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A Bayesian approach for assessing process precision based on multiple samples

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Abstract

Using process capability indices to quantify manufacturing process precision (consistency) and performance, is an essential part of implementing any quality improvement program. Most research works for testing the capability indices have focused on using the traditional distribution frequency approaches. Cheng and Spiring [IIE Trans. 21 (1) 97] proposed a Bayesian procedure for assessing process capability index C_p based on one single sample. In practice, manufacturing information regarding product quality characteristic is often derived from multiple samples, particularly, when a routine-based quality control plan is implemented for monitoring process stability. In this paper, we consider estimating and testing C_p with multiple samples using Bayesian approach, and propose accordingly a Bayesian procedure for capability testing. The posterior probability, p, for which the process under investigation is capable, is derived. The credible interval, a Bayesian analogue of the classical lower confidence interval, is obtained. The results obtained in this paper, are generalizations of those obtained in Cheng and Spiring [IIE Trans. 21 (1), 97]. Practitioners can use the proposed procedure to Cheng and Spiring determine whether their manufacturing processes are capable of reproducing products satisfying the preset precision requirement.

Keywords: Process capability indices; Bayesian approach; Credible interval; Posterior probability; Decision making; Quality control

1. Introduction

Understanding process and quantifying process performance are essential for any successful quality improvement initiative. Process capability analysis has become an important and integrated part in applying statistical process control to continuously improve process quality and productivity. The relationship between the actual process performance and the specification limits may be quantified using appropriate process capability indices. Process capability indices (PCIs), the purpose of which is to provide

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numerical measures of whether or not the ability of a manufacturing process meets a predetermined level of production tolerance, have received considerable research attention and increased usage in process assessments and purchasing decisions in the automotive industry during last decade. The first process capability index appearing in the literature was the precision index C_p, and defined as (Kane, 1986):

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma},$$

where USL is the upper specification limit, LSL is the lower specification limit, and σ is the process standard deviation. The numerator of C_p gives the range over which the process measurements are acceptable. The denominator gives the width of the range over which the process is actually varying. The index C_p was designed to measure the magnitude of the overall process variation relative to the manufacturing tolerance, which is to be used for processes with data that are normally distributed, independent, and in statistical control. Clearly, the index C_p measures the precision/consistency of a process, or the potential to reproduce acceptable products, which provides actual performance measure for centered processes. The index C_p is particularly useful when reducing process variation is the guiding principle for process improvement.

The use of the capability indices was first explored within the automotive industry. Ford Motor Company (1984) has used C_p to track process performance and to reduce process variation. Recently, manufacturing industries have been making an extensive effort to implement statistical process control (SPC) in their plants and supply bases. Process capability measures derived from analyzing SPC data have received increasing usage not only in process performance assessments, but also in the evaluation of purchasing decisions. Capability indices have become the standard tools for quality reporting, particularly, at the management level around the world. Proper understanding and accurate estimation of the capability indices is essential for the company to maintain a capable supplier. The usual practice of judging process capability by evaluating the point estimates of process capability indices, have a flaw since there is no assessment of the error of these estimates. Therefore, a simple point estimate of the index is highly unreliable in making decision in assessing process capability since the estimate does not represent the true index value. When the estimate is greater than a pre-specified value w, say 1.00, or 1.33, it does not guarantee that the index is greater than w and vice versa. It is therefore preferable to obtain an interval estimate for which we can assert, with a reasonable degree of certainty, that it contains the true index value. Existing methods for testing the capability indices have focused on traditional distribution frequency approaches. Examples include Chou and Owen (1989), Chou et al. (1990), Li et al. (1990), Kirmani et al. (1991), Kocherlakota (1992), Pearn et al. (1998), Kotz and Lovelace (1998) and Pearn and Yang (2003). Kotz and Johnson (2002) presented a thorough review for the development of process capability indices in the past 10 years and Spiring et al. (2003) consolidated the research findings of process capability analysis for the period 1990–2002.

Bayesian statistical techniques are an alternative to the frequency approach. These techniques specify a prior distribution for the parameter of interest, in order to obtain the posterior distribution of the parameter. We then could infer about the parameter by only using its posterior distribution given the sample data. It is not difficult to obtain the posterior distribution when a prior distribution is given. Even in the case where the form of the posterior distribution is complicated one can always use numerical methods or Monte Carlo methods (Kalos and Whitlock, 1986) to obtain an approximate point estimate or interval estimate. This is the advantage of the Bayesian approach over the traditional distribution frequency approach.

Cheng and Spiring (1989) proposed a Bayesian procedure for assessing process capability index C_p based on one single sample. In practice, manufacturing information regarding product quality characteristic is often derived from multiple samples rather than one single sample, particularly, when a routine-based quality control plan is implemented for monitoring process stability. In this paper, we consider estimating and testing C_p with multiple samples using Bayesian approach, and propose accordingly Bayesian procedure for capability testing. The posterior probability, p, for which the process under investigation is capable, is derived. The credible interval, a Bayesian analogue of the classical lower confidence interval, is obtained. The results obtained in this paper, generalize those obtained in Cheng and Spiring (1989). Practitioners can use the proposed procedure to determine whether manufacturing processes are capable of reproducing products satisfying the preset precision requirement.

2. Estimating C_p based on multiple samples

If one single sample is given as $\{x_1, x_2, \ldots, x_n\}$, we may consider the natural estimator $\hat{C}_p$ of C_p defined as

$$\hat{C}_p = \frac{\text{USL} - \text{LSL}}{6s},$$

where $s = [\sum_{i=1}^{n}(x_i - \bar{x})^2/(n-1)]^{1/2}$ is the estimator of the process standard deviation σ, which can be obtained from a stable process. Under the assumption of normality, Chou and Owen (1989) obtained the probability density function (PDF) of the natural estimator $\hat{C}_p$, which can be expressed as the following, for $y > 0$:

$$f(y) = 2\frac{\left(\sqrt{(n-1)/2}C_p\right)^{n-1}}{\Gamma[(n-1)/2]}y^{-n}\exp[-(n-1)(C_p)^2(2y^2)^{-1}].$$

Pearn et al. (1998) obtained an unbiased estimator $\widetilde{C}_p = b_{n-1}\widehat{C}_p$ where the correction factor $b_{n-1} = [2/(n-1)]^{1/2}\{\Gamma[(n-1)/2]/\Gamma[(n-2)/2]\}$ and $\Gamma(k) = \int_0^\infty t^{k-1}e^{-t}dt$ is the gamma function. Pearn et al. (1998) also showed that the estimator $\widetilde{C}_p$ is the uniformly minimum variance unbiased estimator (UMVUE) of C_p, which is asymptotically efficient, consistent, and that $n^{1/2}(\widetilde{C}_p - C_p)$ converges to $N(0, C_p^2/2)$ in distribution.

For cases where data are collected as multiple samples, Kirmani et al. (1991) considered m samples each of size n and suggested the following estimator of C_p, where $\bar{x}_i$ is the ith sample mean, and s_i is the ith sample standard deviation:

$$\widehat{C}_p^* = \frac{(\text{USL} - \text{LSL})d_p}{6},$$

where

$$d_p = \sqrt{\frac{m(n-1)-1}{m(n-1)}}\frac{\varepsilon_{m(n-1)-1}}{s_p},$$

$$\varepsilon_{m(n-1)-1} = \sqrt{\frac{2}{m(n-1)-1}}\Gamma\left(\frac{m(n-1)}{2}\right)\left[\Gamma\left(\frac{m(n-1)-1}{2}\right)\right]^{-1},$$

$$s_p^2 = \frac{1}{m(n-1)}\sum_{i=1}^{m}(n-1)s_i^2 = \frac{1}{m}\sum_{i=1}^{m}s_i^2,$$

noting that under normality assumption s_p/σ is distributed as $\chi_{m(n-1)-1}/[m(n-1)-1]^{1/2}$. Therefore, the estimator $\widehat{C}_p^*$ is distributed as

$$\widehat{C}_p^* \sim \frac{\sqrt{m(n-1)-1}\,\varepsilon_{m(n-1)-1}}{\sqrt{\chi^2_{m(n-1)}}}C_p.$$

The estimator $\widehat{C}_p^*$ is unbiased, and its probability density function can be obtained as the following, for $y > 0$, where $k = [m(n-1)-1]\varepsilon^2_{m(n-1)-1}C_p^2$, which can be expressed as a function of C_p:

$$g(y) = \frac{2k^{m(n-1)/2}}{2^{m(n-1)/2}\Gamma[m(n-1)/2]}y^{-[m(n-1)+1]}\exp\left[-\frac{k}{2}\left(\frac{1}{y^2}\right)\right].$$

Furthermore, Pearn and Yang (2003) investigated some statistical properties of $\widehat{C}_p^*$ and showed that $\widehat{C}_p^*$ is the UMVUE of C_p, which is also asymptotically efficient and $(mn)^{1/2}(\widehat{C}_p^* - C_p)$ converges to $N(0, C_p^2/2)$ in distribution. It is easy to verify that $\widehat{C}_p^*$ is consistent. The variance of $\widehat{C}_p^*$ can be calculated as the following (Kirmani et al., 1991):

$$\begin{aligned}\mathrm{Var}(\widehat{C}_p^*) &= E[(\widehat{C}_p^*)^2] - [E(\widehat{C}_p^*)]^2 = (\mathrm{USL}-\mathrm{LSL})^2\varepsilon^2_{m(n-1)-1}\frac{[m(n-1)-1]}{36m(n-1)}E(s_p^2)^{-1} - C_p^2 \\ &= C_p^2\left\{(\varepsilon^2_{m(n-1)-2})^{-1} - 1\right\}.\end{aligned}$$

For multiple samples with variable sample size, we can consider the generalized estimator of C_p defined below. We show that the generalized estimator $\widehat{C}_p^*$ obtained from m samples each of size n_i, remains unbiased. In fact, it can be shown that the unbiased estimator $\widehat{C}_p^*$ is indeed the UMVUE of C_p in the case of multiple samples:

$$\widehat{C}_p^* = b_{\sum_{i=1}^m (n_i-1)} \times \frac{\mathrm{USL}-\mathrm{LSL}}{6s_p}, \quad s_p^2 = \frac{\sum_{i=1}^m (n_i-1)s_i^2}{\sum_{i=1}^m (n_i-1)}, \quad \text{and}$$

$$b_{\sum_{i=1}^m (n_i-1)} = \sqrt{\frac{2}{\sum_{i=1}^m (n_i-1)}}\,\Gamma\left(\frac{\sum_{i=1}^m (n_i-1)}{2}\right)\left[\Gamma\left(\frac{\sum_{i=1}^m (n_i-1)-1}{2}\right)\right]^{-1}.$$

3. Bayesian approach for testing C_p

Cheng and Spiring (1989) proposed a Bayesian procedure for assessing process capability index C_p. Shiau et al. (1999) applied a similar Bayesian approach to index C_{pm}. Shiau et al. (1999) also applied Bayesian method for testing the index C_{pk} but under the restriction that the process mean μ equals to the midpoint of the two specification limits, M. In this case C_{pk} reduces to C_p. However, these research works only focused on cases with one single sample. A common practice of process capability estimation in the manufacturing industry is to first implement a daily-based or weekly-based sample data collection plan for monitoring/controlling the process stability, then to analyze the past "in control" data. It is more practical to develop a procedure for assessing process capability for cases with multiple samples. Therefore, in the following we consider the problem of estimating and testing C_p with multiple samples based on Bayesian approach, and propose accordingly a Bayesian procedure for testing process precision. The posterior probability, p, for which the process under investigation is capable, is derived. A $100p\%$ credible interval is the Bayesian analogue of the classical $100p\%$ confidence interval, where p is the confidence level for the interval. The credible interval covers $100p\%$ of the posterior distribution of the parameter (Berger, 1980). Assuming that the m samples are randomly taken from independent and identically distributed (i.i.d.) $N(\mu, \sigma^2)$, a normal distribution with mean μ and variance σ^2. Denote the measures of the ith sample as

$\mathbf{x}_i = \{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$ with variable sample size n_i, and $\mathbf{X} = \{x_{ij},\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots,\ n_i\}$. Then, the likelihood function for μ and σ can be expressed:

$$L(\mu, \sigma \,|\, \mathbf{X}) = (2\pi\sigma^2)^{\frac{-\sum_{i=1}^{m} n_i}{2}} \times \exp\left\{ -\frac{\sum_{i=1}^{m}\sum_{j=1}^{ni}(x_{ij} - \mu)^2}{2\sigma^2} \right\}.$$

The first step for the Bayesian approach is to find an appropriate prior. Usually, when there is little or no prior information, or there is only one parameter, one of the most widely used non-informative priors is the so-called reference prior, which is a non-informative prior that maximizes the difference between information (entropy) on the parameter provided by the prior and by the posterior. In other words, the reference prior allows the prior to provide information as little as possible about the parameter (see Bernardo and Smith, 1993 for more details). Several priors have been considered in the literature. In practical situation, however, the choice of prior information is hard to justify. Therefore, in this paper we adopt the following non-informative reference prior chosen by Cheng and Spiring (1989):

$$h(\mu, \sigma) = 1/\sigma, \quad -\infty < \mu < \infty, \quad 0 < \sigma < \infty.$$

We note that the parameter space of the prior is infinite, hence the reference prior is improper, which means that it does not integrate to one. However, it is not always a serious problem, since the prior incorporated with ordinary likelihood will lead to proper posterior. Furthermore, the credible interval obtained from a non-informative prior has a more precise coverage probability than that obtained from any other priors. The posterior probability density function (PDF), $f(\mu, \sigma \,|\, \mathbf{X})$ of (μ, σ) may be expressed as the following:

$$f(\mu, \sigma \,|\, \mathbf{X}) \propto L(\mu, \sigma \,|\, \mathbf{X}) \times h(\mu, \sigma) \propto \sigma^{-\left(\sum_{i=1}^{m} n_i + 1\right)} \times \exp\left\{ -\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij} - \mu)^2}{2\sigma^2} \right\}.$$

Also

$$\int_0^{\infty} \int_{-\infty}^{\infty} \sigma^{-\left(\sum_{i=1}^{m} n_i + 1\right)} \times \exp\left\{ -\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij} - \mu)^2}{2\sigma^2} \right\} \mathrm{d}\mu\, \mathrm{d}\sigma$$

$$= \int_0^{\infty} \sigma^{-\left(\sum_{i=1}^{m} n_i + 1\right)} \exp\left(-\frac{1}{\beta\sigma^2} \right) \times \left[\int_{-\infty}^{\infty} \exp\left(-\frac{\sum_{i=1}^{m} n_i(\mu - \bar{\bar{x}})^2}{2\sigma^2} \right) \mathrm{d}\mu \right] \mathrm{d}\sigma = \sqrt{\frac{\pi}{2\sum_{i=1}^{m} n_i}}\,\Gamma(\alpha)\beta^{\alpha}.$$

In order to satisfy the integration property that the probability over PDF is 1, a coefficient of $f(\mu, \sigma \,|\, \mathbf{X})$ can be obtained through some algebraic manipulations. Consequently, the posterior PDF of (μ, σ) can be expressed as

$$f(\mu, \sigma \,|\, \mathbf{X}) = \frac{2\sqrt{\sum_{i=1}^{m} n_i}}{\sqrt{2\pi}\,\Gamma(\alpha)\beta^{\alpha}}\, \sigma^{-\left(\sum_{i=1}^{m} n_i + 1\right)} \times \exp\left(-\frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij} - \mu)^2}{2\sigma^2} \right),$$

where $\alpha = (\sum_{i=1}^{m} n_i - 1)/2$, $\beta = [\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{\bar{x}})^2/2]^{-1}$, $\bar{\bar{x}} = \sum_{i=1}^{m}\sum_{j=1}^{n_i} x_{ij}/\sum_{j=1}^{n_i} x_{ij}$. As mentioned earlier, it is natural to consider the quantity $\Pr\{\text{process is capable}\,|\,\mathbf{X}\}$ in the Bayesian approach. We need the posterior probability $p = \Pr\{C_{\mathrm{p}} > w\,|\,\mathbf{X}\}$ for some fixed positive number w. Therefore, given a pre-specified precision level $w > 0$ and denote $a = (\mathrm{USL}-\mathrm{LSL})/(6w)$, the posterior probability for index C_{p} based on multiple samples that a process is capable is given as

$$p = \Pr\{C_{\mathrm{p}} > w\,|\,\mathbf{X}\} = \Pr\left\{\frac{\mathrm{USL}-\mathrm{LSL}}{6\sigma} > w\middle|\mathbf{X}\right\} = \Pr\left\{\sigma < \frac{\mathrm{USL}-\mathrm{LSL}}{6w}\,|\,\mathbf{X}\right\} = \Pr\{\sigma < a\,|\,\mathbf{X}\}$$

$$= \int_0^a f(\sigma\,|\,\mathbf{X})\,\mathrm{d}\sigma = \int_0^a \frac{2\sigma^{-\sum_{i=1}^{m} n_i}}{\Gamma(\alpha)\beta^{\alpha}} \times \exp\left(-\frac{1}{\beta\sigma^2}\right)\mathrm{d}\sigma.$$

By changing the variable, let $y = (\beta\sigma^2)^{-1}$, then $\mathrm{d}y = -2(\beta\sigma^3)^{-1}\,\mathrm{d}\sigma$, the above posterior probability p expression may be rewritten as

$$p = \int_{1/t}^{\infty} \frac{\sigma^{-\left(\sum_{i=1}^{m} n_i - 3\right)}}{\Gamma(\alpha)\beta^{\alpha-1}} \times \exp(-y)\,\mathrm{d}y = \int_{1/t}^{\infty} \frac{y^{\alpha-1}}{\Gamma(\alpha)} \times \exp(-y)\,\mathrm{d}y = \frac{\Gamma(\alpha, 1/t)}{\Gamma(\alpha)} \tag{1}$$

or, equivalently,

$$p = 1 - G(1/t, \alpha, 1), \tag{2}$$

where $\Gamma(\alpha, 1/t)$ is the value of the incomplete gamma function of $1/t$ with parameter α, $G(1/t, \alpha, 1)$ is the cumulative probability at $1/t$ for the gamma distribution with parameters α and 1, and

$$t = \frac{2\gamma}{\sum_{i=1}^{m}(n_i - 1)}\left(\frac{\widehat{C}_{\mathrm{p}}^*}{w b_{\sum_{i=1}^{m}(n_i-1)}}\right)^2,$$

$$\gamma = \frac{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{x}_i)^2}{\sum_{i=1}^{m}\sum_{j=1}^{n_i}(x_{ij}-\bar{\bar{x}})^2} = \frac{\sum_{i=1}^{m}(n_i-1)s_{\mathrm{p}}^2}{\sum_{i=1}^{m}(n_i-1)s_{\mathrm{p}}^2 + \sum_{i=1}^{m} n_i(\bar{x}_i - \bar{\bar{x}})^2},$$

$$b_{\sum_{i=1}^{m}(n_i-1)} = \sqrt{\frac{2}{\sum_{i=1}^{m}(n_i-1)}}\,\Gamma\left(\frac{\sum_{i=1}^{m}(n_i-1)}{2}\right)\left[\Gamma\left(\frac{\sum_{i=1}^{m}(n_i-1)-1}{2}\right)\right]^{-1}.$$

Note that the posterior probability p depends on m, n_i, γ, w and $\widehat{C}_{\mathrm{p}}^*$ only through m, n_i, γ and $\widehat{C}_{\mathrm{p}}^*/w$. Denoted $C^* = \widehat{C}_{\mathrm{p}}^*/w$. There is a one-to-one correspondence between p and C^* when m and n_i, are given, and by the fact that γ and $\widehat{C}_{\mathrm{p}}^*$ can be calculated from the process data, we find that the minimum value of C^* required to ensure the posterior probability p reaching a certain desirable level, can be useful in assessing process capability. Denote this minimum value as $C^*(p)$. Then, the value $C^*(p)$ satisfies

$$p = \Pr\{C_p > w|\mathbf{X}\} = \Pr\left\{C_p > \frac{\widehat{C}_p^*}{C^*(p)}|\mathbf{X}\right\}.$$

Therefore, a $100p\%$ credible interval for C_p is $[\widehat{C}_p^*/C^*(p), \infty)$, where p is a number between 0 and 1, say 0.95, for 95% confidence interval, which means that the posterior probability that the credible interval contains C_p is p. We say that the process is capable in a Bayesian sense if all the points in this credible interval are greater than a pre-specified value of w, say 1.00 or 1.33. When this happens, we have $p = \Pr\{C_p > w|\mathbf{X}\}$. In other words, to see if a process is capable (with capability level w and confidence level p), we only need to check if $\widehat{C}_p^* > C^*(p) \times w$. For the single sample, that is, $m = 1$, $\gamma = 1$, and $s_p = s$, the results obtained in this paper can be reduced to those obtained in Cheng and Spiring (1989).

4. Decision making for testing C_p

In current practice, process precision is said to be inadequate if $C_p < 1.00$; it indicates that the process is not adequate with respect to the production tolerances. Process precision is said to be marginally capable if $1.00 \leqslant C_p < 1.33$; it indicates that caution needs to be taken regarding the process consistency and some process control is required (usually using R or S control charts). Process precision is said to be satisfactory if $1.33 \leqslant C_p < 1.67$; it indicates that process consistency is satisfactory, material substitution may be allowed, and no stringent precision control is required. Process precision is said to be excellent if $1.67 \leqslant C_p < 2.00$; it indicates that process precision exceeds satisfactory. Finally, process precision is said to be superior if $C_p \geqslant 2.00$.

In recent years, many companies have adopted criteria for evaluating their processes based on process capability objectives that are more stringent than those recommended minimums above. For instance, the ''six-sigma'' program pioneered by Motorola essentially requires that when the process mean is in control, it will not be closer than six standard deviations from the nearest specification limit. Thus, in effect, requires that the process capability ratio will be at least 2.0 (Harry, 1988).

Therefore, it would be desirable to determine a bound which practitioners would be expected to find the true value of the process capability no less than the bound value with certain level of confidence. For users' convenience in applying our Bayesian procedure, we tabulate the minimum values $C^*(p)$ of $\widehat{C}_p^*/w$, for various γ with $m = 2(2)10, 15$, $n_i = n = 10(5)30$ in Tables 1–3 to ensure $p = 0.99$, 0.975, and 0.95, respectively. For example, if $w = 1.33$ is the minimum capability requirement, then for $p = 0.95$, with $m = 10$ of each sample size $n_i = n = 10$ and $\gamma = 0.90$, we can find $C^*(p) = 1.1297$ by checking Table 3. Thus, the minimum value of $\widehat{C}_p^*$ required for a capable process is $C^*(p) \times w = 1.1297 \times 1.33 = 1.5026$. That is, if $\widehat{C}_p^*$ is greater than 1.5026, we say that the process is capable in Bayesian sense.

As a result, to judge if a given process meets the capability requirement, we first determine the pre-specified capability level w, and the confidence level p or the α-risk for the interval. Check the appropriate table or solve Eq. (1) or (2), we may obtain the minimum value of $C^*(p)$ based on given values of p, m sub-samples of size n_i and γ calculated from samples. If the estimated value $\widehat{C}_p^*$ is greater than the critical value $C^*(p) \times w$, then we may conclude that the process meets the capability requirement ($C_p > w$). Otherwise, we do not have sufficient information to conclude that the process meets the present capability requirement. In this case, we would believe that $C_p \leqslant w$. Therefore, the practitioners can easily use the procedure on their in-plant applications to obtain reliable decisions.

We remark that the process must be stable in order to produce a reliable estimate of process capability. If the process is out of control, it will be unreliable to estimate process capability. In these cases the priority is to find and eliminate the assignable causes of variability in order to bring the process in-control.

Table 1
The minimum values of $C^*(p)$ of $\widehat{C}^*_{\mathrm{p}}/w$, with $m = 2(2)10, 15$, $n = 10(5)30$ required to ensure $p = 0.99$

m	γ ($n = 10$)				γ ($n = 15$)				γ ($n = 20$)				γ ($n = 25$)				γ ($n = 30$)			
	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1
2	1.7577	1.6442	1.5502	1.4706	1.6297	1.5244	1.4373	1.3635	1.5601	1.4593	1.3759	1.3053	1.5151	1.4173	1.3362	1.2676	1.4831	1.3873	1.3079	1.2408
4	1.5168	1.4188	1.3376	1.2690	1.4566	1.3625	1.2846	1.2187	1.4208	1.3290	1.2530	1.1887	1.3964	1.3062	1.2316	1.1684	1.3785	1.2895	1.2158	1.1534
6	1.4296	1.3373	1.2608	1.1961	1.3908	1.3010	1.2266	1.1637	1.3665	1.2782	1.2051	1.1433	1.3494	1.2622	1.1900	1.1290	1.3366	1.2503	1.1788	1.1182
8	1.3821	1.2928	1.2189	1.1564	1.3543	1.2668	1.1943	1.1330	1.3359	1.2496	1.1781	1.1177	1.3227	1.2373	1.1665	1.1067	1.3127	1.2279	1.1577	1.0983
10	1.3514	1.2641	1.1918	1.1306	1.3303	1.2444	1.1732	1.1130	1.3158	1.2308	1.1604	1.1008	1.3051	1.2208	1.1510	1.0919	1.2968	1.2131	1.1437	1.0850
15	1.3061	1.2217	1.1518	1.0927	1.2946	1.2110	1.1418	1.0831	1.2856	1.2025	1.1338	1.0755	1.2785	1.1959	1.1275	1.0697	1.2728	1.1906	1.1225	1.0649

Table 2
The minimum values $C^*(p)$ of $\widehat{C}^*_{\mathrm{p}}/w$, with $m = 2(2)10, 15$, $n = 10(5)30$ required to ensure $p = 0.975$

m	γ ($n = 10$)				γ ($n = 15$)				γ ($n = 20$)				γ ($n = 25$)				γ ($n = 30$)			
	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1
2	1.6272	1.5221	1.4350	1.3614	1.5361	1.4369	1.3547	1.2852	1.4848	1.3889	1.3095	1.2422	1.4509	1.3572	1.2796	1.2140	1.4266	1.3345	1.2581	1.1936
4	1.4435	1.3503	1.2731	1.2077	1.4011	1.3106	1.2357	1.1723	1.3748	1.2860	1.2124	1.1503	1.3566	1.2689	1.1964	1.1350	1.3429	1.2562	1.1843	1.1236
6	1.3752	1.2863	1.2128	1.1505	1.3487	1.2616	1.1894	1.1284	1.3312	1.2452	1.1740	1.1137	1.3185	1.2334	1.1628	1.1032	1.3089	1.2243	1.1543	1.0951
8	1.3374	1.2510	1.1795	1.1189	1.3193	1.2341	1.1635	1.1038	1.3064	1.2220	1.1521	1.0930	1.2968	1.2131	1.1437	1.0850	1.2893	1.2061	1.1371	1.0787
10	1.3128	1.2280	1.1578	1.0984	1.2999	1.2159	1.1464	1.0876	1.2900	1.2067	1.1377	1.0793	1.2824	1.1996	1.1310	1.0729	1.2764	1.1939	1.1256	1.0679
15	1.2762	1.1938	1.1255	1.0677	1.2708	1.1887	1.1208	1.0632	1.2652	1.1835	1.1158	1.0586	1.2605	1.1791	1.1117	1.0546	1.2566	1.1755	1.1082	1.0514

Table 3
The minimum values $C^*(p)$ of $\widehat{C}^*_{\mathrm{p}}/w$, with $m = 2(2)10, 15$, $n = 10(5)30$, required to ensure $p = 0.95$

m	γ ($n = 10$)				γ ($n = 15$)				γ ($n = 20$)				γ ($n = 25$)				γ ($n = 30$)			
	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1	0.7	0.8	0.9	1
2	1.5268	1.4282	1.3465	1.2774	1.4622	1.3678	1.2896	1.2234	1.4246	1.3326	1.2564	1.1919	1.3992	1.3089	1.2340	1.1707	1.3808	1.2916	1.2177	1.1552
4	1.3850	1.2956	1.2215	1.1588	1.3561	1.2685	1.1960	1.1346	1.3372	1.2508	1.1793	1.1188	1.3237	1.2382	1.1674	1.1075	1.3135	1.2287	1.1584	1.0989
6	1.3310	1.2450	1.1738	1.1136	1.3141	1.2292	1.1589	1.0995	1.3020	1.2179	1.1482	1.0893	1.2929	1.2094	1.1402	1.0817	1.2858	1.2027	1.1340	1.0758
8	1.3008	1.2168	1.1472	1.0883	1.2903	1.2070	1.1380	1.0796	1.2818	1.1991	1.1305	1.0725	1.2752	1.1928	1.1246	1.0669	1.2698	1.1878	1.1199	1.0624
10	1.2810	1.1983	1.1297	1.0718	1.2746	1.1923	1.1241	1.0664	1.2685	1.1866	1.1187	1.0613	1.2634	1.1818	1.1142	1.0570	1.2592	1.1778	1.1105	1.0535
15	1.2514	1.1706	1.1036	1.0470	1.2509	1.1701	1.1032	1.0466	1.2482	1.1676	1.1008	1.0443	1.2454	1.1650	1.0984	1.0420	1.2429	1.1627	1.0961	1.0399

5. Application example

Liquid crystals have been used in various configurations for display applications. Most of the current displays involve the use of either Twisted Nematic (TN) or Super Twisted Nematic (STN) liquid crystals. The STN-LCD products are used in making PDAs (personal digital assistants), notebook computers, word processors and other peripherals. Due to the advancement of modern manufacturing technology STN-LCD and relatively low production cost, STN-LCDs maintained a competitive advantage in the market.

To illustrate the practicality of our proposed Bayesian approach, we present a case study on a STN-LCD manufacturing process, which located in the Science-Based Industrial Park, Taiwan. This factory manufactures various types of the LCD. For a particular model of the STN-LCD investigated, the upper specification limit, USL, of a glass substrate's thickness is 0.77 mm, the lower specification limit, LSL, of a glass substrate's thickness is 0.63 mm, and the target value is $T = 0.70$ mm. If the product characteristic does not fall within the tolerance (LSL, USL), the lifetime or reliability of the STN-LCD will be discounted. The collected sample data (15 samples each of size 10) are displayed in Table 4.

A $100p\%$ credible interval means the posterior probability that the true capability index lies in this interval is p. Let p be a high probability, say, 0.95. Suppose for this particular process under consideration to be capable, the process index must reach at least a certain level w, say, 1.33. Now, from the process data, we compute the lower bound of the credible interval for the index. The resulting Bayesian testing procedure is simple. That is, if $\widehat{C}_{\mathrm{p}}^{*} > C^{*}(p) \times w$, then we say that the process is capable.

As noted earlier, in order to make the estimation of these capability indices meaningful, we would check if the manufacturing process is under statistical control and the distribution is normal. For those 15 samples of size 10 each, the Shapiro–Wilk test for normality confirms this with p-value > 0.1. That is, it is reasonable to assume that the process data collected from the factory is normally distributed. We then construct the $\overline{X} - S$ charts to check if the process is in control. The $\overline{X} - S$ charts based on the collected samples are displayed in Figs. 1 and 2. The $\overline{X} - S$ control charts show that the process seems to be in-control since all the sample points are within the control limits without any special pattern. Therefore, the basic assumptions are justified so we could proceed with the capability calculations. The calculated sample mean $\bar{x}_i$ and the sample variance s_i^2 for the fifteen samples are summarized in the last two columns of Table 4. Thus, we have $s_{\mathrm{p}}^2 = \sum_{i=1}^{m} s_i^2/m = 0.000158$, $\bar{\bar{x}} = 0.6998$ and $\gamma = 0.869$, $\widehat{C}_{\mathrm{p}}^{*} = b_{\sum_{i=1}^{m}(n_i-1)} \times (\mathrm{USL} - \mathrm{LSL})/(6s_{\mathrm{p}}) = 1.8459$. We run the computer program by solving equation (1) (which is available from authors) to

Table 4
The 15 samples of 10 observations with calculated sample statistics

Sample i	Observations										$\bar{x}_i$	s_i^2
1	0.727	0.701	0.678	0.694	0.713	0.699	0.695	0.696	0.733	0.703	0.7039	0.000267
2	0.677	0.712	0.686	0.689	0.682	0.683	0.709	0.687	0.698	0.699	0.6922	0.000139
3	0.692	0.687	0.685	0.698	0.687	0.698	0.707	0.717	0.702	0.717	0.6990	0.000140
4	0.701	0.702	0.695	0.703	0.682	0.696	0.692	0.720	0.687	0.686	0.6964	0.000120
5	0.700	0.719	0.699	0.697	0.714	0.697	0.683	0.688	0.693	0.714	0.7004	0.000139
6	0.693	0.690	0.709	0.707	0.713	0.701	0.706	0.684	0.695	0.688	0.6986	0.000099
7	0.699	0.722	0.714	0.706	0.694	0.700	0.699	0.704	0.683	0.704	0.7025	0.000113
8	0.708	0.712	0.703	0.721	0.692	0.691	0.678	0.698	0.712	0.713	0.7028	0.000170
9	0.711	0.693	0.677	0.710	0.708	0.702	0.680	0.713	0.711	0.694	0.6999	0.000177
10	0.703	0.686	0.720	0.727	0.714	0.713	0.698	0.713	0.693	0.685	0.7052	0.000208
11	0.693	0.724	0.715	0.708	0.722	0.705	0.710	0.715	0.714	0.694	0.7100	0.000109
12	0.700	0.712	0.686	0.707	0.683	0.699	0.705	0.705	0.691	0.727	0.7015	0.000169
13	0.708	0.696	0.718	0.704	0.678	0.703	0.713	0.694	0.684	0.681	0.6979	0.000188
14	0.686	0.688	0.678	0.701	0.718	0.694	0.688	0.691	0.704	0.689	0.6937	0.000128
15	0.690	0.693	0.673	0.678	0.711	0.684	0.712	0.714	0.694	0.686	0.6935	0.000210

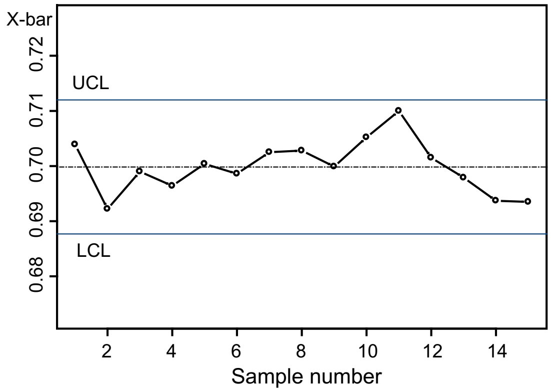

Fig. 1. $\overline{X}$ control chart of the process.

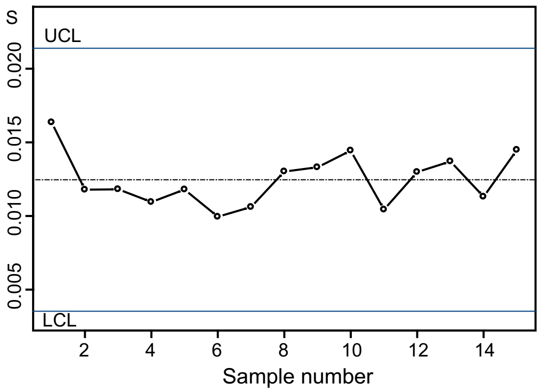

Fig. 2. S control chart of the process.

obtain the critical value $\widehat{C}^*(p) \times w = 1.4938$ based on $p = 0.95$, $m = 15$, $n = 10$. The calculated sample estimator $\widehat{C}_p^* = 1.8456$ is greater than the critical value $\widehat{C}^*(p) \times w = 1.4938$ and the lower confidence bound of C_p is obtained as $\widehat{C}_p^*/C^*(p) = 1.8459/1.1231 = 1.6346$. Therefore, we may conclude, with 95% confidence level, that the process meets the capable precision requirement '$C_p > 1.33$' in this case.

6. Conclusions

Process capability indices establish the relationships between the actual process performance and the manufacturing specifications. Statistical properties of the estimated C_p based on one single sample, have been investigated extensively, but not for multiple samples. For applications where a routine-based data collection plans are implemented, a common practice on process control is to estimate the process precision by analyzing past ''in control'' data. Therefore, the manufacturing information regarding product quality characteristic should be derived from multiple samples rather than one single sample. In this paper, we considered estimating and testing C_p with multiple samples using Bayesian approach, and propose accordingly Bayesian procedure for capability testing. The posterior probability, p, for which the process under investigation is capable, is derived. The credible interval, a Bayesian analogue of the classical lower confidence interval, is obtained. The results obtained in this paper, are generalizations of those obtained in Cheng and Spiring (1989). To make this Bayesian procedure practical for in-plant applications, we tabulated the minimum values of $C^*(p)$ for which the posterior probability p reaches various desirable confidence levels. Subsequently, a real-world case on the STN-LCD manufacturing process, is also investigated using the proposed approach to data collected from the factory.

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