High-temperature ratchets with sawtooth potentials

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The concept of the effective potential is suggested as an efficient instrument to get a uniform analytical description of stochastic high-temperature on-off flashing and rocking ratchets. The analytical representation for the average particle velocity, obtained within this technique, allows description of ratchets with sharp potentials (and potentials with jumps in particular). For sawtooth potentials, the explicit analytical expressions for the average velocity of on-off flashing and rocking ratchets valid for arbitrary frequencies of potential energy fluctuations are derived; the difference in their high-frequency asymptotics is explored for the smooth and cusped profiles, and profiles with jumps. The origin of the difference as well as the appearance of the jump behavior in ratchet characteristics are interpreted in terms of self-similar universal solutions which give the continuous description of the effect. It is shown how the jump behavior in motor characteristics arises from the competition between the characteristic times of the system.

I. INTRODUCTION

In Brownian ratchet theory, the chief and most commonly discussed dependencies are those of an average particle velocity on the frequency of potential energy fluctuations [1]. They usually contain a particular point with a different type of curve behavior to the left and right of it (for rocking ratchets) or a point of maximum (for flashing ratchets), at frequency values in the neighborhood of an inverse characteristic time of the system [2]. These frequency dependencies are very informative for both optimization of ratchet characteristics and understanding physical processes which occur at fluctuations [3–16]. It is evident that cases when such dependencies can be interpreted analytically are very uncommon and, for this reason, of great value. In the present paper, using the high-temperature approach developed in Ref. [17], we derive the explicit uniform analytical expressions for the average velocity of on-off flashing and rocking ratchets which are valid for arbitrary frequencies of potential energy fluctuations. We also explore the effect of large gradients in a potential profile on motion characteristics of a Brownian particle dichotomically driven by a fluctuating sawtooth potential, with a special emphasis on the limiting case of its extremely asymmetric shape, arising with jumps in the potential profile. The analysis of this situation is of particular relevance because the dynamics of a Brownian particle differs essentially in sharp and smooth potentials (with and without jumps, respectively) [10,12,18,19].

It is important to note that a sawtooth potential, though quite simple, plays a prominent role in ratchet theory (see, e.g., Refs. [1–5,7,10,12,18,19]) due to several circumstances. First, only two parameters are enough to characterize its shape: the energy barrier V and the asymmetry coefficient $\kappa = 1 - 2l/L$ (where l is the sawtooth length, $0 < l < L$, and L is the spatial period of the potential). These parameters determine the two main ratchet features: its ability to overcome potential barrier depending on the thermal energy $k_B T$ (k_B is the Boltzmann constant and T is the absolute temperature) and to distinguish right and left directions. Second, with a certain degree of accuracy, a real potential can be approximated by a sawtooth one (see, e.g., Ref. [18]) so that characteristics of any arbitrary Brownian motor can be estimated by means of $V/k_B T$ and κ. Third, a sawtooth potential is described by a piecewise linear function, that leads to a significant simplification of numerical calculations (reducing initial differential equations to the system of linear algebraic equations [2] or using the transfer matrix method [10,12]) or allows to get the resulting analytical expressions if some approximations are applied (as, e.g., the above-mentioned high-temperature one, [17]). And lastly, at present, a sawtooth potential, being "responsible" for ratchet effect, is not only a theoretical idealization, but can be realized experimentally. Here we exemplify, following Ref. [15], such a realization by the experimental scheme of a Brownian ratchet that manipulates charged components within supported lipid bilayers. One side of the patterned bilayers was of a sawtooth shape (a planar surface was the opposite side), the asymmetry of which controlled the amount of the effect. Particularly, the maximum effect was reached for the extremely asymmetric shape of the sawtooth side of the pattern. The same regularity is analyzed in detail in the present paper.

With this insight in mind, we must realize the subtleties in ratchet behavior arising from peculiarities of the potential energy profile, notably from the presence of cusp points in a sawtooth one and of jumps in an extremely asymmetric variant of it. Referring to Refs. [10,12] for more details, note that these subtleties come from the competition between the characteristic times of the system, and the most effective regimes of

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ratchet operation can be reached just as a result of it. It is noteworthy that the instantaneous potential switching, used in the majority of two-state ratchet models, also exemplifies the situation when such a peculiarity, but of the time dependence of the potential energy, leads to the most effective regimes of ratchet operation [13]. So, there is considerable generality in ratchets' behavior arising just from jumps, in coordinate or time dependence of the potential energy. From this point of view, the self-similar functions, obtained in the present paper, which give the continuous description of the effects resulting from the jumplike changes of the potential energy, are particularly important.

The structure of the paper is as follows. In Sec. II we formulate the stochastic model for the overdamped Brownian motion of a particle suitable for uniform description of ratchets of different types (on-off flashing and rocking). Then we introduce the basic quantities of interest as well as the chief mathematical tool, namely, the effective potential depending on the fluctuation frequency of the initial potential. This concept makes possible an analytical calculation of the particle current and average velocity for the high-temperature stochastic (symmetric and dichotomic fluctuations) Brownian motors, applicable for any potential profile and fluctuation frequencies. The expressions for the particle average velocity, containing the effective potential, are the first main result of the paper. Thus we have an instrument, especially effective for potentials with large gradients, for analysis of what happens at competition between characteristic times of the system. In Sec. III we make some points concerning the high-frequency behavior of ratchet systems with smooth potentials, pursuing the aim to indicate the difference between the rocking and flashing ratchet types in frequency asymptotics of the average velocity, and to be prepared to consider sharp (sawtooth) potentials.

Section IV is devoted to the exploration of ratchets with a sawtooth potential of arbitrary asymmetry, based on the technique developed in Sec. II. The increasing of the degree of smoothness of the functions being responsible for the ratchet effect, in terms of the effective potential, is very essential at this point. The main result of this section is the explicit analytical expressions for the average particle velocity of both on-off flashing and rocking ratchet. The frequency dependence of the average velocity as well as its dependence on the asymmetry coefficient is discussed, for both ratchet models, emphasizing the different sensitivity of ratchets to these essential parameters and to the presence of cusp points in the potential profile. The discussion in Sec. V gives, analytically and graphically, the physical interpretation of the origin of jumps in parameters of ratchets with sharp potentials and the continuous analytical description, in terms of self-similar functions, of competition of space and frequency limits. The results are summarized in Sec. VI.

II. MAIN EQUATIONS

Let us consider the overdamped motion of a Brownian particle in a one-dimensional fluctuating potential, $U(x,t) = u(x) + \sigma(t)w(x)$, where x and t denote coordinate and time, respectively, and the function $\sigma(t)$ takes only two values ± 1 and describes a dichotomic process with the rate constants γ_+

and γ_- for direct and reverse transitions. The particle dynamics is determined by the Langevin equation:

$$\zeta \dot{x} = -U'(x,t) + \xi(t). \tag{1}$$

Here we denote the time and coordinate derivative by the dotted and primed symbol, respectively; ζ is the friction coefficient; and $\xi(t)$ is the random force (zero-mean Gaussian white noise with the correlation function $\langle \xi(t)\xi(s) \rangle = 2\zeta k_B T \delta(t-s)$, where k_B is the Boltzmann constant, T is the absolute temperature, and $\delta(t)$ is the delta function).

As the particle can stay only in two states, with the potential profiles $U_\pm(x) = u(x) \pm w(x)$, its dynamics can be also described in terms of the distribution functions $\rho_\pm(x,t)$ [the probability densities to find the particle with the potential energy $U_\pm(x)$ at a point x in a time t] which obey the Smoluchowski equation with an additional term accounting for the particle random interstate transitions [10,20,21]:

$$\frac{\partial}{\partial t}\rho_\pm(x,t) = -\frac{\partial}{\partial x}J_\pm(x,t) \mp [\gamma_+\rho_+(x,t) - \gamma_-\rho_-(x,t)],$$

$$J_\pm(x,t) = -D\frac{\partial}{\partial x}\rho_\pm(x,t) - \beta D U'_\pm(x)\rho_\pm(x,t), \tag{2}$$

where $D = k_B T/\zeta$ is the diffusion coefficient and $\beta = (k_B T)^{-1}$ is the inverse temperature in energy units. In order to get universal results for both on-off flashing and rocking ratchets, we take $w(x) = u(x) \equiv V(x)/2$ for the former ratchet type and $u(x) = V(x)$, $w(x) = Fx$ for the latter one, where $V(x)$ is a spatially periodic function [$V(x+L) = V(x)$; L is the period] and F is the amplitude of the fluctuating force. Thus in our model both ratchet types operate in the same potential profile $V(x)$: It switches on and off for the on-off flashing ratchets and is modified by $\pm Fx$ for the rocking ratchets. With $U'_\pm(x)$ being spatially periodic functions for both ratchet types, the functions $\rho_\pm(x,t)$ are the so-called reduced probability densities, introduced in Ref. [1], which obey the normalization condition $\int_0^L [\rho_+(x,t) + \rho_-(x,t)]dx = 1$, and $J_\pm(x,t)$ are the corresponding reduced probability currents.

For stationary processes [$\partial\rho_\pm(x,t)/\partial t = 0$], as follows from Eq. (2), the sum of the fluxes $J_\pm(x)$ is x independent, and the average velocity of the particle directed motion is given by the relation

$$\langle v \rangle = L[J_+(x) + J_-(x)]. \tag{3}$$

Within the model defined by Eq. (2), the explicit analytical expressions of the average velocity for both ratchet types were derived in Ref. [17], for the case of asymmetric dichotomic processes, using the perturbation theory on small βV_0 [V_0 is the difference between the highest maximum and the lowest minimum of $V(x)$] and βFL (for rocking ratchets). In the present paper, we consider the symmetric dichotomic processes for which the rate constants γ_+ and γ_- are equal to each other, so that the average period of the process (the inverse fluctuation frequency) is $\tau = 2/\gamma_+ = 4/\Gamma$, where Γ is the inverse correlation time entering into the exponential correlation function of dichotomic processes, $\langle \sigma(t)\sigma(t') \rangle = \exp(-\Gamma|t-t'|)$. The mentioned high-temperature expressions

of Ref. [17], adapted for this case, take the form

$$\langle v \rangle_f = \frac{L}{\tau}\Phi_f, \quad \Phi_f = \frac{i}{2L}\beta^3 D^2$$
$$\times \sum_{\substack{qq'(\neq 0) \\ (q+q'\neq 0)}} \frac{k_{q'}k_q k_{q+q'} V_{q'} V_q V_{-q-q'}}{\left(\Gamma + Dk_q^2\right)\left(\Gamma + Dk_{q+q'}^2\right)}, \qquad (4)$$

$$\langle v \rangle_r = -\beta L \zeta^{-1} F^2 \Phi_r, \quad \Phi_r = 2\Phi_f + \Delta\Phi_r,$$
$$\Delta\Phi_r = \frac{2i}{L}\beta^3 D \sum_{\substack{qq'(\neq 0) \\ (q+q'\neq 0)}} \frac{k_{q'} V_{q'} V_q V_{-q-q'}}{\Gamma + Dk_{q+q'}^2}. \qquad (5)$$

Here and hereafter, the indices f and r label the velocities and auxiliary quantities corresponding to the flashing and rocking ratchets, and V_q are the Fourier components of $V(x)$,

$$V(x) = \sum_q V_q \exp(ik_q x), \quad V_q = \frac{1}{L}\int_0^L dx\, V(x)\exp(-ik_q x),$$

$$k_q = \frac{2\pi}{L}q, \quad q = 0, \pm 1, \pm 2, \dots. \qquad (6)$$

Therefore, the average particle velocity for both ratchet types is expressed in a uniform manner via the introduction of the two auxiliary quantities Φ_f and Φ_r. The meaning of Φ_f is the integrated current, which has been widely studied in the theory of flashing ratchets [22], and the quantity Φ_r determines the average velocity of rocking ratchets.

The Fourier representation given by Eqs. (4) and (5) is convenient for obtaining the explicit expressions for the average velocity of ratchets with relatively uncomplicated smooth potentials, such as, for instance, two-sinusoidal potential, commonly used in ratchet theory, since the summation over q and q' in this case is limited to the values $\pm 1, \pm 2$ [17]. For the general case of arbitrary potentials, especially for sharp ones with Fourier components slowly decreasing with increasing q values, it is reasonable to take advantage of the coordinate representation (which simplifies the calculation procedure by means of replacing summation by integration).

We now come to one of the main points of our consideration: The structure of expressions (4) and (5) suggests the idea of introducing the so-called effective potential which we will define by the following relation:

$$\tilde{V}(x) = \Gamma \sum_q \frac{V_q}{\Gamma + Dk_q^2}\exp(ik_q x)$$

$$= \frac{\Gamma}{L}\int_0^L dy\, V(y)\sum_q \frac{\exp[ik_q(x-y)]}{\Gamma + Dk_q^2}$$

$$= \frac{\Gamma}{L}\int_0^L dy\, G(x-y)V(y). \qquad (7)$$

In ratchet theory, the term "effective potential" was used in different meanings: as equivalent of a tilted potential [1]; as a tool to describe particle interactions in ratchet effects [23] or to reduce the two-dimensional problem of coupled Brownian motors to a one-dimensional one [24]; as equivalent of the multistate potentials in the general multistate fluctuation model [25], etc. Unlike the other authors, we use the term "effective potential" for the function $\tilde{V}(x)$, which is in

fact a result of a certain averaging of the initial potential $V(x)$, because this formally introduced quantity describes transformation of the initial potential profile into one that depends on the frequency of fluctuations of $V(x)$; that is, it "includes" a source of driving force, namely, fluctuations of a potential profile. The effective potential coincides with the initial one if this frequency tends to infinity. The Fourier components of the effective and initial potentials are simply related: $\tilde{V}_q = [\Gamma/(\Gamma + Dk_q^2)]V_q$.

The function $G(x-y)$ entering in Eq. (7) is the Green's function of the equation

$$\left(-D\frac{\partial^2}{\partial x^2} + \Gamma\right)G(x-y) = L\delta(x-y), \qquad (8)$$

satisfying the periodic boundary conditions

$$G(L-y) = G(-y), \quad \left.\frac{\partial G(x-y)}{\partial x}\right|_{x=0} = \left.\frac{\partial G(x-y)}{\partial x}\right|_{x=L}. \qquad (9)$$

This function can be written in the form

$$G(x-y) = \sum_q \frac{\exp[ik_q(x-y)]}{\Gamma + Dk_q^2}$$

$$= \frac{z\cosh[z(1-2|x-y|/L)]}{\Gamma \sinh z}, \qquad (10)$$

$$z = \frac{L}{\sqrt{D\tau}},$$

where x and y are assumed to belong to the interval $[0, L]$. One can also say that $G(x-y)$ is the Laplace representation $\tilde{G}(x-y,s) = \int_0^\infty dt\, G(x-y,t)\exp(-st)$ of the Green's function for free diffusion in the interval $[0, L]$, with the periodic boundary conditions, at $s = \Gamma$, since the process considered is characterized by the exponential correlation function $\langle \sigma(t)\sigma(t')\rangle = \exp(-\Gamma|t-t'|)$. The Green's function $G(x-y)$ satisfies the obvious properties

$$\int_0^L dx\, G(x-y) = L/\Gamma, \quad \Gamma G(x-y) \underset{\Gamma\to\infty}{\to} L\delta(x-y), \quad (11)$$

which follow from Eqs. (8)–(10) and will be used below in Sec. III.

Further, using the definition of the effective potential given by Eq. (7) and the auxiliary identities following from it,

$$\frac{d}{dx}\tilde{V}(x) = \frac{\Gamma}{L}\int_0^L dy\, V(y)\frac{d}{dx}G(x-y)$$

$$= -\frac{\Gamma}{L}\int_0^L dy\, V(y)\frac{d}{dy}G(x-y)$$

$$= \frac{\Gamma}{L}\int_0^L dy\, G(x-y)\frac{d}{dy}V(y) \qquad (12)$$

$$= \tilde{V}'(x)$$

$$2\int_0^L dx\, \tilde{V}(x)V(x)V'(x) = -\int_0^L dx\, [V(x)]^2\tilde{V}'(x),$$

we can write the expressions for Φ_f and $\Delta\Phi_r$ in Eqs. (4) and (5) in the following integral form:

$$\Phi_f = \frac{\beta^3 D^2}{2L^2\Gamma^2} \int_0^L dx\, V'(x)[\tilde{V}'(x)]^2,$$

$$\Delta\Phi_r = -\frac{\beta^3 D}{L^2\Gamma} \int_0^L dx\, V^2(x)\tilde{V}'(x). \tag{13}$$

Thus, while in Eqs. (4) and (5) the frequency Γ "sits" inside the sum, in Eq. (13) the main frequency "load" is carried by the function $\tilde{V}'(x)$. The latter means that just the effective potential takes fluctuations into account: The fluctuations of initial potential or tilting force renormalize the initial potential to the effective one. So, we can describe this renormalization, and the quantities of interest are determined by the effective potential. The advantages of this representation will become evident from further consideration.

III. HIGH-FREQUENCY BEHAVIOR FOR A SMOOTH POTENTIAL

The approximation of the high frequency of potential energy fluctuations has been successfully applied in ratchet theory many times [1,26–29], yielding analytical expressions of ratchet characteristics and elucidating the laws under which the frequency dependencies of the average velocity of directed motion drop to zero. The results argue that if the residence time (the inverse frequency) is the smallest time parameter of the system, the velocity tends to zero as the frequency tends to infinity [1]. The presence of sharp links in the potential profile exemplifies a quite unconventional situation when the smallest time parameter is the sliding time tending to zero for jumplike links of the potential. Owing to this fact, the nonzero asymptotics of the flashing ratchet average velocity appears [10]. In the present section, using Eq. (13), we rederive and reanalyze the known high-frequency expressions for average velocity of on-off flashing and rocking ratchets for the case of a smooth potential to lay the groundwork for comparison with the results obtained in this paper for sharp potentials (see Sec. IV).

For a smooth potential $V(x)$ with Fourier components V_q decreasing sufficiently rapidly with increasing q, the product Dk_q^2 in the equations of Sec. II can be neglected in comparison with the quantity Γ if the latter is large enough (high-frequency case). Then, in accordance with Eqs. (7) and (11), we have $\tilde{V}'(x) \approx V'(x)$, and Φ_f in Eq. (13) behaves as Γ^{-2}, so that in the high-frequency limit the average velocity of the on-off flashing ratchet with a smooth potential is determined by the expression

$$\langle v \rangle_f \underset{\tau \to 0}{=} \tau \frac{D^2\beta^3}{32L} \int_0^L dx\, [V'(x)]^3, \tag{14}$$

coinciding with the known one [26,27] [see also Eq. (4.10) in Ref. [1]] in the high-temperature case and tending to zero as τ at $\tau \to 0$.

Further, using two first terms of expansion for the first derivative of the effective potential,

$$\tilde{V}'(x) = V'(x) + (D/\Gamma)V'''(x) + O(\Gamma^{-2}) \tag{15}$$

[where $O(z)$ designates the terms of order z], we have, after integration by parts, the average velocity of the rocking ratchet with a smooth potential:

$$\langle v \rangle_r \underset{\tau \to 0}{=} -\tau^3 \frac{D^4\beta^5 F^2}{64L} \int_0^L dx\, V'(x)[V''(x)]^2, \tag{16}$$

which is, as it should be, in agreement with known formulas (see Refs. [28,29] and Eq. (5.18) in Ref. [1]) rewritten for the high-temperature case. Unlike the result (14), the average velocity (16) tends to zero as τ^3 at $\tau \to 0$ and depends on the second derivative $V''(x)$ of the initial potential. Thus the high-frequency limit for rocking ratchets is more sensitive to the peculiarities of the potential profile than that for flashing ratchets. As we will demonstrate in Sec. IV, a sawtooth potential is a particularly telling illustration of this sensitivity.

Note that the use of the low-frequency limit in its turn does not require any restrictions on the smoothness of the potential. Moreover, we can write the uniform explicit analytical formulas for the average velocity of on-off flashing and rocking ratchets applicable to any temperatures (keeping βFL small for rocking ratchets) [18],

$$\langle v \rangle_f = \frac{L}{\tau}\Phi_f, \quad \langle v \rangle_r = -\beta LF^2\mu\Phi_f + O[(\beta FL)^4],$$

$$\Phi_f = \int_0^L dx[\rho_+(x) - L^{-1}] \int_0^x dy[\rho_-(y) - L^{-1}],$$

$$\rho_\pm(x) = \exp[\pm\beta V(x)]/\int_0^L dx\, \exp[\pm\beta V(x)], \tag{17}$$

$$\mu = \zeta^{-1}L^2\left\{\int_0^L dx\, \exp[\beta V(x)]\int_0^L dx\, \exp[-\beta V(x)]\right\}^{-1}.$$

One makes sure that Eqs. (4), (5), and (13) can be reduced to the high-temperature limit of Eq. (17) since $\Delta\Phi_r \to -\Phi_f$ and $\Phi_r \to \Phi_f$ at $\Gamma \to 0$. It is noteworthy that the uniform representation given by the first line of Eq. (17) remains true even with allowance for small inertial corrections [18,19].

IV. SAWTOOTH POTENTIAL

Now we come to the next main point of our consideration, namely, the application of the concept of the effective potential to the special case of a sawtooth potential which is the best object to exemplify the advantages of this technique.

Let us define a periodic sawtooth potential and its first derivative by the following functions in the interval $[0, L]$ (L is the period):

$$V(x) = \begin{cases} V_0 x/l, & 0 < x < l \\ V_0(L-x)/(L-l), & l < x < L \end{cases}, \quad V'(x) = \begin{cases} V_0/l, & 0 < x < l \\ -V_0/(L-l), & l < x < L \end{cases}. \tag{18}$$

The first derivative of the corresponding effective potential can be easily calculated in accordance with Eqs. (7) and (10):

$$\tilde{V}'(x) = \begin{cases} \frac{V_0}{l} - \frac{V_0 L}{l(L-l)\sinh z} \sinh[z(L-l)/L] \cosh[z(l-2x)/L], & 0 < x < l \\ -\frac{V_0}{L-l} + \frac{V_0 L}{l(L-l)\sinh z} \sinh(zl/L) \cosh[z(L+l-2x)/L], & l < x < L \end{cases}. \qquad (19)$$

The function $\tilde{V}'(x)$ is depicted in Fig. 1. One can make sure that at $l \neq 0$, L and $z \to \infty$ this function tends to the stepwise one $V'(x)$ given by Eq. (18). And on the contrary, with the decrease of z, the derivative $\tilde{V}'(x)$ becomes smoother and at $z \to 0$ takes the uniform form:

$$\frac{L}{V_0 z^2} \tilde{V}'(x) = \begin{cases} \frac{1}{3}\left(1 - \frac{l}{2L}\right) - \frac{2}{lL}\left(x - \frac{l}{2}\right)^2, & 0 < x < l \\ -\frac{1}{6}\left(1 + \frac{l}{L}\right) + \frac{2}{lL}\left(x - \frac{L+l}{2}\right)^2, & l < x < L \end{cases} \qquad (20)$$

(see the frame in Fig. 1). Thus, while the function $V'(x)$ itself undergoes discontinuities at the points $x = 0, l, L$, the "effective" function $\tilde{V}'(x)$ has a discontinuous second derivative at the same points (the degree of smoothness becomes higher), so, we understand how the effective potential changes relative to the initial one.

The explicit representation of the first derivative of the effective potential allows integration in Eq. (13), so we obtain the following final results:

$$\langle v \rangle_f = \frac{L}{\tau}\Phi_f, \quad \langle v \rangle_r = -\beta L \zeta^{-1} F^2 \Phi_r, \quad \Phi_r = 2\Phi_f + \Delta\Phi_r,$$

$$\Phi_f = \frac{(\xi' - \xi)(\beta V_0)^3}{128(\xi'\xi z^2)^2}[6f_1(z,\xi) - 3f_2(z,\xi) + f_1(z,\xi)f_2(z,\xi)],$$

$$\Delta\Phi_r = -\frac{(\xi' - \xi)(\beta V_0)^3}{16(\xi'\xi z^2)^2}[2f_1(z,\xi) - f_2(z,\xi)], \quad \xi = \frac{l}{L},$$
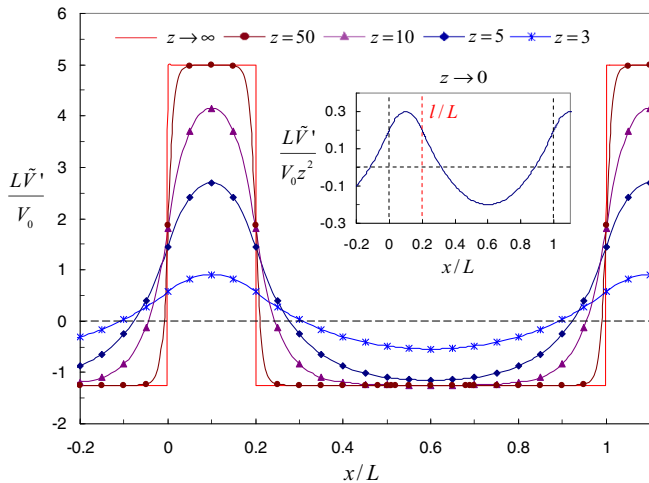
$$\xi' = 1 - \xi, \qquad (21)$$

FIG. 1. The coordinate dependences of the first derivative of the effective potential, Eq. (19), corresponding to the sawtooth potential (18) at $l = 0.2\,L$, for different values of dimensionless frequency parameter $z = L/\sqrt{D\tau}$ (the main plot) and for $z \to 0$ [the uniform normalized representation, Eq. (20), depicted in the inset].

where

$$f_1(z,\xi) = 1 - \frac{\sinh z\xi \sinh z\xi'}{\xi\xi' z \sinh z},$$

$$f_2(z,\xi) = 1 - \frac{\sinh z(\xi - \xi')}{(\xi - \xi')\sinh z} \qquad (22)$$

are the auxiliary functions.

Note that Eqs. (21) and (22) are valid for arbitrary values of the frequency parameter z and the asymmetry coefficient $\kappa = \xi' - \xi = 1 - 2\xi$. Furthermore, they are very informative in point of frequency dependence of the average velocities for both ratchet models as well as of the dependence of these quantities on the asymmetry of the potential. The analysis below provides this information.

The adiabatic regime or low-frequency limit ($z \to 0$) [we noticed its "smooth analog" in connection with Eq. (17)]. This regime of slow periodic driving, introduced by Parrondo in his pioneering work [22], is the simplest one, and moreover, the limit in which both ratchets behave in the same way (its additional advantages are invariance with respect to character of time variations, deterministic or stochastic, of the potential energy [17] and high efficiency of energy conversion [10,13,14]). In order to show the mentioned similar behavior of the ratchets, it will be enough to use the smallness of the z parameter and to expand the functions $f_1(z,\xi)$ and $f_2(z,\xi)$ into series,

$$f_1(z,\xi) = \tfrac{1}{3}\xi\xi'z^2 - \tfrac{1}{45}\xi\xi'(1 + 2\xi\xi')z^4 + O(z^6),$$
$$f_2(z,\xi) = \tfrac{2}{3}\xi\xi'z^2 - \tfrac{2}{45}\xi\xi'(1 + 3\xi\xi')z^4 + O(z^6). \qquad (23)$$

Then the substitution of these expansions into Eqs. (21) gives us the result of an extremely simple form,

$$\Phi_f = \Phi_r = -\Delta\Phi_r = \frac{(\beta V_0)^3}{360}\kappa. \qquad (24)$$

Thus the average velocities $\langle v \rangle_f$ and $\langle v \rangle_r$ of flashing and rocking ratchets are determined by the same factor Φ_f and are proportional to the same (first) degree of the asymmetry coefficient κ. However, though the velocities behave similarly in the sense of sensitivity to the two main parameters of the potential, βV_0 and κ, $\langle v \rangle_f$, unlike $\langle v \rangle_r$, contains an additional time factor. So, the differences between the ratchets are in the opposite motion directions at the same potential asymmetry and in the adiabatic limits: $\langle v \rangle_f \to 0$ as τ^{-1} and $\langle v \rangle_r \to \text{const} \neq 0$ at $\tau \to \infty$ ($z \to 0$). Note that for asymmetric dichotomic processes, such motion reversal can be broken down [17].

The extremely asymmetric potential ($\xi = 0$, $\kappa = 1$). It is the most important limit considered in the present paper, as it shows the jump behavior of the ratchets' characteristics (see discussions in Sec. V). In this case, the expansion of the functions $f_1(z,\xi)$ and $f_2(z,\xi)$ should be made over the small

parameter ξ keeping z finite,

$$f_1(z,\xi) = -(1 - z\coth z)\xi - (1 + 2z^2/3 - z\coth z)\xi^2$$
$$+ O(\xi^3),$$
$$f_2(z,\xi) = -2(1 - z\coth z)\xi - 2(2 + z^2 - 2z\coth z)\xi^2$$
$$+ O(\xi^3). \tag{25}$$

Substitution of Eq. (25) into Eq. (21) gives, in the limit $\xi \to 0$,

$$\Phi_f = \frac{(\beta V_0)^3}{16z^2}\left(\frac{\cosh 2z}{4\sinh^2 z} - \frac{5}{4z}\coth z + \frac{1}{z^2}\right),$$

$$\Phi_r = \frac{(\beta V_0)^3}{16z^2}\left(\frac{\cosh 2z}{2\sinh^2 z} - \frac{2}{3} - \frac{1}{2z}\coth z\right). \tag{26}$$

One can check that these expressions at $z \to 0$ come to the frequency-independent result of Eq. (24) at $\kappa = 1$. In the opposite particular case, $z \to \infty$, the quantities Φ_f and Φ_r have the same power-law frequency dependence, z^{-2}, but with different proportionality factors:

$$\Phi_f = \frac{(\beta V_0)^3}{32z^2}, \quad \Phi_r = \frac{(\beta V_0)^3}{48z^2}. \tag{27}$$

Since $z^2 = L^2/D\tau$, it follows from Eq. (21) that $\langle v \rangle_f \to (D/32L)(\beta V_0)^3 \neq 0$ and $\langle v \rangle_r \propto \Phi_r \propto \tau$ at $\tau \to 0$. Note that such a behavior differs essentially from the results $\langle v \rangle_f \propto \tau$ and $\langle v \rangle_r \propto \tau^3$ of Eqs. (14) and (16) obtained for smooth potentials; that is, the average velocity decreases faster with $\tau \to 0$ (i.e., the motor effect is less) for smooth potentials than it does for those with cusps and of extremely asymmetric form.

High-frequency limit ($z \to \infty$, $\kappa \neq 1$). This describes a regime which is very sensitive to all ratchet parameters: potential features, type of fluctuations (stochastic or deterministic), etc., and moreover has a number of known asymptotics [1,26–29]. Therefore, this regime can serve as some sort of probe allowing understanding of the specificity of a problem.

At $z \to \infty$ and keeping the potential far from extremely asymmetric ($\xi, \xi' \neq 0$), we have approximately $f_1(z,\xi) \approx 1 - (2z\xi\xi')^{-1}$, $f_2(z,\xi) \approx 1$ so that the expressions for Φ_f and Φ_r take the form

$$\Phi_f = \kappa \frac{(\beta V_0)^3}{32(\xi\xi')^2 z^4}\left(1 - \frac{7}{8\xi\xi' z}\right), \quad \Phi_r = \kappa \frac{(\beta V_0)^3}{128(\xi\xi')^3 z^5}. \tag{28}$$

As follows from Eqs. (28) and (21), $\Phi_f \propto \tau^2$ and $\langle v \rangle_f \propto \tau$ which coincides with the result of Eq. (14) obtained for a smooth potential. In its turn, the asymptotic behavior of the average velocity for rocking ratchets differs for a smooth and a sawtooth potential: $\langle v \rangle_r \propto \tau^3$ for the former and $\langle v \rangle_r \propto \tau^{5/2}$ for the latter (due to the presence of cusp points which play an appreciable role, just for rocking ratchets not for flashing, and this is what the sensitivity of ratchets to the potential means). Therefore, the closer the potential is to a sawtooth one with jumps, the more slowly the average velocity of a rocking ratchet tends to zero with $\tau \to 0$ (compare, in addition, the result $\langle v \rangle_r \propto \tau^{5/2}$, obtained here, with $\langle v \rangle_r \propto \tau$ from the previous subsection). Note that the law $\langle v \rangle_r \propto \tau^{5/2}$ was also revealed by the numerical procedure in Ref. [29].

V. DISCUSSION

We start discussion of our results with the graphic analysis of frequency dependences of the average velocities and the auxiliary quantities Φ_f and Φ_r for flashing and rocking ratchets with a sawtooth potential relief (see Fig. 2). At small values of the dimensionless frequency parameter z, we have the adiabatic regime of motors operation and the coincidence of Φ_f and Φ_r values (see also the formulas in the previous sections). It is noteworthy that for larger z values the difference between Φ_f and Φ_r does not become very large and tends to zero at $z \to \infty$ since both quantities tend to zero at this limit (but with different laws of decreasing). That is why one can say that the deep mechanisms governing the operation of both on-off flashing and rocking ratchets are quite similar (showing themselves just by means of the functions $\Phi_{f,r}$, since the velocities $\langle v \rangle_{f,r}$ differ only in the additional frequency factor). At the same time, at high frequencies and not extremely asymmetric profiles, rocking ratchets sense the peculiarities in shape (cusps) of a potential profile while on-off flashing ratchets do not (see the end of Sec. IV). On the other hand, all high-frequency peculiarities in ratchet characteristics become more intensive just for extremely asymmetric profiles. Further, we will come to the causes of this intensification.

To clarify the physical mechanism behind this effect, let us introduce the characteristic diffusion time $\tau_l = l^2/D$ over the characteristic distance l. For a smooth L-periodic potential, we can set $l \sim L$, so that the average period τ of its time fluctuations begins to compete with τ_L if $\tau \lesssim \tau_L$. At high frequencies, the average velocity of rocking ratchets is defined by the higher order of derivative of the potential than the velocity of on-off flashing ratchets [compare
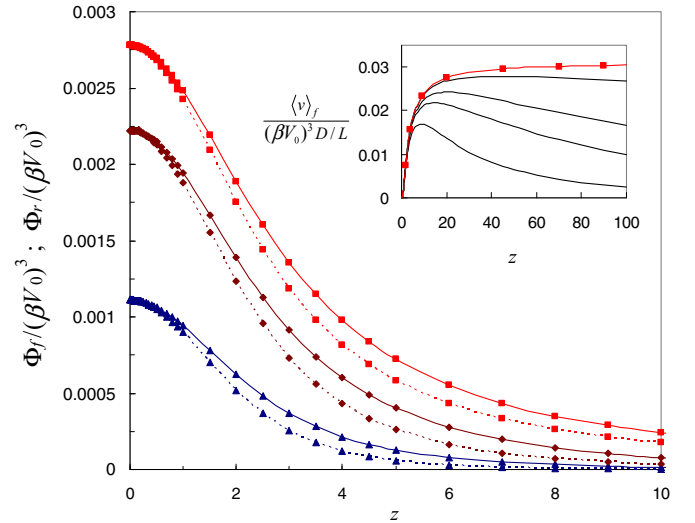

FIG. 2. The frequency dependences of the quantities Φ_f (the solid curves) and Φ_r (the dotted curves) which determine the average velocities of flashing and rocking ratchets, Eq. (21), with a sawtooth potential relief for several values of the parameter ξ: $\xi = 0$ (squares), $\xi = 0.1$ (rhombs), and $\xi = 0.3$ (triangles). The frequency dependence of the normalized average velocity for flashing ratchets is in the inset. The curves, from top to bottom, are in the order of increasing the parameter ξ from $\xi = 0$ (the curve with markers) to $\xi = 0.001, 0.005, 0.01$, and 0.03.

Eqs. (14) and (16)]. That is, as one can see, the difference appears just in the region of this competition. For a sharp L-periodic potential with the small characteristic length l (the length of its sharp link), the characteristic time τ_l can be also small enough so that, addressing small $\tau \leqslant \tau_L$, we expect a different kind of behavior passing from the case $\tau_l < \tau$ to the case $\tau < \tau_l$. Moreover, at $l \to 0$ the whole time region $\tau < \tau_l$ deforms continuously to a point and a jump behavior appears (see below). Returning to a sawtooth potential for which l is the sawtooth length and the parameters $z = L/\sqrt{D\tau}$ and $\xi = l/L$ determine the time and coordinate scale, respectively, we can state that the different behavior at $\tau_l < \tau$ and $\tau < \tau_l$ is governed by a single dimensionless control parameter equal to the product $z\xi = \sqrt{\tau_l/\tau}$. This reduction in number of independent variables results in a universal self-similar description which is known to be of particular use in consideration of many problems [30] as it gives a significant simplification of the analysis, obtaining asymptotics and identifying universality classes ([31]; see also references cited therein). We next focus on application of such a description to our study.

The different high-frequency asymptotic behavior of the average velocities at $\xi = 0$ and $\xi \neq 0$ (at extremely and not extremely asymmetric cases) means that the result depends on the sequence of the limits $\xi \to 0$ and $z \to \infty$. In such a situation, a reasonable way is to analyze the limits $\xi \to 0$, $z \to \infty$ keeping the $z\xi$ value fixed (finite and arbitrary). This problem definition will give us both limits in the framework of a unified continuous description. Indeed, the functions $f_1(z,\xi)$ and $f_2(z,\xi)$ of two variables, defined by Eq. (22), and their limiting behavior can be expressed via the following functions of the single self-similar variable $\lambda = z\xi$:

$$f_1(\lambda) = 1 - \frac{1-e^{-2\lambda}}{2\lambda} \approx \begin{cases} \lambda - 2\lambda^2/3 + \lambda^3/3, & \lambda \ll 1 \\ 1 - (2\lambda)^{-1}, & \lambda \gg 1 \end{cases},$$

$$f_2(\lambda) = 1 - e^{-2\lambda} \approx \begin{cases} 2\lambda - 2\lambda^2 + 4\lambda^3/3, & \lambda \ll 1 \\ 1, & \lambda \gg 1 \end{cases}. \quad (29)$$

One can say that the functions $f_{1,2}(\lambda)$ give the self-similar description of the problem. Indeed, in this case, the expressions (21) for quantities Φ_f and Φ_r are rewritten in a form of self-similar solutions (scaling laws) [30] $\Phi_{f,r} = [(\beta V_0)^3/(128z^2)]F_{f,r}(\lambda)$ where $F_f(\lambda) = \lambda^{-2}[6f_1(\lambda) - 3f_2(\lambda) + f_1(\lambda)f_2(\lambda)]$ and $F_r(\lambda) = 2\lambda^{-2}[-2f_1(\lambda) + f_2(\lambda) + f_1(\lambda)f_2(\lambda)]$ are dimensionless universal scaling functions of the self-similar variable λ. The functions $f_1(\lambda)$ and $f_2(\lambda)$ change from 0 to 1 when λ changes from 0 to ∞ (see the frame in Fig. 3). It is important that this region of λ changes encloses two characteristic regions: of small and large λ values. So the value $\lambda \approx 1$ delimits the regions of different behavior, quite fast increase at $\lambda < 1$ and saturation at $\lambda > 1$ [the functions $F_{f,r}(\lambda)$ also demonstrate the same different behavior, like ξ dependencies of Fig. 3]. For us, the former one is of interest: If z is large enough, the ξ region of the functions' change must be very narrow (Fig. 3) and deforms continuously to a point at $z \to \infty$. The latter means that, at $z \to \infty$, the functions (and, hence, motor characteristics) experience jump discontinuities at the point
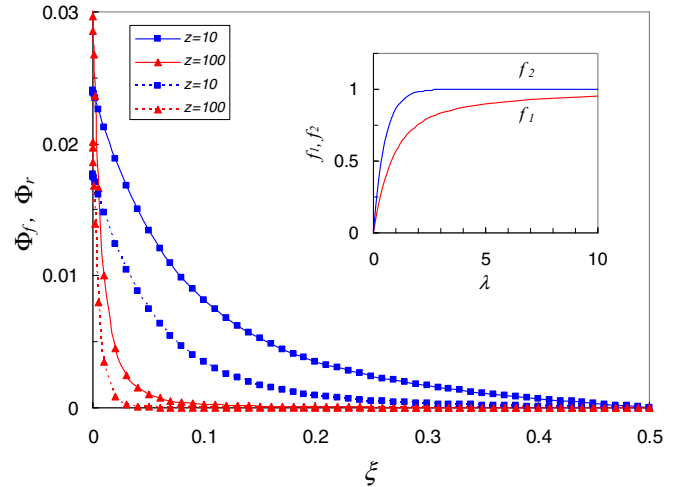

FIG. 3. The behavior of self-similar functions (29) (the inset) and the demonstration of the onset of jump behavior (the main plot): the ξ dependences of the quantities Φ_f and Φ_r (the solid and dotted curves, respectively) for different values of the parameter z: $z = 10$ (squares), $z = 100$ (triangles).

$\xi = 0$, and the maximum motor effect is reached for the extremely asymmetric case.

It is noteworthy that a strong inertial regime with the velocity relaxation time $\tau_v = m/\zeta$ exceeding τ_L is another one in which a jump behavior in ξ dependences of the diffusion transport characteristics arises at $\xi \to 0$ [19]. The analysis of small inertial corrections of the order of τ_v/τ_s ($\tau_s = \zeta l^2/V$ is the sliding time on the l link [18]) shows that this jump behavior is a consequence of the competition between parameters τ_v and τ_s (so, it is in certain analogy to the jump one considered in the present paper) and hence could be expected from the general point of view. The latter means that the interplay of two or more characteristic parameters is needed for observing effects such as these.

VI. CONCLUSIONS

In the theory of Brownian ratchets, the high-temperature approximation allows to represent ratchet average velocities in the form of the double Fourier series over the components of the fluctuating potential [Eqs. (4) and (5)]. Although such a representation is applicable for an arbitrary potential, it cannot be considered as an optimal and illustrative solution. The reason is that it is a useful one for smooth potentials, described by the two first harmonics [17], but is utterly complicated for the analysis of ratchets with sharp potentials. Since such potentials inevitably have links of large gradients, a lot of harmonics should be taken into account, and the series cannot be truncated for potentials with jumps. That is why this technique did not allow analyzing of manifestation of peculiarities of the potential shape in ratchet operation (no universal results were accessible). In this paper, the suggested concept of the effective potential [Eq. (7)] has turned out to be very effective in description of the influence of the fluctuation frequency on its shape; the change of the latter, in its turn, entails changes in ratchet characteristics with frequency. This has allowed avoiding the complexity mentioned above

due to the transition from series to integrals: The effective potential is readily calculated and can be substituted into the integral formulas [Eq. (13)] for the quantities which determine average velocities of both on-off flashing and rocking ratchets. The advantage of the approach is in the possibility of analytical calculations for any potential shape and arbitrary fluctuation frequency, especially for the high frequencies where the ratchets' behavior is very sensitive to the potential shape.

Using the approach developed, we have derived the explicit analytical expressions for the average particle velocity for on-off flashing and rocking ratchets with a sawtooth potential having jumps at its extremely asymmetric case. From these expressions, a similar mode of operation of the ratchets follows, in the sense of their basic frequency behavior. The essential difference appears only at high frequencies τ^{-1}: For on-off flashing ratchets, the average velocity behaves as τ at any smooth potential as well as at a not extremely asymmetric sawtooth one, whereas it takes a constant nonzero value for potentials with jumps; for rocking ratchets, it behaves as τ^3 at any smooth potential and as τ and $\tau^{5/2}$ for sawtooth ones with and without jumps, respectively. Thus the presence of cusp points changes the high-frequency asymptotics for rocking ratchets. There is no such sensitivity to the cusp points for on-off flashing ratchets.

The origin of the changes in asymptotic behavior is elaborated analytically in terms of self-similar functions so that, along with the dependence of the results on the sequence of limits, we have the continuous description of the effect (via the introduction of the self-similar parameter). This description shows that the discussed asymptotic behavior ensues from the competition between the characteristic times of the system: the diffusion time on the width of the sharp link (with large gradient) of the potential and its average fluctuation period.

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