)^T \widetilde{P}_{31} + \widetilde{P}_{31}(A_3 + B_3 F + 10I) + F^T F + I = 0 \quad (57)$$
$$\frac{1}{90^2}(A_3 + B_3 F + 30I)^T \hat{P}_{31}(A_3 + B_3 F + 30I) - \hat{P}_{31} + F^T F + I = 0 \quad (58)$$
$$e^{\mathbf{i}60^o}(A_4 + B_4 F)^T \widetilde{P}_{41} + \widetilde{P}_{41}(A_4 + B_4 F)e^{-\mathbf{i}60^o} + F^T F + I = 0 \quad (59)$$
$$e^{-\mathbf{i}60^o}(A_4 + B_4 F)^T \widetilde{P}_{42} + \widetilde{P}_{42}(A_4 + B_4 F)e^{\mathbf{i}60^o} + F^T F + I = 0 \quad (60)$$
$$\frac{1}{100^2}(A_4 + B_4 F + 30I)^T \hat{P}_{41}(A_4 + B_4 F + 30I) - \hat{P}_{41} + F^T F + I = 0. \quad (61)$$

Suppose matrices L_1, L_2, L_3, L_4, $\widetilde{L}_{11}$, $\widetilde{L}_{21}$, $\widetilde{L}_{22}$, $\widetilde{L}_{23}$, $\widetilde{L}_{31}$, $\hat{L}_{31}$, $\widetilde{L}_{41}$, $\widetilde{L}_{42}$, and $\hat{L}_{41}$ are the positive definite solutions of

$$(A_1 + B_1 F)L_1 + L_1(A_1 + B_1 F)^T + I = 0 \quad (62)$$
$$(A_2 + B_2 F)L_2 + L_2(A_2 + B_2 F)^T + I = 0 \quad (63)$$
$$(A_3 + B_3 F)L_3 + L_3(A_3 + B_3 F)^T + I = 0 \quad (64)$$
$$(A_4 + B_4 F)L_4 + L_4(A_4 + B_4 F)^T + I = 0 \quad (65)$$
$$(A_1 + B_1 F + 3I)\widetilde{L}_{11} + \widetilde{L}_{11}(A_1 + B_1 F + 3I)^T + \rho I = 0 \quad (66)$$
$$(A_2 + B_2 F + 1.5I)\widetilde{L}_{21} + \widetilde{L}_{21}(A_2 + B_2 F + 1.5I)^T + \rho I = 0 \quad (67)$$
$$e^{-\mathbf{i}45^0}(A_2 + B_2 F)\widetilde{L}_{22} + \widetilde{L}_{22}(A_2 + B_2 F)^T e^{i45^0} + \rho I = 0 \quad (68)$$
$$e^{\mathbf{i}45^0}(A_2 + B_2 F)\widetilde{L}_{23} + \widetilde{L}_{23}(A_2 + B_2 F)^T e^{-i45^0} + \rho I = 0 \quad (69)$$
$$(A_3 + B_3 F + 10I)\widetilde{L}_{31} + \widetilde{L}_{31}(A_3 + B_3 F + 10I)^T + \rho I = 0 \quad (70)$$
$$\frac{1}{90^2}(A_3 + B_3 F + 30I)\hat{L}_{31}(A_3 + B_3 F + 30I)^T - \widetilde{L}_{31} + \rho I = 0 \quad (71)$$
$$e^{-\mathbf{i}60^o}(A_4 + B_4 F)\widetilde{L}_{41} + \widetilde{L}_{41}(A_4 + B_4 F)^T e^{\mathbf{i}60^o} + \rho I = 0 \quad (72)$$
$$e^{\mathbf{i}60^o}(A_4 + B_4 F)\widetilde{L}_{42} + \widetilde{L}_{42}(A_4 + B_4 F)e^{-\mathbf{i}60^o} + \rho I = 0 \quad (73)$$
$$\frac{1}{100^2}(A_4 + B_4 F + 30I)\hat{L}_{41}(A_4 + B_4 F + 30I)^T - \hat{L}_{41} + \rho I = 0. \quad (74)$$

From the Theorem 1, we know that the gradient of $J_{\mathrm{aux}}(F)$ at a fixed point F is

$$\begin{aligned}
&F_{grad}(F)\\
&\quad = 2F\Big(L_1+L_2+L_3+L_4+\widetilde{L}_{11}+\widetilde{L}_{21}+\widetilde{L}_{22}+\widetilde{L}_{23}\\
&\qquad +\widetilde{L}_{31}+\widetilde{L}_{41}+\widetilde{L}_{42}\Big)\\
&\qquad +2\left(\frac{1}{90^2}B_3^T\hat{P}_{31}B_3F\hat{L}_{31}+\frac{1}{100^2}B_4^T\hat{P}_{41}B_4F\hat{L}_{41}\right)\\
&\qquad +2B_1^T\left(P_1L_1+\widetilde{P}_{11}\widetilde{L}_{11}\right)\\
&\qquad +2B_2^T\Big(P_2L_2+\frac{1}{2}\Big(2\widetilde{P}_{21}\widetilde{L}_{21}+e^{\mathbf{i}45^\circ}\widetilde{P}_{22}\widetilde{L}_{22}\\
&\qquad +e^{-\mathbf{i}45}\widetilde{P}_{22}^T\widetilde{L}_{22}^T+e^{-\mathbf{i}45^\circ}\widetilde{P}_{23}\widetilde{L}_{23}+e^{\mathbf{i}45^\circ}\widetilde{P}_{23}^T\widetilde{L}_{23}^T\Big)\Big)\\
&\qquad +2B_3^T\Big(P_3L_3+\frac{1}{2}\Big(2\widetilde{P}_{31}\widetilde{L}_{31}+\frac{1}{90^2}\Big(\hat{P}_{31}(A_3\\
&\qquad +A_3^T-60I)\hat{L}_{31}\Big)\Big)\Big)\\
&\qquad +2B_4^T\Big(P_4L_4+\frac{1}{2}\Big(e^{\mathbf{i}60^\circ}\widetilde{P}_{41}\widetilde{L}_{41}\\
&\qquad +e^{-\mathbf{i}60^\circ}\widetilde{P}_{41}^T\widetilde{L}_{41}^T+e^{-\mathbf{i}60^\circ}\widetilde{P}_{42}\widetilde{L}_{42}+e^{\mathbf{i}60^\circ}\widetilde{P}_{42}^T\widetilde{L}_{42}^T\\
&\qquad +\frac{1}{100^2}\Big(\hat{P}_{41}(A_4+A_4^T-60I)\hat{L}_{41}\Big)\Big)\Big).
\end{aligned}$$

The solution algorithms are similar to those presented in the Example 1 and, thus, are omitted here to save space.

The following four cases are considered.

Case 1: The poles' constraints are relaxed ($\rho = 0$). This is the optimal simultaneous stabilization problem considered in [23].

We start the algorithms with the initial guess $F(0) = [0 \quad 0 \quad 0]$. For saving space, we only give the final positive definite solutions P_1, P_2, P_3, and P_4:

$$P_1 = \begin{bmatrix} 0.1619 & 0.0696 & 0.4985\\ 0.0696 & 0.6688 & 0.0838\\ 0.4985 & 0.0838 & 1.6745 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 0.0314 & -0.4403 & 0.2532\\ -0.4403 & 7.2536 & -4.0848\\ 0.2532 & -4.0848 & 2.3371 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 0.2798 & 1.6245 & 0.7637\\ 1.6245 & 11.5476 & 4.2024\\ 0.7637 & 4.2024 & 2.2388 \end{bmatrix}$$
$$P_4 = \begin{bmatrix} 0.1359 & 0.0517 & 0.7651\\ 0.0517 & 0.6329 & 0.1598\\ 0.7651 & 0.1598 & 4.5248 \end{bmatrix}.$$

The resultant optimal feedback gain for the auxiliary minimization problem is

$$F^o = [1.096\,54 \quad 8.315\,10 \quad -4.296\,74].$$

We have

$$\begin{aligned}
\sigma(A_1+B_1F^o) &= \{-3.4369, -10.6577, -253.8434\} \in \Omega_1\\
\sigma(A_2+B_2F^o) &= \{-1.8523, -19.7222, -437.1257\} \in \Omega_2\\
\sigma(A_3+B_3F^o) &= \{-5.5091 \pm \mathbf{i}\times 12.8011, -244.3088\} \notin \Omega_3\\
\sigma(A_4+B_4F^o) &= \{-8.3368, -12.3912, -332.4681\} \notin \Omega_4.
\end{aligned}$$

Moreover, $J(F^o) = 31.4870$.

Case 2: The weighting factor $\rho = 0.000\,01$.

We start the algorithms with the initial guess $F(0) = [0 \quad 0 \quad 0]$. For saving space, we only give the final positive definite solutions P_1, P_2, P_3, and P_4:

$$P_1 = \begin{bmatrix} 0.1620 & 0.0229 & 0.4968\\ 0.0229 & 0.7759 & 0.0013\\ 0.4968 & 0.0013 & 1.6136 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 0.0333 & -0.5048 & 0.2646\\ -0.5048 & 9.1464 & -4.5380\\ 0.2646 & -4.5380 & 2.3992 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 0.2606 & 1.5358 & 0.7045\\ 1.5358 & 10.6698 & 3.7939\\ 0.7045 & 3.7939 & 2.1147 \end{bmatrix}$$
$$P_4 = \begin{bmatrix} 0.1500 & 0.0347 & 0.8431\\ 0.0347 & 0.7437 & 0.1274\\ 0.8431 & 0.1274 & 4.9231 \end{bmatrix}.$$

The resultant optimal feedback gain for the auxiliary minimization problem is

$$F^o = [0.502\,63 \quad 4.298\,37 \quad -0.403\,65].$$

The final solutions of $\widetilde{P}_{11}$, $\widetilde{P}_{21}$, $\widetilde{P}_{22}$, $\widetilde{P}_{23}$, $\widetilde{P}_{31}$, $\hat{P}_{31}$, $\widetilde{P}_{41}$, $\widetilde{P}_{42}$, $\hat{P}_{41}$, L_1, L_2, L_3, L_4, $\widetilde{L}_{11}$, $\widetilde{L}_{21}$, $\widetilde{L}_{22}$, $\widetilde{L}_{23}$, $\widetilde{L}_{31}$, $\hat{L}_{31}$, $\widetilde{L}_{41}$, $\widetilde{L}_{42}$, and $\hat{L}_{41}$ can be easily obtained by solving (53) – (74) with $F = [0.502\,63 \quad 4.298\,37 \quad -0.403\,65]$ and thus are omitted here for the consideration of space.

We can see that

$$\begin{aligned}
\sigma(A_1+B_1F^o) &= \{-3.0107, -22.3397, -67.7474\} \in \Omega_1\\
\sigma(A_2+B_2F^o) &= \{-1.8572, -27.3911, -150.9974\} \in \Omega_2\\
\sigma(A_3+B_3F^o) &= \{-10.1107 \pm \mathbf{i}\times 15.0304, -67.7772\} \in \Omega_3\\
\sigma(A_4+B_4F^o) &= \{-5.3777, -29.4697, -97.2657\} \in \Omega_4.
\end{aligned}$$

All the closed-loop poles of the four systems are located in the desired regions. Moreover, we can expect that $J(F^o) = 32.9924$ will be very close to the (local) infimal value $J(F_{\mathrm{opt}})$ since $\rho = 0.000\,01$ is very small.

Case 3: The weighting factor $\rho = 0.01$.

For saving space, we only give the final positive definite solutions P_1, P_2, P_3, and P_4:

$$P_1 = \begin{bmatrix} 0.1541 & 0.0123 & 0.4700\\ 0.0123 & 1.0012 & -0.0256\\ 0.4700 & -0.0256 & 1.5393 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 0.0316 & -0.4799 & 0.2481\\ -0.4799 & 9.3959 & -4.3015\\ 0.2481 & -4.3015 & 2.2788 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 0.2809 & 1.6853 & 0.7613\\ 1.6853 & 11.7084 & 4.1367\\ 0.7613 & 4.1367 & 2.3943 \end{bmatrix}$$
$$P_4 = \begin{bmatrix} 0.1503 & 0.0301 & 0.8438\\ 0.0301 & 0.8707 & 0.1070\\ 0.8438 & 0.1070 & 4.9461 \end{bmatrix}.$$

The resultant optimal feedback gain for the auxiliary minimization problem is

$$F^o = [0.478\,18 \quad 4.012\,17 \quad 0.422\,14].$$

We have

$$\sigma(A_1+B_1F^o)=\{-3.0636,-31.4349\pm\mathbf{i}\times 21.1090\}\in\Omega_1$$
$$\sigma(A_2+B_2F^o)=\{-1.8668,-35.0226,-111.9374\}\in\Omega_2$$
$$\sigma(A_3+B_3F^o)=\{-12.0661\pm\mathbf{i}\times 20.7483,-37.0123\}\in\Omega_3$$
$$\sigma(A_4+B_4F^o)=\{-5.1120,-48.9671\pm\mathbf{i}\times 19.4258\}\in\Omega_4$$

as desired. Moreover, $J(F^o)=34.7516$.

Case 4: The weighting factor $\rho=1$.

For saving space, we only give the final positive definite solutions P_1, P_2, P_3, and P_4:

$$P_1=\begin{bmatrix}0.1522 & 0.0174 & 0.4641\\ 0.0174 & 1.1956 & -0.0109\\ 0.4641 & -0.0109 & 1.5329\end{bmatrix}$$
$$P_2=\begin{bmatrix}0.0312 & -0.4704 & 0.2454\\ -0.4704 & 9.5874 & -4.2226\\ 0.2454 & -4.2226 & 2.2692\end{bmatrix}$$
$$P_3=\begin{bmatrix}0.2936 & 1.7849 & 0.7981\\ 1.7849 & 12.4458 & 4.4047\\ 0.7981 & 4.4047 & 2.5636\end{bmatrix}$$
$$P_4=\begin{bmatrix}0.1516 & 0.0313 & 0.8521\\ 0.0313 & 0.9662 & 0.1127\\ 0.8521 & 0.1127 & 5.0100\end{bmatrix}.$$

The optimal feedback gain for the auxiliary minimization problem is

$$F^o=[0.496\,99 \quad 4.156\,35 \quad 0.748\,35].$$

We have

$$\sigma(A_1+B_1F^o)=\{-3.0720,-25.4570\pm\mathbf{i}\times 27.2654\}\in\Omega_1$$
$$\sigma(A_2+B_2F^o)=\{-1.8569,-39.7396,-102.5534\}\in\Omega_2$$
$$\sigma(A_3+B_3F^o)=\{-12.2893\pm\mathbf{i}\times 25.1564,-28.3798\}\in\Omega_3$$
$$\sigma(A_4+B_4F^o)=\{-4.9801,-45.7911\pm\mathbf{i}\times 28.6696\}\in\Omega_4$$

as desired. Moreover, $J(F^o)=36.1994$.

Note that our results in case 1 are almost the same as the results presented in [23]. The resultant cost $J(F^o)=31.4870$ is the minimal cost for optimal simultaneous stabilization problem (without regional pole constraints). However, since the constraints on closed-loop poles are not considered, the resultant $\sigma(A_{kc})\not\subset\Omega_k$ for $k=3$ and 4.

The other cases show that the closed-loop poles of each system are assigned to the prespecified region as desired since the constraints on closed-loop poles are considered. The minimal value $J(F^o)$ of the auxiliary minimization problem will be closer to its infimal value $J(F_{\rm opt})$ of the original constraint optimization problem if the weighting factor ρ becomes smaller. No matter how small the weighting factor ρ is, the resultant closed-loop poles of each system will still lie inside the desired regions. Note that the weighting factor in case 2 is very small, we can expect that the infimal solution (may be a local one) of the original problem is very close to $F_{\rm opt}=[0.550\,263 \quad 4.298\,37 \quad -0.403\,65]$. $\Delta\Delta$

V. Conclusions

In this paper, a new method for approximate solving the optimal output feedback simultaneous regional pole placement problem is provided. The constraint region for each system can be the intersection of several open half-planes and/or open disks. Good transient responses of the closed-loop systems can be guaranteed since the closed loop poles are restricted to lie in some desired regions, and good static state responses of the systems are also guaranteed since a quadratic type cost function is minimized. This problem cannot be solved via LMI approach since its admissible solution set may be nonconvex. Based on the barrier method, we instead solve an auxiliary minimization problem to obtain an approximate solution to the original constrained optimization problem. We have shown that the minimum point of the auxiliary cost function does exist if the admissible solution set is nonempty. Moreover, the necessary conditions for which the optimal solution of the auxiliary minimization problem must be satisfied have been derived. Based on gradient method, numerical algorithms have been provided to find the optimal solution.

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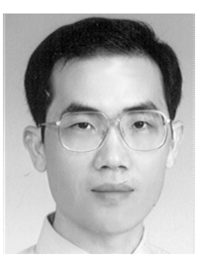

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