SIAM J. MATRIX ANAL. APPL.
Vol. 37, No. 3, pp. 911–932

A POSITIVITY PRESERVING INVERSE ITERATION FOR FINDING THE PERRON PAIR OF AN IRREDUCIBLE NONNEGATIVE THIRD ORDER TENSOR*

CHING-SUNG LIU†, CHUN-HUA GUO‡, AND WEN-WEI LIN§

Abstract. We propose an inverse iterative method for computing the Perron pair of an irreducible nonnegative third order tensor. The method involves the selection of a parameter θ_k in the kth iteration. For every positive starting vector, the method converges quadratically and is positivity preserving in the sense that the vectors approximating the Perron vector are strictly positive in each iteration. It is also shown that $\theta_k = 1$ near convergence. The computational work for each iteration of the proposed method is less than four times (three times if the tensor is symmetric in modes two and three, and twice if we also take the parameter to be 1 directly) that for each iteration of the Ng–Qi–Zhou algorithm, which is linearly convergent for essentially positive tensors.

Key words. inverse iteration, nonnegative tensor, M-matrix, nonnegative matrix, positivity preserving, quadratic convergence, Perron vector, Perron root

AMS subject classifications. 65F15, 65F50

DOI. 10.1137/15M1040128

1. Introduction.

A real-valued mth order n-dimensional tensor $\mathcal{A}$ consists of n^m entries in $\mathbb{R}$, and takes the form

$$\mathcal{A} = (A_{i_1 i_2 \dots i_m}), \quad A_{i_1 i_2 \dots i_m} \in \mathbb{R}, \quad 1 \le i_1, i_2, \dots, i_m \le n.$$

A tensor $\mathcal{A}$ is called nonnegative (positive) if $A_{i_1 i_2 \dots i_m} \ge 0$ ($A_{i_1 i_2 \dots i_m} > 0$) for all $i_1, \dots, i_m$. For various applications of tensors, nonnegative tensors in particular, see [13].

For an n-dimensional column vector $\mathbf{x} = [x_1, x_2, \dots, x_n]^T \in \mathbb{R}^n$, we define an n-dimensional column vector

$$\mathcal{A}\mathbf{x}^{m-1} := \left(\sum_{i_2,\dots,i_m=1}^{n} A_{i\, i_2 \dots i_m} x_{i_2} \dots x_{i_m} \right)_{1 \le i \le n}. \tag{1.1}$$

DEFINITION 1.1 (see [3, 16]). *Let $\mathcal{A}$ be an mth order n-dimensional tensor and $\mathbb{C}$ be the set of all complex numbers. Assume that $\mathcal{A}\mathbf{x}^{m-1}$ is not identically zero. We say that $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is an eigenpair (eigenvalue-eigenvector) of $\mathcal{A}$ if*

$$\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}, \tag{1.2}$$

where $\mathbf{x}^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1}]^T$.

*Received by the editors September 16, 2015; accepted for publication (in revised form) by D. B. Szyld May 31, 2016; published electronically July 19, 2016.
http://www.siam.org/journals/simax/37-3/M104012.html

†Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung 811, Taiwan (chingsungliu@gmail.com). This author was supported in part by the Ministry of Science and Technology in Taiwan.

‡Department of Mathematics and Statistics, University of Regina, Regina, SK S4S 0A2, Canada (Chun-Hua.Guo@uregina.ca). This author was supported in part by an NSERC Discovery Grant and ST Yau Center at Chiao-Da in Taiwan.

§Department of Applied Mathematics, National Chiao Tung University, Hsinchu 300, Taiwan (wwlin@math.nctu.edu.tw). This author was supported in part by the Ministry of Science and Technology, the National Center for Theoretical Sciences, and ST Yau Center at Chiao-Da in Taiwan.

For an irreducible nonnegative tensor $\mathcal{A}$, the Perron–Frobenius theorem [3, Theorem 1.4] states that there are scalar $\lambda_* > 0$ and unit vector $\mathbf{x}_* > 0$ satisfying (1.2), and that $|\lambda| \le \lambda_*$ for every eigenvalue λ of $\mathcal{A}$. The number λ_* is then called the spectral radius of $\mathcal{A}$, denoted by $\rho(\mathcal{A})$, and is also called the Perron root of $\mathcal{A}$. The corresponding positive unit vector $\mathbf{x}_*$ is unique [3] and is called the Perron vector of $\mathcal{A}$. The Perron pair $(\rho(\mathcal{A}), \mathbf{x}_*)$ is needed in several applications. In particular, it is related to measuring higher order connectivity in hypergraphs [7, 8], and determines the probability distribution of higher order Markov chains [16]. By computing $\rho(\mathcal{A})$ for a suitable irreducible nonnegative tensor $\mathcal{A}$, we can also determine whether an irreducible Z-tensor is an M-tensor [21]. The problem of computing the Perron pair has attracted much attention in recent years.

In 2009, Ng, Qi, and Zhou [16] presented a power method for computing $\rho(\mathcal{A})$ for a nonnegative tensor. Later on, Chang, Pearson, and Zhang [2] proved its convergence for primitive tensors. Linear convergence of the algorithm was then proved in [20] for essentially positive tensors with a particular starting vector. Without giving the detailed definitions, we simply mention that essentially positive tensors are primitive tensors, and primitive tensors are irreducible nonnegative tensors. Liu, Zhou, and Ibrahim [14] noted that for any irreducible nonnegative tensor $\mathcal{A}$, one can apply the Ng–Qi–Zhou (NQZ) algorithm to $\mathcal{B} = \mathcal{A} + s\mathcal{I}$, where s is a positive scalar and $\mathcal{I}$ is the unit tensor. Convergence of the algorithm is then guaranteed by [2] since $\mathcal{B}$ is primitive. For a primitive tensor, starting with any positive vector, the NQZ algorithm also produces approximations to the Perron vector that are positive vectors. So the algorithm is positivity preserving. But it is noted in [20] that the rate of convergence could be worse than linear if the tensor is primitive, but not essentially positive.

The Perron pair can also be found by Newton's method [15], which has local quadratic convergence, but is not positivity preserving. Global convergence may be achieved through a line search [15], but this requires additional assumptions and the resulting algorithm is more complicated and is still not positivity preserving. The positivity of approximations is important in applications; if the approximations lose positivity then they may be meaningless and could not be interpreted.

For irreducible nonnegative second order tensors (i.e., matrices), there are fast-converging and positivity preserving methods [17, 5, 9] for computing the Perron pair. Our goal in this paper is to propose a fast-converging and positivity preserving algorithm for computing the Perron pair of an irreducible nonnegative *third order* tensor $\mathcal{A}$, with a detailed convergence analysis. Third order tensors, as the immediate generalization of matrices, have received special attention [11, 12, 18].

In 1971, Noda [17] introduced a positivity preserving method for the nonnegative matrix eigenvalue problem, which has quadratic convergence [5] and is now called the Noda iteration. In this paper, we propose a positivity preserving iteration for nonnegative third order tensors by combining the idea of Newton's method with the idea of the Noda iteration. We, therefore, call the iteration a Newton–Noda iteration (NNI). NNI is an inverse iterative method with variable shifts, and naturally preserves the strict positivity of the Perron vector in its approximations at all iterations for a positive starting vector. The major advantage of NNI is that, for any positive initial vector, it converges quadratically and computes the desired eigenpair correctly. Furthermore, NNI always generates a monotonically decreasing sequence of approximate eigenvalues, converging quadratically to $\rho(\mathcal{A})$, and the computational work (in terms of flop counts) each iteration is less than four times (and sometimes just twice) that for each iteration of the NQZ algorithm.

The paper is organized as follows. In section 2, we introduce some preliminaries and motivation. In section 3, we present an NNI, and prove some basic properties for it. In section 4, we establish its convergence theory, and derive the asymptotic convergence rate precisely. Finally, in section 5 we present some numerical examples illustrating the convergence theory and the effectiveness of NNI, and we make some concluding remarks in section 6.

2. Preliminaries, notation, and motivation.

A real matrix $A = [A_{ij}] \in \mathbb{R}^{n\times k}$ is called nonnegative (positive) if $A_{ij} \geq 0$ ($A_{ij} > 0$) for all i and j. For real matrices A and B of the same size, if $A-B$ is nonnegative (positive), we write $A \geq B$ ($A > B$). A real square matrix A is called a Z-matrix if all its off-diagonal elements are nonpositive. Any Z-matrix A can be written as $sI - B$ with $B \geq 0$; it is called a nonsingular M-matrix if $s > \rho(B)$, and a singular M-matrix if $s = \rho(B)$, where $\rho(\cdot)$ is the spectral radius.

In addition, we denote $|A| = [|A_{ij}|]$, and the superscript T denotes the transpose of a vector or matrix. From now on we use $\mathbf{v}^{(i)}$ (instead of v_i) to represent the ith element of a vector $\mathbf{v}$, since the notation v_i may be confused with a vector sequence $\mathbf{v}_i$. Throughout the paper, we use the 2-norm for vectors and matrices. All vectors are real n-vectors and all matrices are real $n \times n$ matrices, unless specified otherwise.

The following result is well known (see [1, Theorems 6.2.3 and 6.2.7] for example).

THEOREM 2.1. *For a Z-matrix A, the following are equivalent:*
(i) *A is a nonsingular M-matrix.*
(ii) *$A^{-1} \geq 0$.*
(iii) *$A\mathbf{v} > 0$ for some vector $\mathbf{v} > 0$.*
An irreducible Z-matrix is a nonsingular M-matrix if and only if for some $\mathbf{v} > 0$ the vector $A\mathbf{v}$ is nonnegative and nonzero.

The irreducibility of a tensor is a natural generalization of the irreducibility of a matrix.

DEFINITION 2.2 (see [3, 16]). *An mth order n-dimensional tensor $\mathcal{A}$ is called reducible if there exists a nonempty proper index subset $S \subset \{1, 2, \ldots, n\}$ such that*

$$A_{i_1 i_2 \ldots i_m} = 0 \quad \forall\ i_1 \in S,\ \forall\ i_2, \ldots, i_m \notin S.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

For vectors $\mathbf{v} = [\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \ldots, \mathbf{v}^{(n)}]^T$ and $\mathbf{w} = [\mathbf{w}^{(1)}, \mathbf{w}^{(2)}, \ldots, \mathbf{w}^{(n)}]^T$, with $\mathbf{v}^{(i)} \neq 0$ for all i, we define $\frac{\mathbf{w}}{\mathbf{v}}$ to be the n-vector whose ith element is $\frac{\mathbf{w}^{(i)}}{\mathbf{v}^{(i)}}$, and then define

$$\max\left(\frac{\mathbf{w}}{\mathbf{v}}\right) = \max_i \left(\frac{\mathbf{w}^{(i)}}{\mathbf{v}^{(i)}}\right), \quad \min\left(\frac{\mathbf{w}}{\mathbf{v}}\right) = \min_i \left(\frac{\mathbf{w}^{(i)}}{\mathbf{v}^{(i)}}\right).$$

THEOREM 2.3 (see [3, Theorems 1.4 and 4.2]). *Let $\mathcal{A}$ be an irreducible nonnegative tensor of order m and dimension n. Then there exist $\lambda_* > 0$ and a unit vector $\mathbf{x}_* > 0$ such that*

$$\mathcal{A}\mathbf{x}_*^{m-1} = \lambda_* \mathbf{x}_*^{[m-1]}.$$

If λ is an eigenvalue of $\mathcal{A}$, then $|\lambda| \leq \lambda_*$. Denote λ_* by $\rho(\mathcal{A})$. If λ is an eigenvalue with a nonnegative unit eigenvector $\mathbf{x}$, then $\lambda = \rho(\mathcal{A})$ and $\mathbf{x} = \mathbf{x}_*$. Moreover, for any $\mathbf{v} > 0$

$$\min\left(\frac{\mathcal{A}\mathbf{v}^{m-1}}{\mathbf{v}^{[m-1]}}\right) \leq \rho(\mathcal{A}) \leq \max\left(\frac{\mathcal{A}\mathbf{v}^{m-1}}{\mathbf{v}^{[m-1]}}\right).$$

For a third order n-dimensional tensor, (1.1) can be written as

$$\mathcal{A}\mathbf{x}^2 := \begin{bmatrix} \mathbf{x}^T A_1 \mathbf{x} \\ \vdots \\ \mathbf{x}^T A_n \mathbf{x} \end{bmatrix}, \tag{2.1}$$

where the $n \times n$ matrices A_i are given by $\mathcal{A}(i,:,:)$, using the Matlab multidimensional array notation. From (2.1), the nonnegative tensor eigenvalue problem (1.2) can be written as a nonlinear eigenvalue problem, i.e.,

$$A(\mathbf{x})\mathbf{x} = \lambda\mathbf{x},$$

where

$$A(\mathbf{x}) = D(\mathbf{x})^{-1} \begin{bmatrix} \mathbf{x}^T A_1 \\ \vdots \\ \mathbf{x}^T A_n \end{bmatrix}, \quad D(\mathbf{x}) = \begin{bmatrix} \mathbf{x}^{(1)} & & \\ & \ddots & \\ & & \mathbf{x}^{(n)} \end{bmatrix}. \tag{2.2}$$

We will also need

$$B(\mathbf{x}) = D(\mathbf{x})^{-1} \begin{bmatrix} \mathbf{x}^T \left(A_1 + A_1^T\right) \\ \vdots \\ \mathbf{x}^T \left(A_n + A_n^T\right) \end{bmatrix}. \tag{2.3}$$

Note that for each $\mathbf{x} > 0$, $A(\mathbf{x})$ and $B(\mathbf{x})$ are nonnegative and

$$\frac{\mathcal{A}\mathbf{x}^2}{\mathbf{x}^{[2]}} = \frac{A(\mathbf{x})\mathbf{x}}{\mathbf{x}}, \quad B(\mathbf{x})\mathbf{x} = 2A(\mathbf{x})\mathbf{x}. \tag{2.4}$$

LEMMA 2.4. *Let $\mathbf{v}$ be a positive vector and $\mathcal{A}$ be an irreducible nonnegative third order n-dimensional tensor. Then $A(\mathbf{v})$ and $B(\mathbf{v})$ are irreducible nonnegative matrices.*

Proof. If $A(\mathbf{v})$ is a reducible matrix, then there exists a nonempty proper index subset $S \subset \{1, 2, \ldots, n\}$ such that

$$(A(\mathbf{v}))_{ij} = 0 \ \ \forall\, i \in S \ \ \forall\, j \notin S. \tag{2.5}$$

Because $\mathbf{v}$ is a positive vector, from (2.2) and (2.5), it follows that

$$(D(\mathbf{v})A(\mathbf{v}))_{ij} = \begin{bmatrix} \mathbf{v}^T A_1 \\ \vdots \\ \mathbf{v}^T A_n \end{bmatrix}_{ij} = 0 \ \ \forall\, i \in S \ \ \forall\, j \notin S. \tag{2.6}$$

On the other hand,

$$(D(\mathbf{v})A(\mathbf{v}))_{ij} = \sum_{k=1}^{n} A_{ikj}\mathbf{v}^{(k)}. \tag{2.7}$$

Since $\mathcal{A} \geq 0$ and $\mathbf{v} > 0$, by combining (2.6) and (2.7), it follows that

$$A_{ikj} = 0 \ \ \forall\, i \in S \ \ \forall\, j \notin S \ k = 1, \ldots, n,$$

which contradicts the fact that $\mathcal{A}$ is irreducible. Hence, $A(\mathbf{v})$ is an irreducible matrix. Since $B(\mathbf{v}) \geq A(\mathbf{v})$, $B(\mathbf{v})$ is also an irreducible matrix. ∎

THEOREM 2.5. *For any positive unit vector* $\mathbf{v}$, *let* $\overline{\lambda} = \max(\frac{\mathcal{A}\mathbf{v}^2}{\mathbf{v}^{[2]}})$. *If* $\mathbf{v} \neq \mathbf{x}_*$ *(where* $\mathbf{x}_*$ *is the Perron vector of* $\mathcal{A}$*) , then* $\overline{\lambda}I - A(\mathbf{v})$ *and* $2\overline{\lambda}I - B(\mathbf{v})$ *are nonsingular* M*-matrices. If* $\mathbf{v} = \mathbf{x}_*$, *then* $\overline{\lambda}I - A(\mathbf{v})$ *and* $2\overline{\lambda}I - B(\mathbf{v})$ *are singular* M*-matrices, i.e.,* $\rho(A(\mathbf{x}_*)) = \rho(\mathcal{A})$ *and* $\rho(B(\mathbf{x}_*)) = 2\rho(\mathcal{A})$. *Moreover, if* $(2\rho(\mathcal{A})I - B(\mathbf{x}_*))\,\mathbf{q} \geq 0$ *for a unit vector* $\mathbf{q}$, *then* $\mathbf{q} = \pm\mathbf{x}_*$.

Proof. We have by (2.4)

$$\overline{\lambda} = \max\left(\frac{\mathcal{A}\mathbf{v}^2}{\mathbf{v}^{[2]}}\right) = \max\left(\frac{A(\mathbf{v})\mathbf{v}}{\mathbf{v}}\right) \geq \rho(A(\mathbf{v})).$$

Moreover, $\overline{\lambda} = \max(\frac{A(\mathbf{v})\mathbf{v}}{\mathbf{v}}) = \rho(A(\mathbf{v}))$ if and only if $A(\mathbf{v})\mathbf{v} = \rho(A(\mathbf{v}))\mathbf{v}$ (see [1, Exercise 2.2.12]), i.e., $\mathcal{A}\mathbf{v}^2 = \rho(A(\mathbf{v}))\mathbf{v}^{[2]}$, which holds if and only if $\mathbf{v} = \mathbf{x}_*$ by Theorem 2.3. This proves the statements about $A(\mathbf{v})$.

Similarly, we have by (2.4)

$$2\overline{\lambda} = \max\left(\frac{2\mathcal{A}\mathbf{v}^2}{\mathbf{v}^{[2]}}\right) = \max\left(\frac{B(\mathbf{v})\mathbf{v}}{\mathbf{v}}\right) \geq \rho(B(\mathbf{v})),$$

and $2\overline{\lambda} = \max(\frac{B(\mathbf{v})\mathbf{v}}{\mathbf{v}}) = \rho(B(\mathbf{v}))$ if and only if $\mathbf{v} = \mathbf{x}_*$.

For the irreducible singular M-matrix $M = 2\rho(\mathcal{A})I - B(\mathbf{x}_*)$, we have $\mathbf{v} > 0$ such that $M\mathbf{v} = 0$. Given $M\mathbf{q} \geq 0$ for a unit vector $\mathbf{q}$. Suppose $M\mathbf{q} \neq 0$. Then for $s > 0$ large enough, $\mathbf{w} = s\mathbf{v} + \mathbf{q} > 0$ is such that $M\mathbf{w} \geq 0$ and $M\mathbf{w} \neq 0$. Thus M is a nonsingular M-matrix by Theorem 2.1, a contradiction. Therefore, $M\mathbf{q} = 0$ and thus $B(\mathbf{x}_*)\mathbf{q} = 2\rho(\mathcal{A})\mathbf{q}$. Since $B(\mathbf{x}_*)\mathbf{x}_* = 2\rho(\mathcal{A})\mathbf{x}_*$, it follows from the Perron–Frobenius theorem that $\mathbf{q} = \pm\mathbf{x}_*$. ∎

2.1. Motivation. We define two vector-valued functions $\mathbf{r} : \mathbb{R}_+^{n+1} \to \mathbb{R}^n$ and $\mathbf{f} : \mathbb{R}_+^{n+1} \to \mathbb{R}^{n+1}$ as follows:

$$\mathbf{r}(\mathbf{x},\lambda) = \lambda\mathbf{x}^{[2]} - \mathcal{A}\mathbf{x}^2, \quad \mathbf{f}(\mathbf{x},\lambda) = \begin{bmatrix} -\mathbf{r}(\mathbf{x},\lambda) \\ \frac{1}{2}\left(1 - \mathbf{x}^T\mathbf{x}\right) \end{bmatrix}. \tag{2.8}$$

Then the Jacobian of $\mathbf{f}(x,\lambda)$ is given by

$$\mathbf{Jf}(\mathbf{x},\lambda) = -\begin{bmatrix} \mathbf{J_x r}(\mathbf{x},\lambda) & \mathbf{x}^{[2]} \\ \mathbf{x}^T & 0 \end{bmatrix}. \tag{2.9}$$

Here $\mathbf{J_x r}(\mathbf{x},\lambda)$ is the matrix of partial derivatives of $\mathbf{r}(\mathbf{x},\lambda)$ with respect to $\mathbf{x}$, i.e.,

$$\mathbf{J_x r}(\mathbf{x},\lambda) = 2\lambda D(\mathbf{x}) - G(\mathbf{x}), \tag{2.10}$$

where $D(\mathbf{x})$ is defined by (2.2) and

$$G(\mathbf{x}) = \begin{bmatrix} \mathbf{x}^T\left(A_1 + A_1^T\right) \\ \vdots \\ \mathbf{x}^T\left(A_n + A_n^T\right) \end{bmatrix} \tag{2.11}$$

with $A_i = \mathcal{A}(i,:,:)$.

Note that $\mathcal{A}\mathbf{x}^2 = \frac{1}{2}G(\mathbf{x})\mathbf{x}$. It follows from (2.10) that

$$\mathbf{r}(\mathbf{x},\lambda) = \lambda\mathbf{x}^{[2]} - \mathcal{A}\mathbf{x}^2 = \frac{1}{2}\mathbf{J_x r}(\mathbf{x},\lambda)\mathbf{x}. \tag{2.12}$$

We now consider using Newton's method to solve $\mathbf{f}(\mathbf{x},\lambda) = 0$. Given an approximation $(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)$, Newton's method produces the next approximation $(\widehat{\mathbf{x}}_{k+1}, \widehat{\lambda}_{k+1})$ as follows:

$$\mathbf{Jf}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k) \begin{bmatrix} \mathbf{d}_k \\ \delta_k \end{bmatrix} = \begin{bmatrix} \widehat{\lambda}_k \widehat{\mathbf{x}}_k^{[2]} - \mathcal{A}\widehat{\mathbf{x}}_k^2 \\ \frac{1}{2}\left(\widehat{\mathbf{x}}_k^T \widehat{\mathbf{x}}_k - \mathbf{1}\right) \end{bmatrix}, \tag{2.13}$$

$$\widehat{\mathbf{x}}_{k+1} = \widehat{\mathbf{x}}_k + \mathbf{d}_k, \tag{2.14}$$

$$\widehat{\lambda}_{k+1} = \widehat{\lambda}_k + \delta_k. \tag{2.15}$$

Using elimination in (2.13), we find

$$\delta_k = \frac{\frac{1}{2}\left(\widehat{\mathbf{x}}_k^T \widehat{\mathbf{x}}_k - 1\right) - \widehat{\mathbf{x}}_k^T \left(\mathbf{J_x r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)\right)^{-1} \left(\widehat{\lambda}_k \widehat{\mathbf{x}}_k^{[2]} - \mathcal{A}\widehat{\mathbf{x}}_k^2\right)}{\widehat{\mathbf{x}}_k^T \left(\mathbf{J_x r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)\right)^{-1} \widehat{\mathbf{x}}_k^{[2]}}. \tag{2.16}$$

By (2.12) we can simplify (2.16) to

$$\delta_k = \frac{-1}{2\widehat{\mathbf{x}}_k^T \left(\mathbf{J_x r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)\right)^{-1} \widehat{\mathbf{x}}_k^{[2]}}. \tag{2.17}$$

From the first equation of (2.13) we have, using (2.8), (2.9), (2.14), and (2.12),

$$\begin{aligned} 0 &= \mathbf{J_x r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)\left(\widehat{\mathbf{x}}_{k+1} - \widehat{\mathbf{x}}_k\right) + \mathbf{r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k) + \delta_k \widehat{\mathbf{x}}_k^{[2]} \\ &= \mathbf{J_x r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)\widehat{\mathbf{x}}_{k+1} - \frac{1}{2}\mathbf{J_x r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)\widehat{\mathbf{x}}_k + \delta_k \widehat{\mathbf{x}}_k^{[2]} \\ &= \mathbf{J_x r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)\left(\widehat{\mathbf{x}}_{k+1} - \frac{1}{2}\widehat{\mathbf{x}}_k\right) + \delta_k \widehat{\mathbf{x}}_k^{[2]}. \end{aligned}$$

Hence, we have the following linear system

$$\mathbf{J_x r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)\widehat{\mathbf{w}}_k = \widehat{\mathbf{x}}_k^{[2]}, \tag{2.18}$$

where

$$\widehat{\mathbf{w}}_k = \frac{-\mathbf{1}}{\delta_k}\left(\widehat{\mathbf{x}}_{k+1} - \frac{1}{2}\widehat{\mathbf{x}}_k\right), \text{ i.e., } \widehat{\mathbf{x}}_{k+1} = \frac{1}{2}\widehat{\mathbf{x}}_k - \delta_k \widehat{\mathbf{w}}_k. \tag{2.19}$$

This means that $\widehat{\mathbf{x}}_{k+1}$ is a linear combination of $\widehat{\mathbf{x}}_k$ and $\widehat{\mathbf{w}}_k$. Suppose we already have $\widehat{\mathbf{x}}_k > 0$. We would like to guarantee $\widehat{\mathbf{x}}_{k+1} > 0$. What is needed here is that $\mathbf{J_x r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)$ is a nonsingular M-matrix. In this case, $\widehat{\mathbf{w}}_k > 0$ by (2.18) and $\delta_k < 0$ by (2.17), and thus $\widehat{\mathbf{x}}_{k+1} > 0$.

When $\widehat{\mathbf{x}}_k > 0$, the matrix $\mathbf{J_x r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)$ is an irreducible Z-matrix by Lemma 2.4. By (2.12) and Theorem 2.1, it is a nonsingular M-matrix if $\widehat{\lambda}_k \widehat{\mathbf{x}}_k^{[2]} - \mathcal{A}\widehat{\mathbf{x}}_k^2$ is nonnegative and nonzero. This suggests taking $\widehat{\lambda}_k = \max(\frac{\mathcal{A}\widehat{\mathbf{x}}_k^2}{\widehat{\mathbf{x}}_k^{[2]}})$, which is precisely the idea of the Noda iteration [17]. Newton's method does not determine $\widehat{\lambda}_k$ in this way, and it is unlikely that $\mathbf{J_x r}(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)$ will be a nonsingular M-matrix when $(\widehat{\mathbf{x}}_k, \widehat{\lambda}_k)$ is close to $(\mathbf{x}_*, \rho(\mathcal{A}))$ since $\mathbf{J_x}(\mathbf{x}_*, \rho(\mathcal{A}))$ is a singular M-matrix. Indeed, we have examples showing that the sequence $\{\widehat{\mathbf{x}}_k\}$ produced by Newton's method can fail to be positive.

We are thus motivated to present a new algorithm that combines the idea of Newton's method with the idea of the Noda iteration.

3. The NNI and some basic properties.

In this section, we will propose an NNI for computing the spectral radius $\rho(\mathcal{A})$ and the associated eigenvector of an irreducible nonnegative third order tensor $\mathcal{A}$, and then we prove a number of basic properties of the NNI, which will be used to establish its convergence theory in section 4.

3.1. NNI.

Based on (2.18)–(2.19) and the Noda iteration, we propose an NNI which is an inverse iteration, and each iteration consists of four steps:

$$\mathbf{J_x r}(\mathbf{x}_k, \overline{\lambda}_k)\mathbf{w}_k = \mathbf{x}_k^{[2]}, \ \mathbf{y}_k = \mathbf{w}_k / \|\mathbf{w}_k\|, \tag{3.1}$$

$$\widetilde{\mathbf{x}}_{k+1} = \mathbf{x}_k + \theta_k \mathbf{y}_k, \tag{3.2}$$

$$\mathbf{x}_{k+1} = \widetilde{\mathbf{x}}_{k+1} / \|\widetilde{\mathbf{x}}_{k+1}\|, \tag{3.3}$$

$$\overline{\lambda}_{k+1} = \max\left(\frac{\mathcal{A}\mathbf{x}_{k+1}^2}{\mathbf{x}_{k+1}^{[2]}}\right), \tag{3.4}$$

where $\theta_k > 0$ is to be defined later by (3.11).

The following lemma shows that the parameter $\theta_k > 0$ in (3.2) naturally preserves the strict positivity of $\mathbf{x}_k$ at all iterations.

Lemma 3.1. *Let $\mathcal{A}$ be an irreducible nonnegative third order tensor. Given a unit vector $\mathbf{x}_k > 0$, if $\mathbf{x}_k \neq \mathbf{x}_*$ and $\theta_k > 0$, then $\mathbf{y}_k, \mathbf{x}_{k+1} > 0$ and*

$$\overline{\lambda}_{k+1} = \overline{\lambda}_k - \min\left(\frac{\mathbf{h}_k(\theta_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right), \tag{3.5}$$

where

$$\mathbf{h}_k(\theta) = \frac{\theta \mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \theta^2 \mathbf{r}\left(\mathbf{y}_k, \overline{\lambda}_k\right) + \mathbf{r}\left(\mathbf{x}_k, \overline{\lambda}_k\right). \tag{3.6}$$

Proof. By (2.10), $\mathbf{J_x r}(\mathbf{x}_k, \overline{\lambda}_k) = 2\overline{\lambda}_k D(\mathbf{x}_k) - G(\mathbf{x}_k) = D(\mathbf{x}_k)\left(2\overline{\lambda}_k I - B(\mathbf{x}_k)\right)$. So the vector $\mathbf{w}_k$ in (3.1) satisfies

$$\left(2\overline{\lambda}_k I - B(\mathbf{x}_k)\right)\mathbf{w}_k = \mathbf{x}_k. \tag{3.7}$$

Since $\overline{\lambda}_k = \max(\frac{\mathcal{A}\mathbf{x}_k^2}{\mathbf{x}_k^{[2]}})$ and $\mathbf{x}_k \neq \mathbf{x}_*$, we know by Theorem 2.5 that $2\overline{\lambda}_k I - B(\mathbf{x}_k)$ is a nonsingular M-matrix. Thus

$$\mathbf{w}_k = \left(2\overline{\lambda}_k I - B(\mathbf{x}_k)\right)^{-1}\mathbf{x}_k > 0.$$

Then $\mathbf{y}_k > 0$ and $\mathbf{x}_{k+1} > 0$ since $\theta_k > 0$.

By (3.1),

$$\mathbf{J_x r}(\mathbf{x}_k, \overline{\lambda}_k)\mathbf{y}_k = \left(2\overline{\lambda}_k D(\mathbf{x}_k) - G(\mathbf{x}_k)\right)\mathbf{y}_k = \frac{\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|}. \tag{3.8}$$

Therefore,

$$\begin{aligned} &2\overline{\lambda}_k D(\mathbf{x}_k)\mathbf{y}_k - \frac{\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} \\ &= G(\mathbf{x}_k)\mathbf{y}_k = \begin{bmatrix} \mathbf{x}_k^T\left(A_1 + A_1^T\right)\mathbf{y}_k \\ \vdots \\ \mathbf{x}_k^T\left(A_n + A_n^T\right)\mathbf{y}_k \end{bmatrix} = \begin{bmatrix} \mathbf{x}_k^T A_1 \mathbf{y}_k \\ \vdots \\ \mathbf{x}_k^T A_n \mathbf{y}_k \end{bmatrix} + \begin{bmatrix} \mathbf{y}_k^T A_1 \mathbf{x}_k \\ \vdots \\ \mathbf{y}_k^T A_n \mathbf{x}_k \end{bmatrix}. \end{aligned} \tag{3.9}$$

From (3.2) and (3.9), we have

$$
\begin{aligned}
\mathcal{A}\widetilde{\mathbf{x}}_{k+1}^2 &= \begin{bmatrix} \widetilde{\mathbf{x}}_{k+1}^T A_1 \widetilde{\mathbf{x}}_{k+1} \\ \vdots \\ \widetilde{\mathbf{x}}_{k+1}^T A_n \widetilde{\mathbf{x}}_{k+1} \end{bmatrix} = \mathcal{A}\mathbf{x}_k^2 + \theta_k \begin{bmatrix} \mathbf{x}_k^T A_1 \mathbf{y}_k \\ \vdots \\ \mathbf{x}_k^T A_n \mathbf{y}_k \end{bmatrix} + \theta_k \begin{bmatrix} \mathbf{y}_k^T A_1 \mathbf{x}_k \\ \vdots \\ \mathbf{y}_k^T A_n \mathbf{x}_k \end{bmatrix} + \theta_k^2 \mathcal{A}\mathbf{y}_k^2 \\
&= \mathcal{A}\mathbf{x}_k^2 + 2\overline{\lambda}_k \theta_k D(\mathbf{x}_k)\mathbf{y}_k - \frac{\theta_k \mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \theta_k^2 \mathcal{A}\mathbf{y}_k^2 \\
&= \left(\overline{\lambda}_k \mathbf{x}_k^{[2]} + 2\overline{\lambda}_k \theta_k D(\mathbf{x}_k)\mathbf{y}_k + \overline{\lambda}_k \theta_k^2 \mathbf{y}_k^{[2]}\right) - \frac{\theta_k \mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} \\
&\quad + \theta_k^2 \mathcal{A}\mathbf{y}_k^2 - \overline{\lambda}_k \theta_k^2 \mathbf{y}_k^{[2]} + \mathcal{A}\mathbf{x}_k^2 - \overline{\lambda}_k \mathbf{x}_k^{[2]} \\
&= \overline{\lambda}_k \widetilde{\mathbf{x}}_{k+1}^{[2]} - \frac{\theta_k \mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \theta_k^2 \left(\mathcal{A}\mathbf{y}_k^2 - \overline{\lambda}_k \mathbf{y}_k^{[2]}\right) + \mathcal{A}\mathbf{x}_k^2 - \overline{\lambda}_k \mathbf{x}_k^{[2]}.
\end{aligned}
$$

Therefore,

$$
\mathcal{A}\widetilde{\mathbf{x}}_{k+1}^2 = \overline{\lambda}_k \widetilde{\mathbf{x}}_{k+1}^{[2]} - \mathbf{h}_k(\theta_k), \tag{3.10}
$$

where

$$
\mathbf{h}_k(\theta) = \frac{\theta \mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \theta^2 \mathbf{r}\left(\mathbf{y}_k, \overline{\lambda}_k\right) + \mathbf{r}\left(\mathbf{x}_k, \overline{\lambda}_k\right).
$$

From (3.10), it follows that

$$
\overline{\lambda}_{k+1} = \max\left(\frac{\mathcal{A}\mathbf{x}_{k+1}^2}{\mathbf{x}_{k+1}^{[2]}}\right) = \max\left(\frac{\overline{\lambda}_k \widetilde{\mathbf{x}}_{k+1}^{[2]} - \mathbf{h}_k(\theta_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right) = \overline{\lambda}_k - \min\left(\frac{\mathbf{h}_k(\theta_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right). \qquad \square
$$

We next show that $\{\overline{\lambda}_k\}$ is strictly decreasing for suitable θ_k, unless $\mathbf{x}_k = \mathbf{x}_*$ for some k, in which case NNI terminates with $\overline{\lambda}_k = \rho(\mathcal{A})$.

THEOREM 3.2. *Let $\mathcal{A}$ be an irreducible nonnegative third order tensor and $\eta > 0$ be a fixed constant. Given a unit vector $\mathbf{x}_k > 0$, suppose $\mathbf{x}_k \neq \mathbf{x}_*$ and θ_k in* (3.2) *satisfies*

$$
\theta_k = \begin{cases} 1 & \text{if } \mathbf{h}_k(1) \geq \frac{\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|}, \\ \eta_k & \text{otherwise,} \end{cases} \tag{3.11}
$$

where for each k with $\mathbf{h}_k(1) < \frac{\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|}$,

$$
\eta_k = \frac{\eta}{(1+\eta)\|\mathbf{w}_k\|\left(\mu_k - \overline{\lambda}_k\right)} \min\left(\frac{\mathbf{x}_k^{[2]}}{\mathbf{y}_k^{[2]}}\right)
$$

with $\mu_k = \max(\frac{\mathcal{A}\mathbf{y}_k^2}{\mathbf{y}_k^{[2]}})$. Then $0 < \eta_k < 1$ whenever it is defined, $\mathbf{x}_{k+1} > 0$ in (3.3)*, and*

$$
\overline{\lambda}_k > \overline{\lambda}_{k+1} \geq \rho(\mathcal{A}). \tag{3.12}
$$

Proof. By Lemma 3.1, we have

$$\overline{\lambda}_{k+1} = \overline{\lambda}_k - \min\left(\frac{\mathbf{h}_k(\theta_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right).$$

We need to prove $\mathbf{h}_k(\theta_k) > 0$.

From (3.6), we have

$$\begin{aligned}\mathbf{h}_k(\theta) &= \frac{\theta\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \theta^2\mathbf{r}\left(\mathbf{y}_k,\overline{\lambda}_k\right) + \mathbf{r}\left(\mathbf{x}_k,\overline{\lambda}_k\right)\\ &= \frac{\theta\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \theta^2\mathbf{r}\left(\mathbf{y}_k,\mu_k\right) + \mathbf{r}\left(\mathbf{x}_k,\overline{\lambda}_k\right) + \theta^2\left(\overline{\lambda}_k - \mu_k\right)\mathbf{y}_k^{[2]} \qquad (3.13)\\ &= \frac{\theta\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|} + \frac{\theta\eta\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|}\\ &\quad + \theta^2\mathbf{r}\left(\mathbf{y}_k,\mu_k\right) + \mathbf{r}\left(\mathbf{x}_k,\overline{\lambda}_k\right) + \theta^2\left(\overline{\lambda}_k - \mu_k\right)\mathbf{y}_k^{[2]}, \qquad (3.14)\end{aligned}$$

where $\mu_k = \max\left(\frac{\mathcal{A}\mathbf{y}_k^2}{\mathbf{y}_k^{[2]}}\right)$. When $\mu_k \leq \overline{\lambda}_k$,

$$\begin{aligned}\mathbf{h}_k(1) &\geq \frac{\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \mathbf{r}\left(\mathbf{y}_k,\mu_k\right) + \mathbf{r}\left(\mathbf{x}_k,\overline{\lambda}_k\right) \qquad \text{(by (3.13))}\\ &\geq \frac{\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|} > 0.\end{aligned}$$

Whenever $\mathbf{h}_k(1) \geq \frac{\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|}$, we have $\overline{\lambda}_{k+1} < \overline{\lambda}_k$ with $\theta_k = 1$. If $\mathbf{h}_k(1) < \frac{\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|}$, then $\mu_k > \overline{\lambda}_k$ and $\eta_k > 0$ is defined. Suppose $\eta_k \geq 1$, then

$$\frac{\eta}{(1+\eta)\|\mathbf{w}_k\|}\min\left(\frac{\mathbf{x}_k^{[2]}}{\mathbf{y}_k^{[2]}}\right) \geq \left(\mu_k - \overline{\lambda}_k\right)$$

and, thus,

$$\frac{\eta\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|} + \left(\overline{\lambda}_k - \mu_k\right)\mathbf{y}_k^{[2]} \geq 0.$$

It follows from (3.14) that $\mathbf{h}_k(1) \geq \frac{\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|}$, a contradiction. So $\eta_k < 1$. We now have

$$\theta_k = \eta_k = \frac{\eta}{(1+\eta)\|\mathbf{w}_k\|\left(\mu_k - \overline{\lambda}_k\right)}\min\left(\frac{\mathbf{x}_k^{[2]}}{\mathbf{y}_k^{[2]}}\right), \qquad (3.15)$$

which ensures the inequality

$$\frac{\theta_k\eta\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|} \geq \theta_k^2\left(\mu_k - \overline{\lambda}_k\right)\mathbf{y}_k^{[2]}. \qquad (3.16)$$

Substituting (3.16) into (3.14), we obtain

$$\begin{aligned}\mathbf{h}_k(\theta_k) &= \left[\frac{\theta_k \mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|} + \theta_k^2 \mathbf{r}\left(\mathbf{y}_k,\mu_k\right) + \mathbf{r}\left(\mathbf{x}_k,\overline{\lambda}_k\right)\right] \\ &\quad + \left[\frac{\theta_k \eta \mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|} + \theta_k^2\left(\overline{\lambda}_k - \mu_k\right)\mathbf{y}_k^{[2]}\right] \\ &\geq \frac{\theta_k \mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|} + \theta_k^2 \mathbf{r}\left(\mathbf{y}_k,\mu_k\right) + \mathbf{r}\left(\mathbf{x}_k,\overline{\lambda}_k\right).\end{aligned}$$

Therefore,

$$\mathbf{h}_k(\theta_k) \geq \frac{\theta_k \mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|} > 0 \tag{3.17}$$

and then

$$\overline{\lambda}_{k+1} = \overline{\lambda}_k - \min\left(\frac{\mathbf{h}_k(\theta_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right) < \overline{\lambda}_k.$$

By Theorem 2.3 we have $\overline{\lambda}_{k+1} \geq \rho(\mathcal{A})$. □

Based on (3.1)–(3.4) and (3.11), we can present the NNI as Algorithm 3.1. The main computational work in each iteration is in lines $3, 5$, and 8. The computational work in line 8 is the same as that for one iteration of the NQZ algorithm, which is $2n^3$ flops, since the main computational work for one iteration of the NQZ algorithm is just one evaluation of $\mathcal{A}\mathbf{v}^2$ for a positive vector $\mathbf{v}$. If θ_k needs to be determined in step 5, then an additional $2n^3$ flops are needed. But we will see later in this section that we always have $\theta_k = 1$ near convergence. Forming the linear system in step 3 requires $2n^3$ flops in general. But if the tensor $\mathcal{A}$ is symmetric in modes two and three [13], i.e., $A_i = A_i^T$ for all $i = 1, \ldots, n$, then forming the linear system only requires $O(n^2)$ flops. Solving the linear system in step 3 by the Grassmann–Taksar–Heyman (GTH) algorithm [6] will require $\frac{4}{3}n^3$ flops. Therefore, the computational work (in terms of flop counts) in each iteration of NNI is less than four times (and sometimes just twice) that for each iteration of the NQZ algorithm.

The vector $\mathbf{w}_k$ can be computed by the GTH algorithm accurately even near convergence, and is guaranteed to be positive. Therefore, Algorithm 3.1 generates

Algorithm 3.1 NNI

1. Given $\mathbf{x}_0 > 0$ with $\|\mathbf{x}_0\| = 1$, $\overline{\lambda}_0 = \max(\frac{\mathcal{A}\mathbf{x}_0^2}{\mathbf{x}_0^{[2]}})$, $\eta > 0$, and $\mathsf{tol} > 0$.
2. **for** $k = 0, 1, 2, \ldots$
3. Solve the linear system $\mathbf{J_x r}(\mathbf{x}_k, \overline{\lambda}_k)\mathbf{w}_k = \mathbf{x}_k^{[2]}$.
4. Normalize the vector $\mathbf{w}_k$: $\mathbf{y}_k = \mathbf{w}_k/\|\mathbf{w}_k\|$.
5. Compute the scalar θ_k satisfying (3.11).
6. Compute the vector $\widetilde{\mathbf{x}}_{k+1} = \mathbf{x}_k + \theta_k \mathbf{y}_k$.
7. Normalize the vector $\widetilde{\mathbf{x}}_{k+1}$: $\mathbf{x}_{k+1} = \widetilde{\mathbf{x}}_{k+1}/\|\widetilde{\mathbf{x}}_{k+1}\|$.
8. Compute $\overline{\lambda}_{k+1} = \max(\frac{\mathcal{A}\mathbf{x}_{k+1}^2}{\mathbf{x}_{k+1}^{[2]}})$ and $\underline{\lambda}_{k+1} = \min(\frac{\mathcal{A}\mathbf{x}_{k+1}^2}{\mathbf{x}_{k+1}^{[2]}})$.
9. **until** convergence: $|\overline{\lambda}_{k+1} - \underline{\lambda}_{k+1}|/\overline{\lambda}_{k+1} < \mathsf{tol}$.

a positive vector sequence $\{\mathbf{x}_k\}$, so it is a positivity preserving algorithm. In what follows we will prove some properties of $\theta_k, \mathbf{x}_k$, and $\mathbf{y}_k$. These properties will help us to establish the global and quadratic convergence of NNI.

3.2. Some basic properties. Lemma 2.4 shows that $B(\mathbf{x}_*)$ is an irreducible nonnegative matrix. Recall that $B(\mathbf{x}_*)\mathbf{x}_* = \rho(B(\mathbf{x}_*))\mathbf{x}_*$. Then for any orthogonal matrix $\begin{bmatrix} \mathbf{x}_* & V \end{bmatrix}$ direct computation gives

$$\begin{bmatrix} \mathbf{x}_*^T \\ V^T \end{bmatrix} B(\mathbf{x}_*) \begin{bmatrix} \mathbf{x}_* & V \end{bmatrix} = \begin{bmatrix} \rho(B(\mathbf{x}_*)) & \mathbf{c}^T \\ 0 & L \end{bmatrix}.$$

Similarly, for an irreducible nonnegative matrix $B(\mathbf{x}_k)$, let $\mathbf{u}_k$ be the unit length positive eigenvector corresponding to $\rho(B(\mathbf{x}_k))$. Then for any orthogonal matrix $\begin{bmatrix} \mathbf{u}_k & V_k \end{bmatrix}$ it holds that

$$\begin{bmatrix} \mathbf{u}_k^T \\ V_k^T \end{bmatrix} B(\mathbf{x}_k) \begin{bmatrix} \mathbf{u}_k & V_k \end{bmatrix} = \begin{bmatrix} \rho(B(\mathbf{x}_k)) & \mathbf{c}_k^T \\ 0 & L_k \end{bmatrix}. \tag{3.18}$$

When $B(\mathbf{x}_k) \to B(\mathbf{x}_*)$, we have $\mathbf{u}_k \to \mathbf{x}_*$ and choose V_k such that $V_k \to V$, and then we have $\mathbf{c}_k \to \mathbf{c}$ and $L_k \to L$.

Now define

$$\varepsilon_k = \overline{\lambda}_k - \rho(\mathcal{A}), \quad B_k = 2\overline{\lambda}_k I - B(\mathbf{x}_k), \quad \tau_k = 2\overline{\lambda}_k - \rho(B(\mathbf{x}_k)). \tag{3.19}$$

Then from (3.18) we have

$$\begin{bmatrix} \mathbf{u}_k^T \\ V_k^T \end{bmatrix} B_k \begin{bmatrix} \mathbf{u}_k & V_k \end{bmatrix} = \begin{bmatrix} \tau_k & -\mathbf{c}_k^T \\ 0 & \overline{L}_k \end{bmatrix},$$

where $\overline{L}_k = 2\overline{\lambda}_k I - L_k$. For $2\overline{\lambda}_k \neq \rho(B(\mathbf{x}_k))$, it is easy to verify that

$$\begin{bmatrix} \mathbf{u}_k^T \\ V_k^T \end{bmatrix} B_k^{-1} \begin{bmatrix} \mathbf{u}_k & V_k \end{bmatrix} = \begin{bmatrix} \frac{1}{\tau_k} & \mathbf{b}_k^T \\ 0 & \overline{L}_k^{-1} \end{bmatrix} \quad \text{with } \mathbf{b}_k^T = \frac{\mathbf{c}_k^T \overline{L}_k^{-1}}{\tau_k}. \tag{3.20}$$

LEMMA 3.3. *Assume that the sequence* $\{\overline{\lambda}_k, \mathbf{x}_k, \mathbf{y}_k\}$ *is generated by Algorithm* 3.1. *For any subsequence* $\{\mathbf{x}_{k_j}\} \subseteq \{\mathbf{x}_k\}$, *we have the following results:*

(i) *If* $\mathbf{x}_{k_j} \to \mathbf{v}$ *as* $j \to \infty$, *then* $\mathbf{v} > 0$.

(ii) *If* $\mathbf{x}_{k_j} \to \mathbf{x}_*$ *as* $j \to \infty$, *then* $\mathbf{y}_{k_j} \to \mathbf{x}_*$ *as* $j \to \infty$.

Proof. (i) If $\lim_{j\to\infty} \mathbf{x}_{k_j} = \mathbf{v}$, then $\mathbf{v} \geq 0$. Let S be the set of all indices t such that $\lim_{j\to\infty} \mathbf{x}_{k_j}^{(t)} = \mathbf{v}^{(t)} = 0$. Since $\|\mathbf{x}_{k_j}\| = 1$, S is a proper subset of $\{1, 2, \ldots, n\}$. Suppose S is nonempty. Then by the definition of $\overline{\lambda}_k$,

$$\overline{\lambda}_0 \geq \overline{\lambda}_{k_j} = \max\left(\frac{\mathcal{A}\mathbf{x}_{k_j}^2}{\mathbf{x}_{k_j}^{[2]}}\right) \geq \frac{\mathbf{x}_{k_j}^T A_t \mathbf{x}_{k_j}}{\left(\mathbf{x}_{k_j}^{(t)}\right)^2} \text{ for all } t = 1, 2, \ldots, n.$$

Since $\lim_{j\to\infty} \mathbf{x}_{k_j}^{(t)} = 0$ for $t \in S$, it holds that $\lim_{j\to\infty} \mathbf{x}_{k_j}^T A_t \mathbf{x}_{k_j} = \mathbf{v}^T A_t \mathbf{v} = 0$ for $t \in S$. Thus, $A_{tpq} = 0$ for all $t \in S$ and for all $p, q \notin S$, which contradicts the irreducibility of $\mathcal{A}$. Therefore, S is empty and thus $\mathbf{v} > 0$.

(ii) Since $\mathbf{x}_{k_j} \to \mathbf{x}_*$, we have $B(\mathbf{x}_{k_j}) \to B(\mathbf{x}_*)$. Then we have $\mathbf{u}_{k_j} \to \mathbf{x}_*$, $\varepsilon_{k_j} \to 0$, and $\tau_{k_j} \to 0$, where we have used $\rho(B(\mathbf{x}_*)) = 2\rho(\mathcal{A})$. From (3.7) and (3.20), we get

$$\tau_{k_j}\mathbf{w}_{k_j} = \tau_{k_j}B_{k_j}^{-1}\mathbf{x}_{k_j} = \left(\mathbf{u}_{k_j}\mathbf{u}_{k_j}^T + \mathbf{u}_{k_j}\mathbf{c}_{k_j}^T\overline{L}_{k_j}^{-1}V_{k_j}^T + \tau_{k_j}V_{k_j}\overline{L}_{k_j}^{-1}V_{k_j}^T\right)\mathbf{x}_{k_j}.$$

Since $\overline{L}_{k_j}^{-1} \to [\rho(B(\mathbf{x}_*))I - L]^{-1}$ and $\tau_{k_j} \to 0$, we have $\tau_{k_j}V_{k_j}\overline{L}_{k_j}^{-1}V_{k_j}^T \to 0$. Note that $V_{k_j}^T\mathbf{u}_{k_j} = 0$ and $V^T\mathbf{x}_* = \lim_{k\to\infty} V_{k_j}^T\mathbf{u}_{k_j} = 0$. We then get

$$\lim_{j\to\infty} \mathbf{u}_{k_j}\mathbf{c}_{k_j}^T\overline{L}_{k_j}^{-1}V_{k_j}^T\mathbf{x}_{k_j} = \mathbf{x}_*\mathbf{c}^T\left(\rho(B(\mathbf{x}_*))I - L\right)^{-1}V^T\mathbf{x}_* = 0.$$

A combination of the above relations shows that

$$\lim_{j\to\infty}\tau_{k_j}\mathbf{w}_{k_j} = \lim_{j\to\infty}\left(\mathbf{u}_{k_j}\mathbf{u}_{k_j}^T + \mathbf{u}_{k_j}\mathbf{c}_{k_j}^T\overline{L}_{k_j}^{-1}V_{k_j}^T + \tau_{k_j}V_{k_j}\overline{L}_{k_j}^{-1}V_{k_j}^T\right)\mathbf{x}_{k_j} = \mathbf{x}_*.$$

Hence,

$$\lim_{j\to\infty}\mathbf{y}_{k_j} = \lim_{j\to\infty}\frac{\tau_{k_j}\mathbf{w}_{k_j}}{\left\|\tau_{k_j}\mathbf{w}_{k_j}\right\|} = \mathbf{x}_*. \qquad \square$$

LEMMA 3.4. *Assume that the sequence* $\left\{\overline{\lambda}_k, \mathbf{x}_k, \mathbf{y}_k\right\}$ *is generated by Algorithm* 3.1. *Then the sequence* $\{\|\mathbf{w}_k\|\,\|\mathbf{y}_k - \mathbf{x}_k\|\}$ *is bounded, that is, there exists a constant* $M_1 > 0$ *such that*

$$\|\mathbf{w}_k\|\,\|\mathbf{y}_k - \mathbf{x}_k\| \le M_1 \text{ for all } k.$$

Proof. From (2.12),

$$\mathbf{J_x r}(\mathbf{x}_k,\overline{\lambda}_k)\mathbf{y}_k - \mathbf{J_x r}(\mathbf{x}_k,\overline{\lambda}_k)\left(\mathbf{y}_k - \mathbf{x}_k\right) = \mathbf{J_x r}(\mathbf{x}_k,\overline{\lambda}_k)\mathbf{x}_k = 2\mathbf{r}(\mathbf{x}_k,\overline{\lambda}_k) \ge 0.$$

This means

$$\mathbf{J_x r}(\mathbf{x}_k,\overline{\lambda}_k)\mathbf{y}_k \ge \mathbf{J_x r}(\mathbf{x}_k,\overline{\lambda}_k)\left(\mathbf{y}_k - \mathbf{x}_k\right). \tag{3.21}$$

We may assume $\mathbf{x}_k \ne \mathbf{y}_k$. From (3.8) and (3.21), we have

$$\mathbf{x}_k^{[2]} \ge \|\mathbf{w}_k\|\,\|\mathbf{y}_k - \mathbf{x}_k\|\,\mathbf{J_x r}(\mathbf{x}_k,\overline{\lambda}_k)\mathbf{p}_k, \tag{3.22}$$

where $\mathbf{p}_k = (\mathbf{y}_k - \mathbf{x}_k)\,/\,\|\mathbf{y}_k - \mathbf{x}_k\|$ with $\|\mathbf{p}_k\| = 1$. Because $\mathbf{x}_k, \mathbf{y}_k > 0$ and $\|\mathbf{x}_k\| = \|\mathbf{y}_k\| = 1$, we have $\mathbf{p}_k \nleq 0$ and $\mathbf{p}_k \ngeq 0$ for all k.

Suppose $\{\|\mathbf{w}_k\|\,\|\mathbf{y}_k - \mathbf{x}_k\|\}$ is not bounded. Since $\{\mathbf{x}_k\}$ and $\{\mathbf{p}_k\}$ are bounded, we have for some subsequence $\{k_j\}$

$$\lim_{j\to\infty}\left\|\mathbf{w}_{k_j}\right\|\left\|\mathbf{y}_{k_j} - \mathbf{x}_{k_j}\right\| = \infty, \quad \lim_{j\to\infty}\mathbf{x}_{k_j} =: \mathbf{v}, \quad \lim_{j\to\infty}\mathbf{p}_{k_j} =: \mathbf{p}. \tag{3.23}$$

Since $\left\|\mathbf{w}_{k_j}\right\|\left\|\mathbf{y}_{k_j} - \mathbf{x}_{k_j}\right\| \le 2\left\|\mathbf{w}_{k_j}\right\|$, we also have $\lim_{j\to\infty}\left\|\mathbf{w}_{k_j}\right\| = \infty$. By Lemma 3.3 we have $\mathbf{v} > 0$. We now prove $\mathbf{v} = \mathbf{x}_*$.

Since the sequence $\left\{\overline{\lambda}_k\right\}$ is monotonically decreasing and bounded below by $\rho(\mathcal{A})$, $\lim_{j\to\infty} 2\overline{\lambda}_{k_j} = 2\alpha$ exists. By Theorem 2.5, the $2\overline{\lambda}_{k_j}I - B(\mathbf{x}_{k_j})$ are nonsingular M-matrices. Thus $2\alpha I - B(\mathbf{v})$ is an M-matrix, so $2\alpha \ge \rho(B(\mathbf{v}))$. If $2\alpha > \rho(B(\mathbf{v}))$, then

$$\lim_{j\to\infty}\mathbf{w}_{k_j} = \lim_{j\to\infty}\left(2\overline{\lambda}_{k_j}I - B(\mathbf{x}_{k_j})\right)^{-1}\mathbf{x}_{k_j} = \left(2\alpha I - B(\mathbf{v})\right)^{-1}\mathbf{v} =: \mathbf{w} > 0,$$

in contradiction to $\lim_{j\to\infty}\left\|\mathbf{w}_{k_j}\right\| = \infty$. Thus $2\alpha = \rho(B(\mathbf{v}))$. Now

$$\rho(B(\mathbf{v})) = \lim_{j\to\infty} 2\overline{\lambda}_{k_j} = \lim_{j\to\infty} 2\max\left(\frac{\mathcal{A}\mathbf{x}_{k_j}^2}{\mathbf{x}_{k_j}^{[2]}}\right) = 2\max\left(\frac{\mathcal{A}\mathbf{v}^2}{\mathbf{v}^{[2]}}\right) = \max\left(\frac{B(\mathbf{v})\mathbf{v}}{\mathbf{v}}\right).$$

It follows that $B(\mathbf{v})\mathbf{v} = \rho(B(\mathbf{v}))\mathbf{v}$ and then $\mathcal{A}\mathbf{v}^2 = \frac{1}{2}\rho(B(\mathbf{v}))\mathbf{v}^{[2]}$. Thus $\mathbf{v} = \mathbf{x}_*$ by Theorem 2.3.

Now $\lim_{j\to\infty}\mathbf{x}_{k_j} = \mathbf{x}_*$ and $\lim_{j\to\infty}\overline{\lambda}_{k_j} = \alpha = \frac{1}{2}\rho(B(\mathbf{x}_*)) = \rho(\mathcal{A})$ by Theorem 2.5. It follows from (3.22) and (3.23) that

$$\mathbf{J_x r}(\mathbf{x}_*,\rho(\mathcal{A}))\mathbf{p} = \lim_{j\to\infty}\mathbf{J_x r}(\mathbf{x}_{k_j},\overline{\lambda}_{k_j})\mathbf{p}_{k_j} \le \lim_{j\to\infty}\frac{\mathbf{x}_{k_j}^{[2]}}{\left\|\mathbf{w}_{k_j}\right\|\left\|\mathbf{y}_{k_j}-\mathbf{x}_{k_j}\right\|} = 0.$$

Thus $(2\rho(\mathcal{A})I - B(\mathbf{x}_*))\,\mathbf{p} \le 0$. Then we have $\mathbf{p} = \pm\mathbf{x}_*$ by Theorem 2.5. But by the definition of $\mathbf{p}_k$, $\mathbf{p}$ is neither positive nor negative. The contradiction shows that the sequence $\{\|\mathbf{w}_k\|\,\|\mathbf{y}_k - \mathbf{x}_k\|\}$ is bounded. ∎

LEMMA 3.5. *Assume that the sequence* $\left\{\overline{\lambda}_k, \mathbf{x}_k, \mathbf{y}_k\right\}$ *is generated by Algorithm* 3.1. *Then* $\mathbf{h}_k(\theta_k)$ *can be expressed in the form*

$$\mathbf{h}_k(\theta_k) = \frac{2\theta_k\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \mathbf{R}(\theta_k\mathbf{y}_k, \mathbf{x}_k, \overline{\lambda}_k), \tag{3.24}$$

where $\left\|\mathbf{R}(\mathbf{x}, \mathbf{x}_k,\overline{\lambda}_k)\right\| \le M_2\,\|\mathbf{x}-\mathbf{x}_k\|^2$ *for some constant* M_2.

Proof. To prove this, we use Taylor's theorem for the function $\mathbf{r}(\mathbf{x},\overline{\lambda}_k)$ around the point $\mathbf{x}_k$:

$$\mathbf{r}(\mathbf{x},\overline{\lambda}_k) = \mathbf{r}(\mathbf{x}_k,\overline{\lambda}_k) + \mathbf{J_x r}(\mathbf{x}_k,\overline{\lambda}_k)(\mathbf{x}-\mathbf{x}_k) + \mathbf{R}(\mathbf{x}, \mathbf{x}_k,\overline{\lambda}_k), \tag{3.25}$$

where $\left\|\mathbf{R}(\mathbf{x}, \mathbf{x}_k,\overline{\lambda}_k)\right\| \le M_2\,\|\mathbf{x}-\mathbf{x}_k\|^2$ for some constant M_2, noting that $\overline{\lambda}_k \le \overline{\lambda}_0$ for all k.

Therefore, from (3.25), we have

$$\mathbf{r}(\theta_k\mathbf{y}_k,\overline{\lambda}_k) = \mathbf{r}(\mathbf{x}_k,\overline{\lambda}_k) + \mathbf{J_x}(\mathbf{x}_k,\overline{\lambda}_k)(\theta_k\mathbf{y}_k - \mathbf{x}_k) + \mathbf{R}(\theta_k\mathbf{y}_k, \mathbf{x}_k,\overline{\lambda}_k).$$

Since $\mathbf{r}(\mathbf{x}_k,\overline{\lambda}_k) = \frac{1}{2}\mathbf{J_x r}(\mathbf{x}_k,\overline{\lambda}_k)\mathbf{x}_k$ by (2.12), we get

$$\begin{aligned}\mathbf{r}(\theta_k\mathbf{y}_k,\overline{\lambda}_k) &= -\mathbf{r}(\mathbf{x}_k,\overline{\lambda}_k) + \mathbf{J_x r}(\mathbf{x}_k,\overline{\lambda}_k)(\theta_k\mathbf{y}_k) + \mathbf{R}(\theta_k\mathbf{y}_k, \mathbf{x}_k,\overline{\lambda}_k)\\ &= -\mathbf{r}(\mathbf{x}_k,\overline{\lambda}_k) + \frac{\theta_k\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \mathbf{R}(\theta_k\mathbf{y}_k, \mathbf{x}_k,\overline{\lambda}_k).\end{aligned}$$

Hence, noting that $\mathbf{r}(\theta_k\mathbf{y}_k,\overline{\lambda}_k) = \theta_k^2\mathbf{r}\left(\mathbf{y}_k,\overline{\lambda}_k\right)$,

$$\begin{aligned}\mathbf{h}_k(\theta_k) &= \frac{\theta_k\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \theta_k^2\mathbf{r}\left(\mathbf{y}_k,\overline{\lambda}_k\right) + \mathbf{r}\left(\mathbf{x}_k,\overline{\lambda}_k\right)\\ &= \frac{2\theta_k\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \mathbf{R}(\theta_k\mathbf{y}_k, \mathbf{x}_k, \overline{\lambda}_k),\end{aligned}$$

which completes the proof. ∎

LEMMA 3.6. *For the NNI, we have the following.*

(i) $\theta_k = 1$ *if* $\|\mathbf{y}_k - \mathbf{x}_k\| \le \frac{\min(\mathbf{x}_k^{[2]})}{M_1 M_2}$, *where* M_1 *and* M_2 *are as in Lemmas* 3.4 *and* 3.5.

(ii) $\{\theta_k\}$ *is bounded below by some constant* $\xi > 0$, *assuming that* $\mathbf{x}_k \neq \mathbf{x}_*$ *for each* k.

Proof. (i) From Lemma 3.4 and assumption, we have

$$\frac{\mathbf{x}_k^{[2]}}{M_2\|\mathbf{w}_k\|\|\mathbf{y}_k - \mathbf{x}_k\|} \ge \frac{\min\left(\mathbf{x}_k^{[2]}\right)}{M_2\|\mathbf{w}_k\|\|\mathbf{y}_k - \mathbf{x}_k\|}\mathbf{e}$$

$$\ge \frac{\min(\mathbf{x}_k^{[2]})}{M_1 M_2}\mathbf{e} \ge \|\mathbf{y_k} - \mathbf{x_k}\|\,\mathbf{e}, \tag{3.26}$$

where $\mathbf{e} = [1, \ldots, 1]^T$. Then, from (3.26) and Lemma 3.5,

$$\frac{\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} \ge M_2\|\mathbf{y_k} - \mathbf{x_k}\|^2\mathbf{e} \ge \left|\mathbf{R}(\mathbf{y}_k, \mathbf{x}_k, \overline{\lambda}_k)\right|. \tag{3.27}$$

Substituting (3.27) into (3.24), we obtain

$$\mathbf{h}_k(1) = \frac{2\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} + \mathbf{R}(\mathbf{y}_k, \mathbf{x}_k, \overline{\lambda}_k) \ge \frac{\mathbf{x}_k^{[2]}}{\|\mathbf{w}_k\|} > \frac{\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|},$$

which means $\theta_k = 1$.

(ii) From (3.11), we recall that

$$\theta_k = \begin{cases} 1 & \text{if } \mathbf{h}_k(1) \ge \frac{\mathbf{x}_k^{[2]}}{(1+\eta)\|\mathbf{w}_k\|}, \\ \eta_k & \text{otherwise,} \end{cases}$$

where $\eta_k = \frac{\eta}{(1+\eta)\|\mathbf{w}_k\|(\mu_k - \overline{\lambda}_k)}\min(\frac{\mathbf{x}_k^{[2]}}{\mathbf{y}_k^{[2]}}) < 1$ with $\mu_k = \max(\frac{\mathcal{A}\mathbf{y}_k^2}{\mathbf{y}_k^{[2]}}) > \overline{\lambda}_k$. Suppose θ_k is not bounded below by $\xi > 0$. Since $\mathbf{x}_k$ is bounded, we can find a subsequence $\{k_j\}$ such that

$$\lim_{j\to\infty}\theta_{k_j} = 0, \quad \lim_{j\to\infty}\mathbf{x}_{k_j} =: \mathbf{v}. \tag{3.28}$$

Note that $\mathbf{v} > 0$ by Lemma 3.3.

As in the proof of Lemma 3.4, we have $\lim_{j\to\infty} 2\overline{\lambda}_{k_j} = 2\alpha \ge \rho(B(\mathbf{v}))$.

If $2\alpha > \rho(B(\mathbf{v}))$, then $\lim_{j\to\infty}\mathbf{w}_{k_j} =: \mathbf{w} > 0$, as in the proof of Lemma 3.4. In this case,

$$\lim_{j\to\infty}\mathbf{y}_{k_j} = \lim_{j\to\infty}\frac{\mathbf{w}_{k_j}}{\|\mathbf{w}_{k_j}\|} =: \mathbf{y} > 0.$$

If η_k is defined only on a finite subset of $\{k_j\}$, then $\theta_{k_j} = 1$ except for a finite number of j values, contradicting $\lim_{j\to\infty}\theta_{k_j} = 0$. If η_k is defined on an infinite subset $\{k_{j_i}\}$ of $\{k_j\}$, then

$$\lim_{i\to\infty}\eta_{k_{j_i}} = \frac{\eta}{(1+\eta)\|\mathbf{w}\|(\mu - \alpha)}\min\left(\frac{\mathbf{v}}{\mathbf{y}}\right)^2 > 0,$$

where $\lim_{i\to\infty} \mu_{k_{j_i}} = \mu > \alpha$ since $\eta_{k_{j_i}} < 1$. This is contradictory to $\lim_{j\to\infty} \theta_{k_j} = 0$. Thus $2\alpha = \rho(B(\mathbf{v}))$, and $\mathbf{v} = \mathbf{x}_*$ as in the proof of Lemma 3.4. Hence, $\lim_{j\to\infty} \mathbf{x}_{k_j} = \mathbf{x}_*$ and by Lemma 3.3 $\lim_{j\to\infty} \mathbf{y}_{k_j} = \mathbf{x}_*$, which means $\mathbf{x}_{k_j}$ and $\mathbf{y}_{k_j}$ are close enough for j sufficiently large. Therefore, from (i), $\theta_{k_j} = 1$ for j large enough, a contradiction to $\lim_{j\to\infty} \theta_{k_j} = 0$. ∎

This Lemma shows that if $\mathbf{x}_k$ and $\mathbf{y}_k$ are close enough, then the parameter θ_k in (3.2) can easily be determined, i.e., $\theta_k = 1$.

COROLLARY 3.7. *If* $\mathbf{x}_k = \mathbf{y}_k$ *for any* $k \geq 0$ *in Algorithm* 3.1, then $\mathbf{x}_k = \mathbf{x}_*$.

Proof. If $\mathbf{x}_k = \mathbf{y}_k$, then $\theta_k = 1$ by Lemma 3.6, and it is easily seen from Algorithm 3.1 that $\overline{\lambda}_{k+1} = \overline{\lambda}_k$. By Theorem 3.2, we have $\mathbf{x}_k = \mathbf{x}_*$. ∎

4. Convergence analysis.

In this section, we prove that the convergence of the NNI is global and quadratic, assuming that $\mathbf{x}_k \neq \mathbf{x}_*$ for each k.

4.1. Global convergence of the NNI.

Theorem 3.2 shows that the sequence $\{\overline{\lambda}_k\}$ is strictly decreasing and bounded below by $\rho(\mathcal{A})$, and hence converges. We now show that the limit of $\overline{\lambda}_k$ is precisely $\rho(\mathcal{A})$.

THEOREM 4.1. *Let* $\mathcal{A}$ *be an irreducible nonnegative third order tensor and the sequence* $\{\overline{\lambda}_k\}$ *is generated by Algorithm* 3.1. *Then the monotonically decreasing sequence* $\{\overline{\lambda}_k\}$ *converges to* $\rho(\mathcal{A})$, *and* $\{\mathbf{x}_k\}$ *from Algorithm* 3.1 *converges to the positive eigenvector* $\mathbf{x}_*$ *corresponding to* $\rho(\mathcal{A})$.

Proof. From (3.5), (3.17), and Lemma 3.6, we have

$$\begin{aligned} \overline{\lambda}_k - \overline{\lambda}_{k+1} &= \min\left(\frac{\mathbf{h}_k(\theta_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right) \geq \min\left(\frac{\theta_k \mathbf{x}_k^{[2]}}{(1+\eta)\,\|\mathbf{w}_k\|\,\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right) \\ &\geq \min\left(\frac{\xi \mathbf{x}_k^{[2]}}{(1+\eta)\,\|\mathbf{w}_k\|\,\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right). \end{aligned} \tag{4.1}$$

Since $\theta_k \leq 1$ by construction, we have $\|\widetilde{\mathbf{x}}_{k+1}\| = \|\mathbf{x}_k + \theta_k \mathbf{y}_k\| \leq 2$. It follows from (4.1) that $\lim_{k\to\infty} \|\mathbf{w}_k\|^{-1} \min(\mathbf{x}_k^{[2]}) = 0$.

Suppose $\min(\mathbf{x}_k^{[2]})$ is not bounded below by a positive constant. Then there exists a subsequence $\{k_j\}$ such that $\lim_{j\to\infty} \min(\mathbf{x}_{k_j}^{[2]}) = 0$. Since $\|\mathbf{x}_{k_j}\| = 1$, we may assume that $\lim_{j\to\infty} \mathbf{x}_{k_j} = \mathbf{v}$ exists. Then $\lim_{j\to\infty} \min(\mathbf{x}_{k_j}^{[2]}) = \min(\mathbf{v}^{[2]}) = 0$. This is a contradiction since $\mathbf{v} > 0$ by Lemma 3.3. Therefore, $\min(\mathbf{x}_k^{[2]})$ is bounded below by a positive constant, and thus $\lim_{k\to\infty} \|\mathbf{w}_k\|^{-1} = 0$.

Let $\mathbf{v}$ be any limit point of $\{\mathbf{x}_k\}$, with $\lim_{j\to\infty} \mathbf{x}_{k_j} = \mathbf{v}$. If $\lim_{j\to\infty} 2\overline{\lambda}_{k_j} > \rho(B(\mathbf{v}))$, then (as in the proof of Lemma 3.4) $\{\mathbf{w}_{k_j}\}$ is bounded, a contradiction. So $\lim_{j\to\infty} 2\overline{\lambda}_{k_j} = \rho(B(\mathbf{v}))$, which implies $\mathbf{v} = \mathbf{x}_*$, again as in the proof of Lemma 3.4. Therefore, $\mathbf{x}_*$ is the only limit point of the bounded sequence $\{\mathbf{x}_k\}$. Thus $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}_*$ and it follows that $\lim_{k\to\infty} \overline{\lambda}_k = \max\left(\frac{\mathcal{A}\mathbf{x}_*^2}{\mathbf{x}_*^{[2]}}\right) = \rho(\mathcal{A})$. ∎

4.2. Quadratic convergence of the NNI.

The proof of quadratic convergence uses a number of technical results in section 3 about the NNI. It also exploits a connection between the NNI and Newton's method. So we start with the following result about Newton's method.

THEOREM 4.2. *Let* $\mathbf{f}(\mathbf{x},\lambda)$ *be defined by* (2.8) *such that* $\mathbf{f}(\mathbf{x}_*,\rho(\mathcal{A}))=0$. *Then*
(i) $\mathbf{Jf}(\mathbf{x}_*,\rho(\mathcal{A}))$ *is nonsingular;*
(ii) $\mathbf{Jf}(\mathbf{x},\lambda)$ *satisfies a Lipschitz condition at* $(\mathbf{x}_*,\rho(\mathcal{A}))$.
Let $\left\{\mathbf{x}_k,\overline{\lambda}_k\right\}$ *be generated by the NNI. Then there is a constant* β *such that for all* $\left(\mathbf{x}_k,\overline{\lambda}_k\right)$ *sufficiently close to* $(\mathbf{x}_*,\rho(\mathcal{A}))$

$$\left\|\begin{bmatrix}\widehat{\mathbf{x}}_{k+1}\\ \widehat{\lambda}_{k+1}\end{bmatrix}-\begin{bmatrix}\mathbf{x}_*\\ \rho(\mathcal{A})\end{bmatrix}\right\|\le\beta\left\|\begin{bmatrix}\mathbf{x}_k\\ \overline{\lambda}_k\end{bmatrix}-\begin{bmatrix}\mathbf{x}_*\\ \rho(\mathcal{A})\end{bmatrix}\right\|^2, \tag{4.2}$$

where $\left\{\widehat{\mathbf{x}}_{k+1},\widehat{\lambda}_{k+1}\right\}$ *is generated by the Newton step* (2.13)–(2.15) *from* $\left\{\mathbf{x}_k,\overline{\lambda}_k\right\}$, *instead of* $\left\{\widehat{\mathbf{x}}_k,\widehat{\lambda}_k\right\}$.

Proof. (i) Recall that $\mathbf{Jf}(\mathbf{x},\lambda)$ is given by (2.9). Let $\left(\mathbf{z}^T,\zeta\right)^T\in\mathbb{R}^{n+1}$ be such that

$$0=\mathbf{Jf}(\mathbf{x}_*,\rho(\mathcal{A}))\begin{bmatrix}\mathbf{z}\\ \zeta\end{bmatrix}=\begin{bmatrix}-\mathbf{J_x r}(\mathbf{x}_*,\rho(\mathcal{A})) & -\mathbf{x}_*^{[2]}\\ -\mathbf{x}_*^T & 0\end{bmatrix}\begin{bmatrix}\mathbf{z}\\ \zeta\end{bmatrix}.$$

We need to show $\left(\mathbf{z}^T,\zeta\right)=0$. Since $\mathbf{J_x r}(\mathbf{x}_*,\rho(\mathcal{A}))$ is defined by (2.10), premultiplying both sides by $\operatorname{diag}(\,(\mathbf{x}_*)^T,1)^{-1}$ yields

$$0=\begin{bmatrix}B(\mathbf{x}_*)-2\rho(\mathcal{A})I & -\mathbf{x}_*\\ -\mathbf{x}_*^T & 0\end{bmatrix}\begin{bmatrix}\mathbf{z}\\ \zeta\end{bmatrix}. \tag{4.3}$$

Since $B(\mathbf{x}_*)$ is an irreducible nonnegative matrix by Lemma 2.4, we assume that $\mathbf{x}_\ell$ is the left Perron vector of $B(\mathbf{x}_*)$. Premultiplying the first equation in (4.3) by $\mathbf{x}_\ell^T$, we obtain

$$\mathbf{x}_\ell^T\left[B(\mathbf{x}_*)-2\rho(\mathcal{A})I\right]\mathbf{z}-\mathbf{x}_\ell^T\mathbf{x}_*\zeta=0.$$

Since $\mathbf{x}_\ell^T B(\mathbf{x}_*)=\mathbf{x}_\ell^T 2\rho(\mathcal{A})$ and $\mathbf{x}_\ell>0$, we get $\zeta=0$. The first equation of (4.3) then becomes

$$\left[B(\mathbf{x}_*)-2\rho(\mathcal{A})I\right]\mathbf{z}=0.$$

Then by the Perron–Frobenius theorem for a nonnegative irreducible matrix, $\mathbf{z}=s\mathbf{w}$ with $\mathbf{w}>0$. From the second equation of (4.3) we have $\mathbf{x}_*^T\mathbf{z}=0$. So $s=0$ and then $\mathbf{z}=0$. Hence $\left(\mathbf{z}^T,\zeta\right)=0$.

(ii) Let $\mathcal{N}$ be a neighborhood of $(\mathbf{x}_*,\rho(\mathcal{A}))$. From the definition of $\mathbf{Jf}(\mathbf{x},\lambda)$, for any $(\mathbf{x},\lambda)\in\mathcal{N}$, we have

$$\mathbf{Jf}(\mathbf{x},\lambda)-\mathbf{Jf}(\mathbf{x}_*,\rho(\mathcal{A}))=\begin{bmatrix}\mathbf{J_x r}(\mathbf{x}_*,\rho(\mathcal{A}))-\mathbf{J_x r}(\mathbf{x},\lambda) & \mathbf{x}_*^{[2]}-\mathbf{x}^{[2]}\\ \mathbf{x}_*^T-\mathbf{x}^T & 0\end{bmatrix}. \tag{4.4}$$

Direct computation yields

$$\begin{aligned}\mathbf{J_x r}(\mathbf{x}_*,\rho(\mathcal{A}))-\mathbf{J_x r}(\mathbf{x},\lambda)&=[2\rho(\mathcal{A})D(\mathbf{x}_*)-G(\mathbf{x}_*)]-[2\lambda D(\mathbf{x})-G(\mathbf{x})]\\ &=[G(\mathbf{x})-G(\mathbf{x}_*)]-[2\lambda D(\mathbf{x})-2\rho(\mathcal{A})D(\mathbf{x}_*)]\\ &=\mathbf{J_x r}(\mathbf{x}_*-\mathbf{x},\rho(\mathcal{A}))-2\left(\rho(\mathcal{A})-\lambda\right)D(\mathbf{x}).\end{aligned} \tag{4.5}$$

Substituting (4.5) into (4.4) and using basic properties of matrix and vector norms, we obtain the conclusion (ii): There is a constant κ such that

$$\|\mathbf{Jf}(\mathbf{x},\lambda)-\mathbf{Jf}(\mathbf{x}_*,\rho(\mathcal{A}))\|\le\kappa\left\|\begin{bmatrix}\mathbf{x}\\ \lambda\end{bmatrix}-\begin{bmatrix}\mathbf{x}_*\\ \rho(\mathcal{A})\end{bmatrix}\right\|\quad\text{for all }(\mathbf{x},\lambda)\in\mathcal{N}.$$

The inequality (4.2) is a basic result of Newton's method after (i) and (ii) have been verified; see [10, Theorem 5.1.2], for example. ∎

However, the inequality (4.2) itself does not imply

$$\left|\widehat{\lambda}_{k+1} - \rho(\mathcal{A})\right| \le c\left|\overline{\lambda}_k - \rho(\mathcal{A})\right|^2 \tag{4.6}$$

for some constant $c > 0$. We need to establish first a relationship between $\|\mathbf{x}_k - \mathbf{x}_*\|$ and $\left|\overline{\lambda}_k - \rho(\mathcal{A})\right|$.

THEOREM 4.3. *Let $\{\mathbf{x}_k, \overline{\lambda}_k\}$ be generated by the NNI. Then there are constants $c_1, c_2 > 0$ such that $c_1\|\mathbf{x}_k - \mathbf{x}_*\| \le \left|\overline{\lambda}_k - \rho(\mathcal{A})\right| \le c_2\|\mathbf{x}_k - \mathbf{x}_*\|$ for all $k \ge 0$.*

Proof. From (2.8) we have

$$\frac{\mathcal{A}\mathbf{x}^2}{\mathbf{x}^{[2]}} = \lambda\mathbf{e} - \frac{\mathbf{r}(\mathbf{x},\lambda)}{\mathbf{x}^{[2]}} = \rho(\mathcal{A})\mathbf{e} - \frac{\mathbf{r}(\mathbf{x},\rho(\mathcal{A}))}{\mathbf{x}^{[2]}}.$$

Thus the Fréchet derivative of $\frac{\mathcal{A}\mathbf{x}^2}{\mathbf{x}^{[2]}}$ is given by

$$-D(\mathbf{x})^{-2}\mathbf{J}_{\mathbf{x}}\mathbf{r}(\mathbf{x},\rho(\mathcal{A})) + 2D(\mathbf{x})^{-3}D(\mathbf{r}(\mathbf{x},\rho(\mathcal{A})).$$

Then by Taylor's theorem

$$\frac{\mathcal{A}\mathbf{x}_k^2}{\mathbf{x}_k^{[2]}} - \frac{\mathcal{A}\mathbf{x}_*^2}{\mathbf{x}_*^{[2]}} = -D(\mathbf{x}_*)^{-2}\mathbf{J}_{\mathbf{x}}\mathbf{r}(\mathbf{x}_*,\rho(\mathcal{A}))(\mathbf{x}_k - \mathbf{x}_*) + O(\|\mathbf{x}_k - \mathbf{x}_*\|^2).$$

Now

$$\begin{aligned}\left|\overline{\lambda}_k - \rho(\mathcal{A})\right| &= \max\left(\frac{\mathcal{A}\mathbf{x}_k^2}{\mathbf{x}_k^{[2]}} - \frac{\mathcal{A}\mathbf{x}_*^2}{\mathbf{x}_*^{[2]}}\right) \le \left\|\frac{\mathcal{A}\mathbf{x}_k^2}{\mathbf{x}_k^{[2]}} - \frac{\mathcal{A}\mathbf{x}_*^2}{\mathbf{x}_*^{[2]}}\right\| \\ &\le \left(\|D(\mathbf{x}_*)^{-2}\mathbf{J}_{\mathbf{x}}\mathbf{r}(\mathbf{x}_*,\rho(\mathcal{A}))\| + 1\right)\|\mathbf{x}_k - \mathbf{x}_*\|\end{aligned}$$

for k large enough, and the existence of c_2 follows readily.

On the other hand,

$$\begin{aligned}\left|\overline{\lambda}_k - \rho(\mathcal{A})\right| &= \max\left(\frac{\mathcal{A}\mathbf{x}_k^2}{\mathbf{x}_k^{[2]}} - \frac{\mathcal{A}\mathbf{x}_*^2}{\mathbf{x}_*^{[2]}}\right) \\ &\ge \max\left(-D(\mathbf{x}_*)^{-2}\mathbf{J}_{\mathbf{x}}\mathbf{r}(\mathbf{x}_*,\rho(\mathcal{A}))(\mathbf{x}_k - \mathbf{x}_*)\right) - c_3\|\mathbf{x}_k - \mathbf{x}_*\|^2\end{aligned}$$

for some $c_3 > 0$. Let $\mathbf{q}_k = (\mathbf{x}_k - \mathbf{x}_*) / \|\mathbf{x}_k - \mathbf{x}_*\|$ with $\|\mathbf{q}_k\| = 1$. Since $\mathbf{x}_k, \mathbf{x}_* > 0$ and $\|\mathbf{x}_k\| = \|\mathbf{x}_*\| = 1$, we know that $\mathbf{q}_k \not\le 0$ and $\mathbf{q}_k \not\ge 0$ for all k. To show the existence of c_1 in the theorem, we need to show that $\max\left(-D(\mathbf{x}_*)^{-2}\mathbf{J}_{\mathbf{x}}\mathbf{r}(\mathbf{x}_*,\rho(\mathcal{A}))\mathbf{q}_k\right) \ge c_4$ for some $c_4 > 0$. Suppose that such a constant c_4 does not exist. Then there is subsequence $\{k_j\}$ such that $\lim \mathbf{q}_{k_j} = \mathbf{q}$ with $\mathbf{q}$ neither positive nor negative, $\|\mathbf{q}\| = 1$, and $\max\left(-D(\mathbf{x}_*)^{-2}\mathbf{J}_{\mathbf{x}}\mathbf{r}(\mathbf{x}_*,\rho(\mathcal{A}))\mathbf{q}\right) = 0$. Now $\mathbf{J}_{\mathbf{x}}\mathbf{r}(\mathbf{x}_*,\rho(\mathcal{A}))\mathbf{q} = (2\rho(\mathcal{A})I - B(\mathbf{x}_*))\,\mathbf{q} \ge 0$. By Theorem 2.5 we have $\mathbf{q} = \pm\mathbf{x}_*$, a contradiction. ∎

We are now ready to prove the quadratic convergence of the NNI.

THEOREM 4.4. *Assume $\{\mathbf{x}_k, \overline{\lambda}_k\}$ is generated by the NNI. Then $\overline{\lambda}_k$ converges to $\rho(\mathcal{A})$ quadratically and $\mathbf{x}_k$ converges to $\mathbf{x}_*$ quadratically.*

Proof. Since the NNI has global convergence, we assume that $(\mathbf{x}_k, \overline{\lambda}_k)$ is sufficiently close to $(\mathbf{x}_*, \rho(\mathcal{A}))$. Let $\{\widehat{\mathbf{x}}_{k+1}, \widehat{\lambda}_{k+1}\}$ be generated by the Newton step (2.13)–(2.15) from $\{\mathbf{x}_k, \overline{\lambda}_k\}$, instead of $\{\widehat{\mathbf{x}}_k, \widehat{\lambda}_k\}$, and assume that (4.2) holds.

From (2.15), (2.17), and (3.1), we now have

$$\widehat{\lambda}_{k+1} = \overline{\lambda}_k + \delta_k = \overline{\lambda}_k - \frac{1}{2\mathbf{x}_k^T \mathbf{w}_k} = \overline{\lambda}_k - \frac{1}{2\left\|\mathbf{w}_k\right\| \mathbf{x}_k^T \mathbf{y}_k}$$

and then

$$\widehat{\lambda}_{k+1} - \rho(\mathcal{A}) = \overline{\lambda}_k - \rho(\mathcal{A}) - \frac{1}{2\left\|\mathbf{w}_k\right\| \mathbf{x}_k^T \mathbf{y}_k}.$$

By (4.2) and Theorem 4.3, the inequality (4.6) holds with $c = \beta(1 + 1/c_1^2)$. It follows that for $\varepsilon_k = \overline{\lambda}_k - \rho(\mathcal{A})$

$$\varepsilon_k - \frac{1}{2\left\|\mathbf{w}_k\right\| \mathbf{x}_k^T \mathbf{y}_k} = O(\varepsilon_k^2)$$

and then

$$\left\|\mathbf{w}_k\right\| = \frac{1}{2\mathbf{x}_k^T \mathbf{y}_k \varepsilon_k (1 - O(\varepsilon_k))}.$$

In particular, $\lim_{k\to\infty} \varepsilon_k \left\|\mathbf{w}_k\right\| = \frac{1}{2}$.

From (3.5), we have

$$\varepsilon_{k+1} = \varepsilon_k - \min\left(\frac{\mathbf{h}_k(\theta_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right). \tag{4.7}$$

By Theorem 4.1 and Lemmas 3.3 and 3.6, we have $\theta_k = 1$ for k large enough, and from (3.24) we have

$$\frac{\mathbf{h}_k(1)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}} = \frac{2\mathbf{x}_k^{[2]}}{\left\|\mathbf{w}_k\right\| \widetilde{\mathbf{x}}_{k+1}^{[2]}} + \frac{\mathbf{R}(\mathbf{y}_k, \mathbf{x}_k, \overline{\lambda}_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}.$$

Now for some j dependent on k

$$\varepsilon_{k+1} = \varepsilon_k - \min\frac{\mathbf{h}_k(1)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}} = \varepsilon_k - \frac{2(\mathbf{x}_k^{(j)})^2}{\left\|\mathbf{w}_k\right\| (\widetilde{\mathbf{x}}_{k+1}^{(j)})^2} - \frac{\mathbf{R}(\mathbf{y}_k, \mathbf{x}_k, \overline{\lambda}_k)^{(j)}}{(\widetilde{\mathbf{x}}_{k+1}^{(j)})^2}. \tag{4.8}$$

From Lemmas 3.5 and 3.4, it follows that

$$\left\|\mathbf{R}(\mathbf{y}_k, \mathbf{x}_k, \overline{\lambda}_k)\right\| \le M_2 \left\|\mathbf{y}_k - \mathbf{x}_k\right\|^2 \le M_2 M_1^2 \left\|\mathbf{w}_k\right\|^{-2}.$$

We then have

$$-\frac{\mathbf{R}(\mathbf{y}_k, \mathbf{x}_k, \overline{\lambda}_k)^{(j)}}{(\widetilde{\mathbf{x}}_{k+1}^{(j)})^2} \le \frac{\left\|\mathbf{R}(\mathbf{y}_k, \mathbf{x}_k, \overline{\lambda}_k)\right\|}{(\widetilde{\mathbf{x}}_{k+1}^{(j)})^2} \le \frac{4M_2 M_1^2}{3(\min x_*)^2}\varepsilon_k^2 \tag{4.9}$$

for k large enough. We also have

$$\varepsilon_k - \frac{2(\mathbf{x}_k^{(j)})^2}{\left\|\mathbf{w}_k\right\| (\widetilde{\mathbf{x}}_{k+1}^{(j)})^2} = \varepsilon_k - \frac{2(\mathbf{x}_k^{(j)})^2}{(\widetilde{\mathbf{x}}_{k+1}^{(j)})^2} 2\mathbf{x}_k^T \mathbf{y}_k \varepsilon_k (1 - O(\varepsilon_k)). \tag{4.10}$$

Note that $\|\mathbf{y}_k - \mathbf{x}_k\| \leq M_1 \|\mathbf{w}_k\|^{-1} \leq 3M_1\varepsilon_k$ for k large enough. Thus $\mathbf{x}_k^T\mathbf{y}_k = 1 + \mathbf{x}_k^T(\mathbf{y}_k - \mathbf{x}_k) = 1 + O(\varepsilon_k)$ and

$$\frac{4(\mathbf{x}_k^{(j)})^2}{(\widetilde{\mathbf{x}}_{k+1}^{(j)})^2} = \frac{(2\mathbf{x}_k^{(j)})^2}{(2\mathbf{x}_k^{(j)} + \mathbf{y}_k^{(j)} - \mathbf{x}_k^{(j)})^2} = \left(1 + \frac{1}{2\mathbf{x}_k^{(j)}}(\mathbf{y}_k - \mathbf{x}_k)^{(j)}\right)^{-2} = 1 + O(\varepsilon_k).$$

It follows from (4.10) that

$$\varepsilon_k - \frac{2(\mathbf{x}_k^{(j)})^2}{\|\mathbf{w}_k\| (\widetilde{\mathbf{x}}_{k+1}^{(j)})^2} = O(\varepsilon_k^2).$$

It then follows from (4.8) and (4.9) that $\varepsilon_{k+1} \leq d\varepsilon_k^2$ for some constant d. Thus $\overline{\lambda}_k$ converges to $\rho(\mathcal{A})$ quadratically. It follows from Theorem 4.3 that $\mathbf{x}_k$ converges to $\mathbf{x}_*$ quadratically. ∎

Since we use $\overline{\lambda}_k - \underline{\lambda}_k$ in the stopping criterion in Algorithm 3.1, the following result is also relevant.

THEOREM 4.5. *Assume* $\{\overline{\lambda}_k, \underline{\lambda}_k, \mathbf{x}_k\}$*is generated by the NNI. Then* $\overline{\lambda}_k - \underline{\lambda}_k$ *converges to* 0 *quadratically.*

Proof. From (3.10), we have

$$\underline{\lambda}_{k+1} = \overline{\lambda}_k - \max\left(\frac{\mathbf{h}_k(\theta_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right),$$

where $\underline{\lambda}_k = \min\left(\frac{\mathcal{A}\widetilde{\mathbf{x}}_k^2}{\widetilde{\mathbf{x}}_k^{[2]}}\right)$. Then

$$\begin{aligned}\overline{\lambda}_{k+1} - \underline{\lambda}_{k+1} &= \overline{\lambda}_{k+1} - \overline{\lambda}_k + \max\left(\frac{\mathbf{h}_k(\theta_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right) \\ &= \varepsilon_{k+1} - \left(\varepsilon_k - \max\left(\frac{\mathbf{h}_k(\theta_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right)\right).\end{aligned}$$

As in the proof of Theorem 4.4, we now have $\varepsilon_k - \max\left(\frac{\mathbf{h}_k(\theta_k)}{\widetilde{\mathbf{x}}_{k+1}^{[2]}}\right) = O(\varepsilon_k^2)$. Thus for some constant $c > 0$, $\overline{\lambda}_{k+1} - \underline{\lambda}_{k+1} \leq c\varepsilon_k^2 = c(\overline{\lambda}_k - \rho(\mathcal{A}))^2 \leq c(\overline{\lambda}_k - \underline{\lambda}_k)^2$. ∎

5. Numerical experiments.

In this section, we present some numerical results to support our theory for the NNI, and to illustrate its effectiveness. We compare the NNI with the NQZ method [16]. All numerical tests were performed on an Intel (R) Core (TM) i7 CPU 4770@ 3.4 GHz with 16 GB memory using Matlab R2013*a* with machine precision $\varepsilon = 2.22 \times 10^{-16}$ under Microsoft Windows 7 64-bit. Throughout the experiments, the initial vector is $\mathbf{x}_0 = \frac{1}{\sqrt{n}}[1, \ldots, 1]^T \in \mathbb{R}^n$, which is precisely the one used in [20] to prove the linear convergence of the NQZ algorithm. We also take $\eta = 0.1$ for the NNI. But we found that the choice of η has no significant effect on the performance of the NNI. For both methods, we terminate the iteration when one of the following conditions is satisfied:

1. $k \geq 10000$.
2. $\left(\overline{\lambda}_k - \underline{\lambda}_k\right)/\overline{\lambda}_k \leq 10^{-13}$, where $\underline{\lambda}_k = \min(\frac{\mathcal{A}\mathbf{x}_k^2}{\mathbf{x}_k^{[2]}})$.

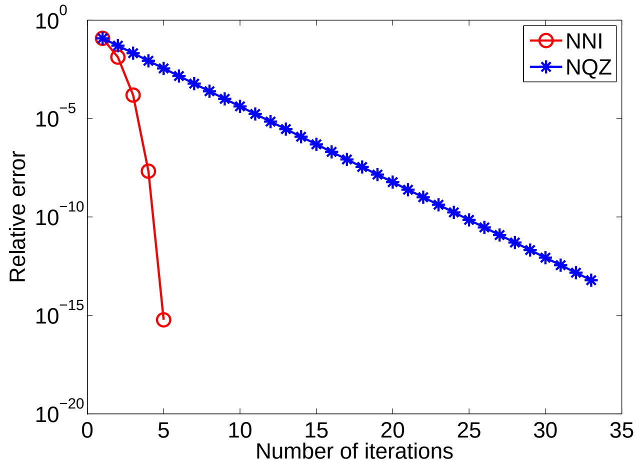

FIG. 1. *The relative error versus the number of iterations for Example* 1.

Note that $\left(\overline{\lambda}_k - \underline{\lambda}_k\right)/\overline{\lambda}_k$ is an upper bound for $\left(\overline{\lambda}_k - \rho\left(\mathcal{A}\right)\right)/\overline{\lambda}_k$. For simplicity, we call $\left(\overline{\lambda}_k - \underline{\lambda}_k\right)/\overline{\lambda}_k$ the relative error in step k.

We first apply the NNI and NQZ to compute the Perron pair of a transition probability tensor arising from a higher order Markov chain. A probability distribution of the higher order Markov chain is then obtained by normalizing the Perron vector to a positive vector with unit 1-norm [16].

EXAMPLE 1. *Consider the transition probability tensor $\mathcal{P}$ of order 3 and dimension 3 given by:*

$$\mathcal{P}(1,:,:) = \begin{bmatrix} 0.9000 & 0.6700 & 0.6604 \\ 0.3340 & 0.1040 & 0.0945 \\ 0.3106 & 0.0805 & 0.0710 \end{bmatrix},$$

$$\mathcal{P}(2,:,:) = \begin{bmatrix} 0.0690 & 0.2892 & 0.0716 \\ 0.6108 & 0.8310 & 0.6133 \\ 0.0754 & 0.2956 & 0.0780 \end{bmatrix},$$

$$\mathcal{P}(3,:,:) = \begin{bmatrix} 0.0310 & 0.0408 & 0.2680 \\ 0.0552 & 0.0650 & 0.2922 \\ 0.6140 & 0.6239 & 0.8510 \end{bmatrix}.$$

The data here are obtained from the occupational mobility of physicists data in [19].

For Example 1, Figure 1 depicts how the relative error evolves versus the number of iterations for the NQZ and NNI, respectively. It indicates that the NQZ converges linearly and the NNI converges quadratically. Note that the NQZ and NNI use 33 and 5 iterations, respectively, to achieve the desired accuracy.

We then apply the NNI and NQZ to compute the Perron pair of a perturbation of the third order n-dimensional signless Laplacian tensor [7, 8].

EXAMPLE 2. *Consider the third order n-dimensional signless Laplacian tensor $\mathcal{B} = \mathcal{D} + \mathcal{C}$ of a connected hypergraph* [7, 8]*, where $\mathcal{D}$ is the diagonal tensor with diagonal element $d_{i,i,i}$ equal to the degree of vertex i for each i, and $\mathcal{C}$ is the adjacency*

Table 1
Numerical results for Example 2(a).

Tensor $\mathcal{A}$	NNI		NQZ	
n	Iter	Err	Iter	Err
20	5	3.17e-15	37	9.64e-14
50	5	6.34e-15	38	8.79e-14
100	4	2.73e-14	38	6.53e-14
200	4	4.87e-14	37	3.22e-14

Table 2
Numerical results for Example 2(b).

Tensor $\mathcal{A}$	NNI		NQZ	
n	Iter	Err	Iter	Err
20	8	2.56e-15	131	9.86e-14
50	9	6.63e-15	513	9.50e-14
100	10	2.81e-14	1313	9.82e-14
200	11	4.23e-14	3033	9.54e-14

tensor defined in [4, 7, 8]. *Let* $E_1 = \{(i, j, j+1)\}$ *for* $i = 1, 2, 3$ *and* $j = i+1, \ldots, n-1$. *We consider two hypergraphs:*

(a) *The edge set of the hypergraph is given by* $E \setminus E_1$, *where* E *is the edge set of the complete* 3*-uniform hypergraph* [7, 8].

(b) *The edge set of the hypergraph is* E_1 *itself.*

Since the tensor $\mathcal{B}$ is reducible, we follow the common approach (see [21], for example) of obtaining a nearby irreducible tensor by letting $\mathcal{A} = \mathcal{B} + 10^{-8}\mathcal{E}$, where $\mathcal{E}$ is the tensor with all entries equal to 1, and then apply the NNI and NQZ to the irreducible nonnegative tensor $\mathcal{A}$.

Tables 1 and 2 report the results obtained by the NQZ and NNI, for Examples 2(a) and 2(b), respectively. In the tables, n specifies the dimension, "Iter"denotes the number of iterations to achieve convergence, "Err"denotes the relative error when the iterative methods are terminated. From the tables, we see that the number of iterations for the NNI is at most 11, clearly indicating its quadratic convergence.

6. Conclusion. We have presented an efficient method for computing the Perron pair of an irreducible nonnegative third order tensor, by combining the idea of Newton's method with the idea of the Noda iteration, and we have called it an NNI. The iterative method has several very nice features: It is positivity preserving in its computation of the positive Perron vector, and its convergence is global and quadratic. The structure of the new algorithm is still very simple, although its convergence analysis is rather involved for the third order tensor. We are currently working on the more challenging problem of designing a proper NNI for higher-order tensors and providing a rigorous convergence analysis.

REFERENCES

[1] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Classics Appl. Math. 9, SIAM, Philadelphia, 1994.

[2] K.-C. Chang, K. J. Pearson, and T. Zhang, *Primitivity, the convergence of the NQZ method, and the largest eigenvalue for nonnegative tensors*, SIAM J. Matrix Anal. Appl., 32 (2011), pp. 806–819.

[3] K. C. Chang, K. Pearson, and T. Zhang, *Perron–Frobenius theorem for nonnegative tensors*, Commun. Math. Sci., 6 (2008), pp. 507–520.

[4] J. Cooper and A. Dutle, *Spectra of uniform hypergraphs*, Linear Algebra Appl., 436 (2012), pp. 3268–3292.

[5] L. Elsner, *Inverse iteration for calculating the spectral radius of a non-negative irreducible matrix*, Linear Algebra Appl., 15 (1976), pp. 235–242.

[6] W. K. Grassmann, M. J. Taksar, and D. P. Heyman, *Regenerative analysis and steady-state distributions for Markov chains*, Oper. Res., 33 (1985), pp. 1107–1116.

[7] S. Hu and L. Qi, *The Laplacian of a uniform hypergraph*, J. Comb. Optim., 29 (2015), pp. 331–366.

[8] S. Hu, L. Qi, and J. Xie, *The largest Laplacian and signless Laplacian H-eigenvalues of a uniform hypergraph*, Linear Algebra Appl., 469 (2015), pp. 1–27.

[9] Z. Jia, W.-W. Lin, and C.-S. Liu, *A positivity preserving inexact Noda iteration for computing the smallest eigenpair of a large irreducible M-matrix*, Numer. Math., 130 (2015), pp. 645–679.

[10] C. T. Kelley, *Iterative Methods for Linear and Nonlinear Equations*, Front. Appl. Math. 16, SIAM, Philadelphia, 1995.

[11] M. E. Kilmer, K. Braman, N. Hao, and R. C. Hoover, *Third-order tensors as operators on matrices: A theoretical and computational framework with applications in imaging*, SIAM J. Matrix Anal. Appl., 34 (2013), pp. 148–172.

[12] M. E. Kilmer and C. D. Martin, *Factorization strategies for third-order tensors*, Linear Algebra Appl., 435 (2011), pp. 641–658.

[13] T. G. Kolda and B. W. Bader, *Tensor decompositions and applications*, SIAM Rev., 51 (2009), pp. 455–500.

[14] Y. Liu, G. Zhou, and N. F. Ibrahim, *An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor*, J. Comput. Appl. Math., 235 (2010), pp. 286–292.

[15] Q. Ni and L. Qi, *A quadratically convergent algorithm for finding the largest eigenvalue of a nonnegative homogeneous polynomial map*, J. Global Optim., 61 (2015), pp. 627–641.

[16] M. Ng, L. Qi, and G. Zhou, *Finding the largest eigenvalue of a nonnegative tensor*, SIAM J. Matrix Anal. Appl., 31 (2009), pp. 1090–1099.

[17] T. Noda, *Note on the computation of the maximal eigenvalue of a non-negative irreducible matrix*, Numer. Math., 17 (1971), pp. 382–386.

[18] L. Qi, F. Wang, and Y. Wang, *Z-eigenvalue methods for a global polynomial optimization problem*, Math. Program., 118 (2009), pp. 301–316.

[19] A. E. Raftery, *A model of high-order Markov chains*, J. R. Stat. Soc. Ser. B. Stat. Methodol., 47 (1985), pp. 528–539.

[20] L. Zhang and L. Qi, *Linear convergence of an algorithm for computing the largest eigenvalue of a nonnegative tensor*, Numer. Linear Algebra Appl., 19 (2012), pp. 830–841.

[21] L. Zhang, L. Qi, and G. Zhou, *M-tensors and some applications*, SIAM J. Matrix Anal. Appl., 35 (2014), pp. 437–452.