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Research

A Bayesian Approach to Obtain a Lower Bound for the C_{pm} Capability Index

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The Taguchi capability index C_{pm}, which incorporates the departure of the process mean from the target value, has been proposed to the manufacturing industry for measuring manufacturing capability. A Bayesian procedure has been considered for testing process performance assuming $\mu = T$, which was generalized without assuming $\mu = T$. Statistical properties of the natural estimator of the index C_{pm} for normal processes have been investigated extensively. However, the investigation was restricted to processes with symmetric tolerances. Recently, a generalized C_{pm}, referred to as C''_{pm}, was proposed to cover processes with asymmetric tolerances. Under the normality assumption, the statistical properties of the estimated C''_{pm} including the exact sampling distribution, the rth moment, expected value, variance, and the mean-squared error were obtained. In this paper, we use a Bayesian approach to obtain the interval estimation for the generalized Taguchi capability index C''_{pm}. Consequently, the manufacturing capability testing can be performed for quality assurance. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: asymmetric tolerances; process capability index

1. INTRODUCTION

Process capability indices (PCIs), whose purpose is to provide numerical measures on whether a manufacturing process is capable of reproducing items satisfying the quality requirements preset by the engineer (or the product designer), have recently been the research focus in quality assurance and engineering statistics literature. Examples include Boyles[1], Bordignon and Scagliarini[2], Borges and Ho[3], Chang *et al.*[4], Hoffman[5], Nahar *et al.*[6], Noorossana[7], Pearn *et al.*[8], Pearn and Lin[9], Zimmer *et al.*[10], Lee *et al.*[11]. Kotz and Johnson[12] and Spiring *et al.*[13] provided a rather complete list of the PCI research papers from the

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past 10 years. The two basic capability indices $C_{\rm p}$ and $C_{\rm pk}$, are defined in the following (Kane[14]):

$$C_{\rm p} = \frac{\text{USL} - \text{LSL}}{6\sigma} \tag{1}$$

$$C_{\rm pk} = \min\left\{\frac{\text{USL} - \mu}{3\sigma}, \frac{\mu - \text{LSL}}{3\sigma}\right\} \tag{2}$$

where USL and LSL are the upper and the lower specification limits, respectively, μ is the process mean, and σ is the process standard deviation. The index $C_{\rm p}$ only reflects the magnitude of the process variation relative to the specification tolerance, and is therefore used to measure process precision. The index $C_{\rm pk}$ takes into account process variation as well as the location of the process mean. The natural estimators of $C_{\rm p}$ and $C_{\rm pk}$ can be obtained by substituting the sample mean $\bar{X} = \sum_{i=1}^{n} X_i/n$ for μ and the sample variance $S_{n-1}^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/(n-1)$ for σ^2 in expressions (1) and (2). Chou and Owen[15], Pearn *et al.*[16] and Kotz *et al.*[17] investigated the statistical properties and the sampling distributions of the natural estimators of $C_{\rm p}$ and $C_{\rm pk}$.

Boyles[1] noted that $C_{\rm pk}$ is a yield-based index. In fact, the design of $C_{\rm pk}$ is independent of the target value T, which can fail to account for process targeting (the ability to cluster around the target). For this reason, Chan *et al.*[18] introduced the index $C_{\rm pm}$ to take the process targeting issue into consideration. The index $C_{\rm pm}$ is defined as follows:

$$C_{\rm pm} = \frac{\text{USL} - \text{LSL}}{6\sqrt{\sigma^2 + (\mu - T)^2}} \tag{3}$$

We note that the index $C_{\rm pm}$ is not originally designed to provide an exact measure on the number of non-conforming items. However, $C_{\rm pm}$ includes the process departure $(\mu - T)^2$ (rather than 6σ alone) in the denominator of the definition to reflect the degree of process targeting. Chan *et al.*[18] proposed a Bayesian procedure under the restriction that $\mu = T$, and investigated the statistical properties of the sampling distribution of $C_{\rm pm}$. Boyles[1] suggested a maximum likelihood estimator (MLE) of $C_{\rm pm}$ by substituting the sample mean $\bar{X} = \sum_{i=1}^{n} X_i/n$ for μ and the MLE of σ^2, $S_n^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/n$, in expression (3). Based on the suggested MLE of $C_{\rm pm}$, Boyles[1] proposed two approximate $100(1-\alpha)\%$ lower confidence bounds using the normal and chi-square distributions for $C_{\rm pm}$ from the distribution frequency point of view. Pearn *et al.*[16] investigated the statistical properties of the MLE estimator of $C_{\rm pm}$. Shiau *et al.*[19] proposed a Bayesian procedure based on the MLE of $C_{\rm pm}$ without the restriction $\mu = T$ on the process mean μ. Their results generalized those discussed in Chan *et al.*[18].

Pearn *et al.*[16] proposed the process capability index $C_{\rm pmk}$, which combines the merits of the three earlier indices $C_{\rm p}$, $C_{\rm pk}$ and $C_{\rm pm}$. The index $C_{\rm pmk}$ alerts the user when the process variance increases and/or the process mean deviates from its target value, which is designed to monitor the normal and the near-normal processes. The index $C_{\rm pmk}$, referred to as the third-generation capability index and is defined as follows:

$$C_{\rm pmk} = \min\left\{\frac{\text{USL} - \mu}{3\sqrt{\sigma^2 + (\mu - T)^2}}, \frac{\mu - \text{LSL}}{3\sqrt{\sigma^2 + (\mu - T)^2}}\right\} \tag{4}$$

We remark that those indices presented above are designed to monitor the performance for normal and near-normal processes with symmetric tolerances, which are shown to be inappropriate for cases with asymmetric tolerances.

2. A GENERALIZATION OF $C_{\rm pm}$ FOR ASYMMETRIC TOLERANCES

A process is said to have asymmetric tolerances if the upper tolerance, $d_{\rm U} = \text{USL} - T$, is unequal to the lower tolerance, $d_{\rm L} = T - \text{LSL}$, where T is the preset target value (a known constant). To handle processes with asymmetric tolerances, Chen *et al.*[20] considered a generalization of the Taguchi capability index C_{pm}. The generalization, referred to as C''_{pm}, is defined as follows:

$$C''_{\mathrm{pm}} = \frac{d^*}{3\sqrt{\sigma^2 + A^2}} \tag{5}$$

where $d^* = \min\{d_{\mathrm{U}}, d_{\mathrm{L}}\}$, $A = \max\{d(\mu - T)/d_{\mathrm{U}}, d(T - \mu)/d_{\mathrm{L}}\}$. Clearly, if the preset target value $T = m = (\mathrm{USL} + \mathrm{LSL})/2$ (symmetric case), then $d^* = d = (\mathrm{USL} - \mathrm{LSL})/2$, $A = |\mu - T|$, and the generalization C''_{pm} reduces to the original index C_{pm}. The factor A in the definition ensures that the generalization C''_{pm} obtains its maximal value at $\mu = T$ (process is on-target) regardless of whether the tolerances are symmetric ($T = m$) or asymmetric ($T \neq m$). An estimator of C''_{pm} considered by Chen *et al.*[20] is defined as

$$\hat{C}''_{\mathrm{pm}} = \frac{d^*}{3\sqrt{S_n^2 + \hat{A}^2}} \tag{6}$$

where $\hat{A} = \max\{d(\bar{X} - T)/d_{\mathrm{U}}, d(T - \bar{X})/d_{\mathrm{L}}\}$ and $S_n^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2/n$. We note that if the production tolerance is symmetric, then $\hat{A}$ may be simplified as $|\bar{X} - T|$ and the estimator $\hat{C}''_{\mathrm{pm}}$ becomes the MLE of C_{pm} discussed by Boyles[1]. Chen *et al.*[20] investigated the statistical properties of the estimated C''_{pm}. They obtained the exact distribution and the formulae for the rth moment, expected value, variance, and the mean-squared error under the normality assumption.

We note that the natural estimator $\hat{C}''_{\mathrm{pm}}$ can be rewritten as

$$\hat{C}''_{\mathrm{pm}} = \frac{C}{3\sqrt{K + Y}} \tag{7}$$

where $C = n^{1/2}d^*/\sigma$, $K = nS_n^2/\sigma^2$, and $Y = [\max\{(d/d_{\mathrm{U}})Z, -(d/d_{\mathrm{L}})Z\}]^2$ with $Z = n^{1/2}(\bar{X} - T)/\sigma$. On the assumption of normality, the statistic K is distributed as χ^2_{n-1}, Z is distributed as $N(\delta_0, 1)$, $\delta_0 = n^{1/2}(\mu - T)/\sigma$, and the probability density function of Y is

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(\frac{1}{d_1} f_Z(-\sqrt{y}/d_1) + \frac{1}{d_2} f_Z(\sqrt{y}/d_2)\right), \quad y > 0 \tag{8}$$

where $d_1 = d/d_{\mathrm{L}}$ and $d_2 = d/d_{\mathrm{U}}$. Therefore, the probability density function of $\hat{C}''_{\mathrm{pm}}$ can be expressed as

$$f_{\hat{C}''_{\mathrm{pm}}}(x) = \frac{C^3}{27x^4}\int_0^1 \frac{1}{\sqrt{t}} f_K\left(\frac{C^2(1-t)}{9x^2}\right)\left\{\frac{1}{d_1} f_Z\left(-\frac{C\sqrt{t}}{3xd_1}\right) + \frac{1}{d_2} f_Z\left(\frac{C\sqrt{t}}{3xd_2}\right)\right\} \mathrm{d}t, \quad x > 0 \tag{9}$$

Chen *et al.*[20] showed that the statistic Z^2 follows a non-central chi-square distribution with one degree of freedom and non-centrality parameter δ_0^2. The distribution of Y is a weighted non-central chi-square distribution with one degree of freedom and non-centrality parameter δ_0^2, under the assumption of normality. The probability density function of Y, in an alternative form, may be expressed as

$$f_Y(y) = \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty}\left\{\frac{(\sqrt{2}\delta_0)^j}{j!}\Gamma\left(\frac{1+j}{2}\right)\sum_{i=1}^{2}\frac{(-1)^{ij}}{d_i^2} f_{Y_j}\left(y/d_i^2\right)\right\}, \quad y > 0 \tag{10}$$

Table I. $C^*(p^*)$ for $p^* = 0.90$ with $d/d_L = 5/6$ and $d/d_U = 5/4$

	$\delta = (\bar{x} - T)/s_{n-1}$				
n	0.0	0.5	1.0	1.5	2.0
5	2.4149	2.1576	1.7691	1.5361	1.4014
10	1.6450	1.5605	1.3998	1.2933	1.2270
15	1.4464	1.4006	1.2968	1.2223	1.1740
20	1.3520	1.3227	1.2448	1.1854	1.1460
25	1.2959	1.2754	1.2123	1.1620	1.1280
30	1.2582	1.2431	1.1897	1.1455	1.1153
35	1.2309	1.2195	1.1728	1.1331	1.1057
40	1.2100	1.2012	1.1596	1.2330	1.0981
45	1.1935	1.1866	1.1489	1.1154	1.0919
50	1.1801	1.1747	1.1401	1.1087	1.0867
55	1.1689	1.1646	1.1326	1.1031	1.0823
60	1.1594	1.1560	1.1261	1.0983	1.0785
65	1.1513	1.1486	1.1205	1.0940	1.0752
70	1.1441	1.1420	1.1156	1.0903	1.0722
75	1.1379	1.1362	1.1117	1.0870	1.0696
80	1.1323	1.1311	1.1072	1.0840	1.0672
85	1.1273	1.1264	1.1036	1.0812	1.0651
90	1.1228	1.1222	1.1004	1.0788	1.0632
95	1.1187	1.1184	1.0974	1.0765	1.0614
100	1.1149	1.1149	1.0947	1.0744	1.0597
110	1.1083	1.1087	1.0899	1.0707	1.0568
120	1.1027	1.1033	1.0857	1.0675	1.0542
130	1.0978	1.0987	1.0820	1.0647	1.0520
140	1.0935	1.0946	1.0788	1.0622	1.0500
150	1.0897	1.0909	1.0759	1.0599	1.0482
160	1.0863	1.0877	1.0733	1.0579	1.0466
170	1.0833	1.0847	1.0709	1.0561	1.0452
180	1.0805	1.0820	1.0688	1.0544	1.0439
190	1.0780	1.0796	1.0668	1.0529	1.0426
200	1.0757	1.0773	1.0650	1.0515	1.0415

where $\lambda = \delta_0^2$ and Y_j is distributed as χ^2_{1+j}. Therefore, the probability density function of $\hat{C}''_{pm}$, in an alternative form, can be expressed as

$$f_{\hat{C}''_{pm}}(x) = \frac{2^{1-n/2} C^n x^{-(n+1)}}{3^n \Gamma((n-1)/2)} \frac{e^{-\lambda/2}}{2\sqrt{\pi}} \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\delta_0 C}{3x}\right)^j \times \sum_{i=1}^{2} (-1)^{ij} \left(d_i^{-(j+1)} \int_0^1 (1-y)^{(n-3)/2} y^{(j-1)/2} \exp\left\{ \frac{-C^2}{18x^2}(1 - y + d_i^{-2} y) \right\} \mathrm{d}y \right), \quad x > 0 \tag{11}$$

3. A BAYESIAN PROCEDURE BASED ON C''_{pm}

Most existing PCI research works in testing the manufacturing capability are based on the traditional distribution frequency approach. Shiau *et al.* [19] proposed a Bayesian approach for assessing process capability by finding a $100p\%$ credible interval, which covers $100p\%$ of the posterior distribution for the index C_{pm}. Assuming that

Table II. $C^*(p^*)$ for $p^* = 0.95$ with $d/d_L = 5/6$ and $d/d_U = 5/4$

	$\delta = (\bar{x} - T)/s_{n-1}$				
n	0.0	0.5	1.0	1.5	2.0
5	2.9665	2.6081	2.0771	1.7539	1.5651
10	1.8483	1.7380	1.5293	1.3892	1.3014
15	1.5789	1.5215	1.3883	1.2912	1.2280
20	1.4537	1.4182	1.3186	1.2416	1.1902
25	1.3800	1.3561	1.2759	1.2104	1.1663
30	1.3309	1.3140	1.2458	1.1886	1.1494
35	1.2955	1.2832	1.2236	1.1722	1.1367
40	1.2686	1.2594	1.2063	1.1594	1.1268
45	1.2473	1.2405	1.1924	1.1490	1.1187
50	1.2301	1.2250	1.1808	1.1404	1.1119
55	1.2157	1.2119	1.1711	1.1331	1.1062
60	1.2035	1.2008	1.1627	1.1267	1.1013
65	1.1931	1.1912	1.1554	1.1212	1.0969
70	1.1839	1.1828	1.1490	1.1164	1.0931
75	1.1759	1.1753	1.1433	1.1121	1.0897
80	1.1688	1.1686	1.1381	1.1082	1.0866
85	1.1624	1.1626	1.1335	1.1046	1.0839
90	1.1566	1.1572	1.1293	1.1014	1.0813
95	1.1514	1.1523	1.1255	1.0985	1.0790
100	1.1466	1.1477	1.1220	1.0958	1.0769
110	1.1382	1.1397	1.1157	1.0910	1.0731
120	1.1310	1.1328	1.1103	1.0869	1.0698
130	1.1248	1.1268	1.1055	1.0832	1.0669
140	1.1193	1.1215	1.1014	1.0800	1.0643
150	1.1145	1.1168	1.0976	1.0771	1.0620
160	1.1102	1.1126	1.0943	1.0745	1.0600
170	1.1063	1.1088	1.0912	1.0722	1.0581
180	1.1027	1.1054	1.0884	1.0700	1.0564
190	1.0995	1.1022	1.0859	1.0680	1.0548
200	1.0966	1.0993	1.0836	1.0662	1.0534

$\{X_1, X_2, \ldots, X_n\}$ is a random sample taken from $N(\mu, \sigma^2)$, a normal distribution with mean μ and variance σ^2, Shiau *et al.*[19] adopted the prior $\pi(\mu, \sigma) = 1/\sigma$ and derived the posterior probability density function $f(\mu, \sigma|\mathbf{x})$ of (μ, σ) as follows:

$$f(\mu, \sigma|\mathbf{x}) = \sqrt{\frac{2n}{\pi}} \frac{\sigma^{-(n+1)}}{\beta^\alpha \Gamma(\alpha)} \exp\left\{-\frac{\sum_{i=1}^n (x_i - \mu)^2}{2\sigma^2}\right\} \tag{12}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $-\infty < \mu < \infty$, $0 < \sigma < \infty$, $\alpha = (n-1)/2$, $\beta = 2[(n-1)S_{n-1}^2]^{-1}$. Given a pre-specified capability level $\omega > 0$, the posterior probability based on index C_{pm} that a process with symmetric tolerances is capable is given as (Shiau *et al.*[19]):

$$p = \int_0^t \frac{\Phi\,(b_1(y) + b_2(y)) - \Phi\,(b_1(y) - b_2(y))}{\gamma^\alpha y^{\alpha+1} \Gamma(\alpha)} \exp\left(-\frac{1}{\gamma y}\right) \mathrm{d}y \tag{13}$$

where Φ is the cumulative distribution function of the standard normal distribution, $b_1(y) = \sqrt{(2/y)[1 - (1/\gamma)]}$, $b_2(y) = \sqrt{n[(t/y) - 1]}$, $\delta = |\bar{x} - T|/s_{n-1}$, $\gamma = 1 + [(n\delta^2)/(n-1)]$, $t = 2\hat{C}_{pm}^2/(n\omega^2)$, $\hat{C}_{pm} = (\mathrm{USL} - \mathrm{LSL})/\{6[\sum_{i=1}^n (X_i - T)^2/n]^{1/2}\}$.

Table III. $C^*(p^*)$ for $p^* = 0.975$ with $d/d_L = 5/6$ and $d/d_U = 5/4$

	$\delta = (\bar{x} - T)/s_{n-1}$				
n	0.0	0.5	1.0	1.5	2.0
5	3.6036	3.1265	2.4314	2.0040	1.7525
10	2.0572	1.9176	1.6584	1.4839	1.3745
15	1.7098	1.6390	1.4758	1.3566	1.2790
20	1.5521	1.5093	1.3878	1.2937	1.2311
25	1.4604	1.4321	1.3342	1.2548	1.2012
30	1.3997	1.3801	1.2973	1.2278	1.1804
35	1.3562	1.3423	1.2700	1.2077	1.1648
40	1.3232	1.3132	1.2488	1.1920	1.1526
45	1.2973	1.2901	1.2317	1.1793	1.1427
50	1.2763	1.2711	1.2177	1.1688	1.1345
55	1.2588	1.2552	1.2058	1.1599	1.1275
60	1.2440	1.2417	1.1956	1.1522	1.1215
65	1.2314	1.2300	1.1867	1.1455	1.1163
70	1.2203	1.2198	1.1789	1.1396	1.1116
75	1.2106	1.2107	1.1720	1.1344	1.1075
80	1.2020	1.2026	1.1658	1.1297	1.1038
85	1.1943	1.1954	1.1602	1.1254	1.1004
90	1.1873	1.1888	1.1551	1.1216	1.0974
95	1.1810	1.1828	1.1505	1.1180	1.0946
100	1.1753	1.1773	1.1462	1.1148	1.0920
110	1.1652	1.1677	1.1387	1.1090	1.0874
120	1.1565	1.1593	1.1321	1.1040	1.0835
130	1.1491	1.1521	1.1264	1.0996	1.0800
140	1.1425	1.1457	1.1214	1.0957	1.0769
150	1.1367	1.1400	1.1169	1.0922	1.0742
160	1.1315	1.1350	1.1128	1.0891	1.0717
170	1.1269	1.1304	1.1092	1.0863	1.0694
180	1.1226	1.1262	1.1058	1.0837	1.0674
190	1.1188	1.1224	1.1028	1.0813	1.0655
200	1.1153	1.1189	1.1000	1.0792	1.0638

Posterior probability

If the production tolerance is asymmetric (USL $-\, T \neq T\, -$ LSL), then the posterior probability based on index C''_{pm} that a process with asymmetric tolerances is capable is given as follows:

$$p^* = \begin{cases} \displaystyle\int_0^{t^*} \frac{\Phi(b_1(y) + b_U^*(y)) - \Phi(b_1(y) - b_L^*(y))}{\gamma^\alpha y^{\alpha+1}\Gamma(\alpha)} \exp\left(-\frac{1}{\gamma y}\right) \mathrm{d}y, & \text{for } \bar{x} < T \\ \displaystyle\int_0^{t^*} \frac{\Phi(b_1(y) + b_L^*(y)) - \Phi(b_1(y) - b_U^*(y))}{\gamma^\alpha y^{\alpha+1}\Gamma(\alpha)} \exp\left(-\frac{1}{\gamma y}\right) \mathrm{d}y, & \text{for } \bar{x} > T \end{cases} \tag{14}$$

where $b_L^*(y) = (d_L/d)\sqrt{n[(t^*/y) - 1]}$, $b_U^*(y) = (d_U/d)\sqrt{n[(t^*/y) - 1]}$, $t^* = 2[d^*/(3\omega)]^2[ns_n^2 + n(\bar{x} - T)^2]^{-1}$, $\hat{C}''_{pm} = (d^*/3)[s_n^2 + (d/d_L)^2(T - \bar{x})^2]^{-1/2}$ for $\bar{x} < T$, $\hat{C}''_{pm} = (d^*/3)[s_n^2 + (d/d_U)^2(\bar{x} - T)^2]^{-1/2}$ for $\bar{x} > T$, $d^* = d_U$ for USL $-\, T < T\, -$ LSL and $d^* = d_L$ for USL $-\, T > T\, -$ LSL. The derivation of (14) is given in Appendix A. We note that if the production tolerance is symmetric (USL $-\, T = T\, -$ LSL), then $d^* = d$, $t^* = t$ and $b_L^*(y) = b_U^*(y) = b_2(y)$ implies that $p^* = p$. Consequently, the posterior probability based on index C_{pm} derived by Shiau *et al.*[19] is a special case of $T = m$ (symmetric tolerance).

It is rather complicated and computationally inefficient to calculate p^* from (14). However, for fixed sample size n and given parameter value δ, there is a one-to-one correspondence between p^* and $C^* = \hat{C}''_{pm}/\omega$.

Table IV. $C^*(p^*)$ for $p^* = 0.99$ with $d/d_L = 5/6$ and $d/d_U = 5/4$

	$\delta = (\bar{x} - T)/s_{n-1}$				
n	0.0	0.5	1.0	1.5	2.0
5	4.6131	3.9473	2.9953	2.4036	2.0521
10	2.3472	2.1639	1.8335	1.6115	1.4727
15	1.8841	1.7932	1.5890	1.4405	1.3442
20	1.6801	1.6262	1.4754	1.3293	1.2823
25	1.5635	1.5284	1.4073	1.3099	1.2444
30	1.4870	1.4632	1.3610	1.2761	1.2183
35	1.4326	1.4159	1.3270	1.2510	1.1990
40	1.3917	1.3799	1.3007	1.2316	1.1839
45	1.3596	1.3513	1.2797	1.2160	1.1717
50	1.3336	1.3279	1.2624	1.2030	1.1616
55	1.3121	1.3084	1.2478	1.1921	1.1531
60	1.2940	1.2918	1.2353	1.1828	1.1458
65	1.2785	1.2774	1.2245	1.1746	1.1394
70	1.2650	1.2649	1.2150	1.1675	1.1337
75	1.2531	1.2538	1.2066	1.1611	1.1287
80	1.2426	1.2440	1.1990	1.1554	1.1242
85	1.2332	1.2351	1.1922	1.1502	1.1202
90	1.2248	1.2271	1.1861	1.1455	1.1165
95	1.2171	1.2198	1.1804	1.1412	1.1131
100	1.2102	1.2131	1.1753	1.1373	1.1100
110	1.1979	1.2014	1.1661	1.1303	1.1045
120	1.1874	1.1913	1.1582	1.1243	1.0997
130	1.1784	1.1825	1.1513	1.1190	1.0955
140	1.1705	1.1748	1.1452	1.1143	1.0918
150	1.1635	1.1679	1.1398	1.1101	1.0885
160	1.1572	1.1618	1.1349	1.1064	1.0855
170	1.1516	1.1562	1.1305	1.1030	1.0828
180	1.1465	1.1512	1.1265	1.0999	1.0804
190	1.1419	1.1466	1.1228	1.0970	1.0781
200	1.1376	1.1424	1.1194	1.0944	1.0760

Since the estimator $\hat{C}''_{pm}$ can be calculated from the collected sample data, we could then find the minimum value $C^*(p^*)$ of C^* required to ensure the posterior probability p^* reaching a certain desirable level, which is useful for practical applications in assessing process capability. For engineers to conveniently apply our Bayesian procedure based on C''_{pm} in their factory applications, we calculate some $C^*(p^*)$ for various values of sample sizes n and $\delta = (\bar{x} - T)/s_{n-1}$.

In Tables I–VIII, we tabulate $C^*(p^*)$ for an example process with asymmetric tolerance under the specifications $d/d_L = 5/6$, and $d/d_U = 5/4$ for $p^* = 0.90$, 0.95, 0.975, 0.99, respectively. We observe that the $C^*(p^*)$ value decreases as n increases for each fixed p^* and δ. We also note that for each fixed n and p^*, the $C^*(p^*)$ value decreases as positive δ value increases and increases as negative δ value increases. In fact, the entries in these tables are values of $C^*(p^*)$ that satisfy $P\{C''_{pm} > [\hat{C}''_{pm}/C^*(p^*)]\mathbf{x}\} = p^*$, and $[\hat{C}''_{pm}/C^*(p^*), \infty)$ is a $100p^*\%$ credible interval (similar to the $100p^*\%$ confidence interval using the distribution frequency approach) for C''_{pm}. Hence, the posterior probability that the credible interval contains the true C''_{pm} value is p^*. In our Bayesian approach, we say that a process with asymmetric production tolerance is capable if all the points fall within this credible interval are greater than a pre-specified value of ω. When this occurs, we have P {process with asymmetric production tolerance is capable$|\mathbf{x}\} > p^*$. Therefore, to test whether a process is capable or not (with capability level ω and credible level p^*), we only need to check if $\hat{C}''_{pm} > \omega C^*(p^*)$.

For the special case in which $\mu = T$, the formula for $C''_{pm} = (d^*/3)(\sigma^2 + A^2)^{-1/2}$ reduces to $C''_{pm} = d^*/(3\sigma)$ and so we could use the estimator $\hat{C}''_{pm} = d^*/(3\hat{\tau})$, where $\hat{\tau}^2 = \sum_{i=1}^{n} (X_i - T)^2/n$. We note that

Table V. $C^*(p^*)$ for $p^* = 0.90$ with $d/d_L = 5/6$ and $d/d_U = 5/4$

	$\delta = (\bar{x} - T)/s_{n-1}$				
n	−0.25	−0.5	−1.0	−1.5	−2.0
5	2.3256	2.1840	1.8757	1.6407	1.4845
10	1.6002	1.5437	1.4195	1.3189	1.2490
15	1.4156	1.3807	1.3003	1.2325	1.1842
20	1.3285	1.3033	1.2424	1.1898	1.1516
25	1.2769	1.2569	1.2073	1.1635	1.1313
30	1.2422	1.2256	1.1833	1.1454	1.1172
35	1.2171	1.2029	1.1656	1.1319	1.1067
40	1.1979	1.1854	1.1520	1.1248	1.0985
45	1.1827	1.1715	1.1411	1.1131	1.0919
50	1.1704	1.1601	1.1321	1.1061	1.0864
55	1.1600	1.1506	1.1246	1.1003	1.0818
60	1.1513	1.1425	1.1181	1.0953	1.0778
65	1.1437	1.1355	1.1125	1.0909	1.0743
70	1.1371	1.1294	1.1076	1.0871	1.0712
75	1.1313	1.1240	1.1033	1.0837	1.0685
80	1.1261	1.1192	1.0994	1.0806	1.0661
85	1.1215	1.1148	1.0959	1.0779	1.0639
90	1.1173	1.1109	1.0928	1.0754	1.0619
95	1.1134	1.1074	1.0899	1.0731	1.0600
100	1.1099	1.1041	1.0873	1.0710	1.0584
110	1.1038	1.0983	1.0826	1.0673	1.0554
120	1.0985	1.0934	1.0786	1.0641	1.0528
130	1.0939	1.0891	1.0751	1.0613	1.0505
140	1.0899	1.0853	1.0720	1.0589	1.0485
150	1.0863	1.0820	1.0692	1.0567	1.0467
160	1.0831	1.0790	1.0667	1.0547	1.0451
170	1.0802	1.0763	1.0645	1.0529	1.0437
180	1.0776	1.0738	1.0625	1.0513	1.0424
190	1.0752	1.0716	1.0606	1.0498	1.0411
200	1.0731	1.0695	1.0589	1.0484	1.0400

$n(C''_{pm}/\hat{C}''_{pm})^2 = \sum_{i=1}^{n} (X_i - T)^2/\sigma^2$ is distributed as $\chi^2(n)$, the ordinary chi-square distribution with n degrees of freedom. The posterior probability for a capable process that is on target with asymmetric tolerance, $C''_{pm} = d^*/(3\sigma)$, can be expressed as $p^* = P\{C''_{pm} > \omega|\mathbf{x}\} = P\{\chi^2(n) > n/C^{*2}\}$, where $C^* = d^*/(3\omega\hat{\tau})$. Thus, to compute p^*, we only need to check the commonly available chi-square tables for the posterior probability p^*. If p^* is greater than a desirable level, say 90 or 95%, then we may claim that the process is capable (in a Bayesian sense) with 90 or 95% confidence. We note that our computational results have been verified to be identical to those in Shiau *et al.*[19] for symmetric tolerance cases with $USL - T = T - LSL$, and identical to those in Chan *et al.*[18] for on-target cases with $\mu = T$.

4. APPLICATION OF THE PROCEDURE

To demonstrate how we may apply the proposed procedure to the actual data and judge whether the process is capable, we consider the following case taken from a microelectronic manufacturing factory making current transmitters. The process investigated is one that makes a monolithic 4–20 mA, two-wire current transmitter integrated circuit (2WCT IC) designed for bridge input signals. This device provides complete bridge excitation, instrumentation amplifier, linear circuitry, and the current output circuitry necessary for high-impedance strain

Table VI. $C^*(p^*)$ for $p^* = 0.95$ with $d/d_L = 5/6$ and $d/d_U = 5/4$

	$\delta = (\bar{x} - T)/s_{n-1}$				
n	−0.25	−0.5	−1.0	−1.5	−2.0
5	2.8546	2.6647	2.2410	1.9132	1.6928
10	1.7931	1.7199	1.5572	1.4246	1.3320
15	1.5414	1.4967	1.3930	1.3049	1.2418
20	1.4253	1.3931	1.3152	1.2472	1.1977
25	1.3572	1.3320	1.2686	1.2122	1.1706
30	1.3118	1.2909	1.2369	1.1882	1.1519
35	1.2791	1.2612	1.2138	1.1706	1.1381
40	1.2543	1.2385	1.1960	1.1569	1.1273
45	1.2346	1.2204	1.1818	1.1459	1.1187
50	1.2186	1.2057	1.1701	1.1369	1.1115
55	1.2053	1.1934	1.1604	1.1293	1.1054
60	1.1940	1.1829	1.1520	1.1228	1.1003
65	1.1842	1.1739	1.1448	1.1171	1.0957
70	1.1757	1.1660	1.1384	1.1122	1.0918
75	1.1682	1.1590	1.1328	1.1077	1.0882
80	1.1616	1.1528	1.1278	1.1038	1.0851
85	1.1556	1.1473	1.1233	1.1002	1.0822
90	1.1502	1.1422	1.1192	1.0970	1.0796
95	1.1453	1.1377	1.1155	1.0941	1.0773
100	1.1408	1.1335	1.1121	1.0914	1.0751
110	1.1329	1.1261	1.1061	1.0866	1.0712
120	1.1261	1.1197	1.1009	1.0825	1.0679
130	1.1203	1.1142	1.0964	1.0788	1.0650
140	1.1151	1.1094	1.0924	1.0757	1.0624
150	1.1105	1.1050	1.0889	1.0728	1.0601
160	1.1064	1.1012	1.0857	1.0702	1.0580
170	1.1027	1.0977	1.0828	1.0680	1.0561
180	1.0994	1.0946	1.0802	1.0659	1.0544
190	1.0964	1.0917	1.0778	1.0640	1.0529
200	1.0935	1.0891	1.0756	1.0622	1.0514

gage sensors. The instrumentation amplifier can be used over a wide range of gain, accommodating a variety of input signals and sensors. Linear circuitry consists of a second, fully independent instrumentation amplifier that controls the bridge excitation voltage. It provides the second-order correction to the transfer function, typically achieving a 20 : 1 improvement in nonlinearity, even with low-cost transducers. Total unadjusted error of the complete current transmitter, including the linearized bridge is low enough to permit use without adjustment in many applications such as industrial process control, factory automation, SCADA remote data acquisition, weighting systems, and accelerometers.

The total unadjusted error of the 2WCT IC is an essential product characteristic, which has significant impact to product quality. The unadjusted error has an USL, of 14 μA, target value T is set to 6 μA, and the LSL is set to -6 μA. Therefore, the factory engineers have been recommended to use C_{pm} for such applications with asymmetric tolerance for determining whether products meet specifications and taking action to improve the process if necessary. The calculation shows that

$$d = 10, \quad d_U = 8, \quad d_L = 12, \quad d^* = 8,$$
$$n = 100, \quad \bar{x} = 7.5599, \quad s_{n-1} = 1.5599,$$
$$(\bar{x} - T)/s_{n-1} = 1, \quad \hat{C}''_{pm} = 1.07$$

Table VII. $C^*(p^*)$ for $p^* = 0.975$ with $d/d_L = 5/6$ and $d/d_U = 5/4$

	$\delta = (\bar{x} - T)/s_{n-1}$				
n	−0.25	−0.5	−1.0	−1.5	−2.0
5	3.4684	3.2253	2.6707	2.2356	1.9399
10	1.9917	1.9009	1.6981	1.5323	1.4161
15	1.6655	1.6108	1.4835	1.3752	1.2974
20	1.5187	1.4797	1.3848	1.3018	1.2412
25	1.4337	1.4032	1.3263	1.2579	1.2072
30	1.3775	1.3523	1.2871	1.2280	1.1840
35	1.3372	1.3157	1.2585	1.2062	1.1669
40	1.3067	1.2878	1.2366	1.1894	1.1536
45	1.2826	1.2657	1.2192	1.1759	1.1430
50	1.2631	1.2477	1.2049	1.1649	1.1342
55	1.2469	1.2327	1.1930	1.1556	1.1269
60	1.2331	1.2200	1.1828	1.1477	1.1205
65	1.2213	1.2090	1.1740	1.1408	1.1150
70	1.2110	1.1994	1.1664	1.1347	1.1102
75	1.2019	1.1910	1.1595	1.1294	1.1059
80	1.1938	1.1834	1.1535	1.1246	1.1021
85	1.1866	1.1767	1.1480	1.1203	1.0986
90	1.1801	1.1706	1.1431	1.1164	1.0955
95	1.1742	1.1651	1.1386	1.1128	1.0926
100	1.1687	1.1600	1.1345	1.1096	1.0900
110	1.1592	1.1511	1.1272	1.1038	1.0853
120	1.1511	1.1434	1.1209	1.0988	1.0813
130	1.1440	1.1368	1.1155	1.0944	1.0778
140	1.1378	1.1309	1.1108	1.0906	1.0747
150	1.1323	1.1258	1.1064	1.0872	1.0719
160	1.1273	1.1211	1.1026	1.0841	1.0694
170	1.1229	1.1169	1.0991	1.0813	1.0672
180	1.1189	1.1132	1.0960	1.0788	1.0651
190	1.1152	1.1097	1.0931	1.0765	1.0632
200	1.1118	1.1065	1.0905	1.0744	1.0615

Consider $\omega = 1$ and $p^* = 0.95$. From Table II or running the Mathcad program as displayed in Appendix B, we find that $C^*(p^*) = 1.1220$, which implies that the minimum value of $\hat{C}''_{pm}$ is equal to $\omega C^*(p^*) = 1.1220$. Since $1.07 < 1.1220$, we claim that this process is incapable in a Bayesian sense with 95% confidence.

5. *CONCLUSION*

Existing developments and applications of the Taguchi capability index C_{pm} have focused on processes with symmetric tolerances. The generalization C''_{pm}, which incorporates the departure of the process mean from the target value and the magnitude of the process variation, was proposed to the manufacturing industry for handling processes with asymmetric tolerances. However, the sampling distribution of the natural estimator $\hat{C}''_{pm}$ considered by Chen *et al.*[20] is rather complicated to deal with computationally for obtaining an interval estimation of C''_{pm}. Under the assumption of a non-informative prior we obtained a simple Bayesian procedure for process capability assessment, which allows one to proceed with the Bayesian credible interval estimation for C''_{pm}, which is similar to the classical confidence interval using the distribution frequency approach. This Bayesian procedure can provide an efficient alternative to the classical distribution frequency approach in assessing process capability for asymmetric tolerances.

Table VIII. $C^*(p^*)$ for $p^* = 0.99$ with $d/d_L = 5/6$ and $d/d_U = 5/4$

	$\delta = (\bar{x} - T)/s_{n-1}$				
n	−0.25	−0.5	−1.0	−1.5	−2.0
5	4.4445	4.1214	3.3653	2.7617	2.3462
10	2.2682	2.1531	1.8938	1.6815	1.5323
15	1.8308	1.7624	1.6033	1.4678	1.3703
20	1.6402	1.5919	1.4745	1.3718	1.2967
25	1.5317	1.4942	1.3997	1.3155	1.2532
30	1.4607	1.4300	1.3501	1.2778	1.2238
35	1.4102	1.3840	1.3143	1.2504	1.2024
40	1.3722	1.3492	1.2870	1.2295	1.1859
45	1.3424	1.3218	1.2654	1.2128	1.1727
50	1.3182	1.2996	1.2477	1.1991	1.1619
55	1.2982	1.2811	1.2330	1.1877	1.1528
60	1.2813	1.2654	1.2205	1.1779	1.1451
65	1.2668	1.2520	1.2097	1.1695	1.1384
70	1.2542	1.2402	1.2003	1.1621	1.1325
75	1.2431	1.2299	1.1920	1.1556	1.1272
80	1.2332	1.2207	1.1846	1.1497	1.1226
85	1.2244	1.2125	1.1779	1.1445	1.1184
90	1.2164	1.2051	1.1719	1.1397	1.1145
95	1.2092	1.1983	1.1664	1.1354	1.1111
100	1.2027	1.1922	1.1614	1.1314	1.1079
110	1.1911	1.1813	1.1526	1.1244	1.1022
120	1.1812	1.1720	1.1450	1.1183	1.0973
130	1.1726	1.1640	1.1384	1.1131	1.0931
140	1.1651	1.1569	1.1326	1.1084	1.0893
150	1.1584	1.1506	1.1274	1.1043	1.0860
160	1.1524	1.1450	1.1228	1.1006	1.0830
170	1.1471	1.1400	1.1186	1.0973	1.0803
180	1.1422	1.1354	1.1148	1.0942	1.0778
190	1.1378	1.1312	1.1114	1.0914	1.0755
200	1.1338	1.1274	1.1082	1.0889	1.0734

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APPENDIX A. DERIVATION OF EXPRESSION (14)

Given $\omega > 0$, the posterior probability

$$\begin{aligned}
p^* = P\{C''_{pm} > \omega|\mathbf{x}\} &= P\left\{\frac{d^*}{3\sqrt{\sigma^2 + A^2}} > \varpi \,\middle|\, \mathbf{x}\right\}\\
&= P\left\{A^2 + \sigma^2 < \left(\frac{d^*}{3\varpi}\right)^2 \,\middle|\, \mathbf{x}\right\} = \int_0^{a^*}\int_{T-g_L(\sigma)}^{T+g_U(\sigma)} f(\mu, \sigma|\mathbf{x})\,d\mu\,d\sigma\\
&= \int_0^{a^*}\int_{T-g_L(\sigma)}^{T+g_U(\sigma)} \sqrt{\frac{2n}{\pi}}\left\{\frac{\exp[-1/(\beta\sigma^2)]}{\sigma^{n+1}\beta^\alpha\Gamma(\alpha)}\right\}\exp\left[-\frac{n}{2}\left(\frac{\bar{x}-\mu}{\sigma}\right)^2\right]d\mu\,d\sigma\\
&= \int_0^{a^*}\left\{\frac{2\exp[-1/(\beta\sigma^2)]}{\sigma^n\beta^\alpha\Gamma(\alpha)}\right\}\left\{\frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_{T-g_L(\sigma)}^{T+g_U(\sigma)}\exp\left[-\frac{n}{2}\left(\frac{\mu-\bar{x}}{\sigma}\right)^2\right]d\mu\right\}d\sigma
\end{aligned}$$

Let $z = \sqrt{n}(\mu - \bar{x})/\sigma$, then $dz = (\sqrt{n}/\sigma)\,d\mu$

$$\begin{aligned}
&\int_0^{a^*}\left\{\frac{2\exp[-1/(\beta\sigma^2)]}{\sigma^n\beta^\alpha\Gamma(\alpha)}\right\}\left\{\frac{\sqrt{n}}{\sqrt{2\pi}\sigma}\int_{T-g_L(\sigma)}^{T+g_U(\sigma)}\exp\left[-\frac{n}{2}\left(\frac{\mu-\bar{x}}{\sigma}\right)^2\right]d\mu\right\}d\sigma\\
&\quad= \int_0^{a^*}\left\{\frac{2\exp[-1/(\beta\sigma^2)]}{\sigma^n\beta^\alpha\Gamma(\alpha)}\right\}\left\{\frac{1}{\sqrt{2\pi}}\int_{\sqrt{n}[T-g_L(\sigma)-\bar{x}]/\sigma}^{\sqrt{n}[T+g_U(\sigma)-\bar{x}]/\sigma}\exp\left[-\frac{z^2}{2}\right]dz\right\}d\sigma
\end{aligned}$$

$$= \int_0^{a^*} \left\{ \frac{2\exp[-1/(\beta\sigma^2)]}{\sigma^n \beta^\alpha \Gamma(\alpha)} \right\} \left\{ \Phi\left(\frac{T - \bar{x} + g_U(\sigma)}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{T - \bar{x} - g_L(\sigma)}{\sigma/\sqrt{n}}\right) \right\} d\sigma$$

If $\bar{x} < T$, then

$$P^* = \int_0^{a^*} \left\{ \frac{2\exp[-1/(\beta\sigma^2)]}{\sigma^n \beta^\alpha \Gamma(\alpha)} \right\} \left\{ \Phi\left(\frac{T - \bar{x} + g_U(\sigma)}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{T - \bar{x} - g_L(\sigma)}{\sigma/\sqrt{n}}\right) \right\} d\sigma$$

If $\bar{x} > T$, then

$$\begin{aligned} P^* &= \int_0^{a^*} \left\{ \frac{2\exp[-1/(\beta\sigma^2)]}{\sigma^n \beta^\alpha \Gamma(\alpha)} \right\} \left\{ \Phi\left(\frac{T - \bar{x} + g_U(\sigma)}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{T - \bar{x} - g_L(\sigma)}{\sigma/\sqrt{n}}\right) \right\} d\sigma \\ &= \int_0^{a^*} \left\{ \frac{2\exp[-1/(\beta\sigma^2)]}{\sigma^n \beta^\alpha \Gamma(\alpha)} \right\} \left\{ \Phi\left(-\frac{\bar{x} - T - g_U(\sigma)}{\sigma/\sqrt{n}}\right) - \Phi\left(-\frac{\bar{x} - T + g_L(\sigma)}{\sigma/\sqrt{n}}\right) \right\} d\sigma \\ &= \int_0^{a^*} \left\{ \frac{2\exp[-1/(\beta\sigma^2)]}{\sigma^n \beta^\alpha \Gamma(\alpha)} \right\} \left\{ \left[1 - \Phi\left(\frac{\bar{x} - T - g_U(\sigma)}{\sigma/\sqrt{n}}\right)\right] - \left[1 - \Phi\left(\frac{\bar{x} - T + g_L(\sigma)}{\sigma/\sqrt{n}}\right)\right] \right\} d\sigma \\ &= \int_0^{a^*} \left\{ \frac{2\exp[-1/(\beta\sigma^2)]}{\sigma^n \beta^\alpha \Gamma(\alpha)} \right\} \left\{ \Phi\left(\frac{\bar{x} - T + g_L(\sigma)}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{\bar{x} - T - g_U(\sigma)}{\sigma/\sqrt{n}}\right) \right\} d\sigma \end{aligned}$$

where $a^* = d^*/(3\omega)$, $g_L(\sigma) = (d_L/d)(a^{*2} - \sigma^2)^{1/2}$, $g_U(\sigma) = (d_U/d)(a^{*2} - \sigma^2)^{1/2}$, $\alpha = (n-1)/2$, $\beta = 2[(n-1)s_{n-1}^2]^{-1}$, Φ is the cumulative distribution function of the standard normal distribution.

Let $y = \beta'\sigma^2$, $\beta' = 2/\sum_{i=1}^n (x_i - T)^2$, $b_1(y) = \sqrt{2/y}\sqrt{1 - (1/\gamma)}$, $\gamma = 1 + [n/(n-1)]\delta^2$, $\delta = (\bar{x} - T)/s_{n-1}$, $b_L^*(y) = (d_L/d)\sqrt{n[(t^*/y) - 1]}$, $b_U^*(y) = (d_U/d)\sqrt{n[(t^*/y) - 1]}$, $t^* = 2[d^*/(3\omega)]^2[ns_n^2 + n(\bar{x} - T)^2]^{-1}$.

The posterior probability based on index C''_{pm} that a process with asymmetric or symmetric tolerances is capable is given as

$$p^* = \int_0^{t^*} \left\{ \frac{\exp[-1/(\gamma y)]}{y^{\alpha+1}\gamma^\alpha \Gamma(\alpha)} \right\} \{\Phi[b_1(y) + b_U^*(y)] - \Phi[b_1(y) - b_L^*(y)]\} dy \qquad \text{(A1)}$$

for $\bar{x} < T$, and

$$p^* = \int_0^{t^*} \left\{ \frac{\exp[-1/(\gamma y)]}{y^{\alpha+1}\gamma^\alpha \Gamma(\alpha)} \right\} \{\Phi[b_1(y) + b_L^*(y)] - \Phi[b_1(y) - b_U^*(y)]\} dy \qquad \text{(A2)}$$

for $\bar{x} > T$.

APPENDIX B. MATHCAD PROGRAM FOR THE $C^*(p^*)$ VALUES

Input the values of parameters:

$$p := 0.95, \quad n := 100, \quad w := 1, \quad d_{\mathrm{U}} := 8, \quad d_{\mathrm{L}} := 12, \quad d := 10, \quad \delta := 1$$

$$\alpha = \frac{n-1}{2}, \quad \gamma = 1 + \frac{n}{n-1}\delta^2,$$

$$t(C_{\mathrm{p}m}) = \left[\frac{(n-1) + (n\delta^2)(d/d_{\mathrm{U}})^2}{(n-1) + (n\delta^2)}\right]\frac{(2C_{\mathrm{p}m}^2)}{nw^2}, \quad b(y) = \left(\sqrt{\frac{2}{y}}\right)\left(\frac{\delta^2}{\delta^2 + (n-1)/n}\right)^{1/2}$$

$$b_{\mathrm{L}}(y, C_{\mathrm{p}m}) = \left(\frac{d_{\mathrm{L}}}{d}\right)\left[\sqrt{n}\left(\frac{t(C_{\mathrm{p}m})}{y} - 1\right)^{1/2}\right], \quad b_{\mathrm{U}}(y, C_{\mathrm{p}m}) = \left(\frac{d_{\mathrm{U}}}{d}\right)\left[\sqrt{n}\left(\frac{t(C_{\mathrm{p}m})}{y} - 1\right)^{1/2}\right],$$

$$\mathrm{LNGA}(n) = \begin{cases} \displaystyle\sum_{i-1}^{n-1} \ln(i) & \text{if } \bmod(2n, 2) = 0 \\ \displaystyle\left(\left(\ln(\Gamma(n - \mathrm{floor}(n))) + \sum_{i=n-\mathrm{floor}(n)}^{n-1} \ln i\right)\right) & \text{otherwise} \end{cases}$$

Set the initial value for solving:

$$C_{\mathrm{p}m} = 1.07$$

Given

$$p = \int_0^{C_{\mathrm{p}m}} \exp\left[\frac{-1}{\gamma y}(\alpha+1)\ln(y) - \mathrm{LNGA}(\alpha) - \alpha\ln(y)\right](\mathrm{pnorm}(b_1(y) + b_{\mathrm{L}}(y, c_{\mathrm{p}m}, 0, 1)$$
$$- \mathrm{pnorm}(b_1(y) - b_{\mathrm{U}}(yC_{\mathrm{p}m}, 0, 1))\,\mathrm{d}y$$

The calculated value $C^*(p^*)$ based on above setting:

$$\mathrm{Find}(C_{\mathrm{p}m}) = 1.121\,953\,93$$

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