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Quasi semisymmetric designs with extremal conditions

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Abstract

A finite incidence structure $\Pi = (\mathscr{X}, \mathscr{B})$ is called a quasi-semisymmetric design (QSSD) with nexus α if there exist positive integers λ, μ, and α such that any two distinct points are in 0 or λ common blocks, any two distinct blocks are incident with 0 or μ common points, and for each nonincident point-block pair (x, B), there are exactly α blocks B' with $x \in B'$ and $B' \cap B \neq \emptyset$. Symmetric designs, semisymmetric designs, and partial λ-geometries are among such structures. In this paper, in addition to some general properties, we study the existence conditions for QSSDs with $\mu = \lambda - 1 \geqslant 2$ and the properties of QSSDs satisfying the following extremal condition: if B_1 and B_2 are two blocks with a nonempty intersection, then there are another $\lambda - 2$ blocks $B_3, \ldots, B_\lambda$ such that $\bigcap_{1 \leqslant i \leqslant \lambda} B_i = B_1 \cap B_2$. We show that $\alpha \geqslant (\lambda^2(\mu - 1) + \lambda)/\mu$ under such a condition, and QSSDs with equality are classified whenever $\mu = \lambda$ or $\mu = \lambda - 1$ following a classification of affine polar spaces by Cohen and Shult (*Geometraic Dedicata* 35 (1990), 43–76).

0. Introduction

Motivated by the study of the geometric structures associated with the half dual polar graph $D_{n,n}(q)$ and the alternating forms garphs $\mathrm{Alt}(n, q)$ (Fu and Huang, 1995; Huang and Laurent, 1993; Huang and Pan, 1988), we consider some specific conditions over incidence structures:

(QSS1) every two distinct points are in 0 or λ common blocks,

(QSS2) every two distinct blocks intersect in 0 or μ points,

(QSS3) if $\lambda = 1$, then there are constants k and r such that every block contains k points and every point is on r blocks,

(QSS4) if (x, B) is a nonincident pair of point x and block B, then there are exactly α blocks of x intersecting B.

Let λ, μ, and α be positive integers. A finite incidence structure $\Pi = (\mathscr{X}, \mathscr{B})$ is called a *quasi-semisymmetric design* (abbreviated 'QSSD') for λ, μ if conditions (QSS1)–(QSS3) are satisfied, and Π is called a quasi semisymmetric design for λ, μ with *nexus* α if conditions (QSS1)–(QSS4) are satisfied. Clearly, $\lambda = 1$ if and only if $\mu = 1$, and

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hence Π is a semilinear space or a partial linear space (see Brouwer et al., 1989, for the definition). Condition (QSS3) is necessary to ensure the k-uniformity and r-regularity of Π (i.e., every block of Π contains k points, and every point of Π is in r blocks). An example that satisfies (QSS1) and (QSS2) with $\lambda = \mu = 1$ but does not satisfy (QSS3) is given in Huang and Pan (1988). Partial geometries, first studied by Bose, are examples of QSSDs with $\lambda = \mu = 1$, and partial λ-geometries, introduced by Cameron and Drake (1980) are QSSDs with $\lambda = \mu$.

QSSDs with multiple intersections, i.e., $\lambda \geqslant \mu \geqslant 2$, are treated in this paper. Basic properties, associated combinatorial structures, some examples constructed from vector spaces, and some existence conditions for QSSDs with $\mu = \lambda - 1 \geqslant 2$ are described in Section 1. Two extremal conditions are introduced in Section 2; we shall show that $\alpha = (\lambda^2(\mu - 1) + \lambda)/\mu$ under these extremal conditions. QSSDs satisfying the extermal conditions with $\mu = \lambda$ and $\mu = \lambda - 1$ are classified in Section 3 following a classification of affine polar spaces by Cohen and Shult in (1990).

1. Basic properties and some necessary conditions

In this section, we study basic properties and associated combinatorial structures such as the block graphs and point graphs of QSSDs. We also study some existence conditions for QSSDs with $\mu = \lambda - 1 \geqslant 2$.

Lemma 1.1 (Lin, 1992). *If $\Pi = (\mathscr{X}, \mathscr{B})$ is a QSSD for λ and μ, where λ, $\mu \geqslant 2$, then Π is k-uniform and r-regular for some positive integers k and r, and has no repeated blocks.*

Lemma 1.1 shows that each block can be identified with the set of points it contains. Let $v = |\mathscr{X}|$ and $b = |\mathscr{B}|$, and denote Π by QSSD$(v, k, [\lambda], [\mu])$. It is obvious that the dual incidence structure of Π is a QSSD$(b, r, [\mu], [\lambda])$, and hence, without loss of generality, we may assume that $\lambda \geqslant \mu$.

For the rest of this paper, we assume that $\Pi = (\mathscr{X}, \mathscr{B})$ is a QSSD$(v, k, [\lambda], [\mu])$ with nexus α, where $\lambda \geqslant \mu \geqslant 2$. Two points are called *collinear* if they are in a common block. For $x \in \mathscr{X}$, define $x^{\perp} = \{x\} \cup \{y \in \mathscr{X} \mid y \text{ and } x \text{ are collinear}\}$. An incident pair (x, B) of point x and block B is called a *flag*. For a QSSD$(v, k, [\lambda], [\mu])$ with nexus α, the condition (QSS4) is equivalent to the following dual condition:

(QSS4′) if (x, B) is a nonflag, then x is collinear with exactly β points of B.

For a nonflag (x, B), counting the number of flags (y, A) with $x \in A$ and $y \in B$ shows that $\beta\lambda = \alpha\mu$. The following lemma shows that v, b, and r are functions of k, λ, and μ.

Lemma 1.2. (i) $v = (k(r - 1)(k - \mu)/\alpha\mu) + k$,

(ii) $b = (r(k - 1)(r - \lambda)/\beta\lambda) + r$, *and*

(iii) $(k - 1)(\lambda - 1) = (r - 1)(\mu - 1)$.

Proof. Fix a block B, and count the set $\{(x, y, A) \mid A \in \mathscr{B}, x \notin B, y \in B, x, y \in A\}$ in two ways. By condition (QSS4), each point $x \notin B$ is in α blocks A with $|A \cap B| = \mu$.

On the other hand, each point $y \in B$ is in another $r-1$ blocks A with $|A \backslash B| = k - \mu$. Therefore, $(v-k)\alpha\mu = k(r-1)(k-\mu)$, and hence (i) follows. (ii) is obtained by a dual argument. To prove (iii), fix a flag (x,B), and count the number of flags (y,A) with $y \neq x$, $A \neq B$, and x, $y \in A \cap B$. □

Let $A_1, \ldots, A_\alpha$ be the blocks of x that intersect B, and let $\Pi_{x,B}$ be the incidence structure $(x^\perp \cap B, \{A_i \cap B \mid 1 \leqslant i \leqslant \alpha\})$. Observe that each point $z \in x^\perp \cap B$ is in λ members of $\{A_i \cap B \mid 1 \leqslant i \leqslant \alpha\}$, and each $|A_i \cap B|$ is μ. We thus have the following result:

Lemma 1.3. *$\Pi_{x,B}$ is λ-regular and μ-uniform.*

The regularity of QSSDs is reflected in some graph structures. The *block graph* of a QSSD $\Pi = (\mathscr{X}, \mathscr{B})$ is defined on the block set $\mathscr{B}$ such that two blocks A and B are adjacent if and only if $A \cap B$ is nonempty. The *point graph* of Π is defined on the point set $\mathscr{X}$ such that two points x and y are adjacent if and only if x and y are in a common block. Recall that a *strongly regular graph* is a simple connected b_0-regular graph of order n with the following property: the number of vertices adjacent to both x and y is a constant a if x and y are themselves adjacent, and is the constant c otherwise, for some nonnegative integers n, b_0, a, and c.

Theorem 1.4. *Let $\Pi = (\mathscr{X}, \mathscr{B})$ be a QSSD$(v,k,[\lambda],[\mu])$, $\mu \geqslant 2$, with nexus α.*

(i) *The block graph of Π is a strongly regular graph with the following parameters:*

$$n = (r(k-1)(r-\lambda)/\alpha\mu) + r, \qquad b_0 = k(r-1)/\mu,$$

$$a = (k(\alpha-1) + \mu(r-\alpha-1))/\mu, \qquad c = k\alpha/\mu.$$

(ii) *The point graph of Π is a strongly regular graph with the following parameters:*

$$n = (k(r-1)(k-\mu)/\alpha\mu) + k, \qquad b_0 = r(k-1)/\lambda,$$

$$a = (r(\beta-1) + \lambda(k-\beta-1))/\lambda, \qquad c = r\beta/\lambda.$$

Proof. (i) $n = b$, given in Lemma 1.2. To compute b_0, fix a block B and observe that the number of flags (x,A) with $A \neq B$, $x \in A \cap B$, is $k(r-1) = b_0\mu$. To compute a, let A and B be two fixed blocks with a nonempty intersection, and count the set

$$\mathscr{R} = \{(x,y,C) \mid C \in \mathscr{B}, C \neq A, B, \text{and } x \in C \cap A, y \in C \cap B\}$$

in two ways. We have

$$\begin{aligned} a\mu^2 &= \sum_{x \in A \backslash B} (\alpha-1)\mu + \sum_{x \in A \cap B} (r-2)\mu \\ &= (k-\mu)(\alpha-1)\mu + (r-2)\mu^2 \\ &= k(\alpha-1)\mu + (r-\alpha-1)\mu^2. \end{aligned}$$

To compute c, let A and B be two fixed blocks with an empty intersection. Counting the set $\mathscr{R}$ in two ways shows that $c\mu^2 = k\alpha\mu$. (ii) is obtained by a dual argument. □

The integrality condition for strongly regular graphs (see Cameron, 1978, for more details) provides some necessary conditions on parameters.

Proposition 1.5. (i) *$\alpha\mu(k + \mu(r - \alpha - 1))$ is a divisor of $kr(k-1)(r-\lambda)$;*
(ii) *$\alpha\mu(r + \lambda(k - \beta - 1))$ is a divisor of $rk(r-1)(k-\mu)$.*

Some classes of examples of QSSDs are given below. Example (iii) was treated in (Cameron and Drake, 1980). Example (iv) will be treated in detail in Section 3.

Examples. (i) Let V be a vector space of dimension m over GF(q). Let $\mathscr{X} = \begin{bmatrix} V \\ 2 \end{bmatrix}$, the set of all two-dimensional subspaces of V, and $\mathscr{B} = \{[B] | B \in \begin{bmatrix} V \\ 1 \end{bmatrix}\}$, where $[B] = \{x \in \begin{bmatrix} V \\ 2 \end{bmatrix} | B \subseteq x\}$. Then $(\mathscr{X}, \mathscr{B})$ is a

$$QSSD\left(\frac{(q^m-1)(q^{m-1}-1)}{(q^2-1)(q-1)}, \frac{q^{m-1}-1}{q-1}, [1], [1]\right)$$

with nexus $\alpha = q+1$, and the point graph of $(\mathscr{X}, \mathscr{B})$ is a strongly regular graph with parameters $n = [(q^m-1)(q^{m-1}-1)/(q^2-1)(q-1)]$, $b_0 = (q+1)(q^{m-1}-q)/(q-1)$, $a = (q^{m-1}-1)/(q-1) + q^2 - 2$, and $c = (q+1)^2$.

(ii) Let V be a vector space of dimension $m+2$ over GF(q). Fix an m-dimensional subspace W of V, and let $\mathscr{X} = \{x \in \begin{bmatrix} V \\ 2 \end{bmatrix} | x \cap W = 0\}$ and $\mathscr{B} = \{[B] \in \begin{bmatrix} V \\ 1 \end{bmatrix} | B \cap W = 0\}$, where $[B] = \{x \in \mathscr{X} | B \subseteq x\}$. Then $(\mathscr{X}, \mathscr{B})$ is a QSSD$(q^{2m}, q^m, [1], [1])$ with nexus $\alpha = q$, and the point graph of $(\mathscr{X}, \mathscr{B})$ is a strongly regular graph with parameters $n = q^{2m}$, $b_0 = (q+1)(q^m-1)$, $a = q^m + q^2 - q - 2$, and $c = q(q+1)$.

(iii) Let V be an eight-dimensional vector space over GF(q) with a quadratic form of Witt index 4. The set $\mathscr{S}$ of all maximal totally isotropic subspaces (of dimension 4) can be partitioned into two families with the property that $x, y \in \mathscr{S}$ belong to the same family if and only if the codimension of $x \cap y$ is even. The incidence structure $(\mathscr{X}, \mathscr{B})$, where $\mathscr{X}$ is the set of all isotropic 1-subspaces and $\mathscr{B}$ is one family of $\mathscr{S}$, is a QSSD$((q^3+1)(q^2+1)(q+1), (q^4-1)/(q-1), [q+1], [q+1])$ with nexus $\alpha = q^2 + q + 1$. The point graph of $(\mathscr{X}, \mathscr{B})$ is a strongly regular graph with parameters $n = (q^3+1)(q^2+1)(q+1)$, $b_0 = (q^2+1)(q^3+q^2+q)$, $a = (q^4-1)/(q-1) + q^2(q^2+q) - 2$, and $c = (q^2+1)(q^2+q+1)$.

(iv) Let V be a four-dimensional vector space over GF(q). Denote by $\mathscr{X}$ the set of all alternating bilinear forms defined over V, and let $\mathscr{B} = \{[A] \mid A \in \begin{bmatrix} V \\ 3 \end{bmatrix}\}$, where $[A]$ is $\{f \in \mathscr{X} \mid \mathrm{Rad}(f) \subseteq A\}$ with translations (see Huang and Laurent, 1993, for more details). The incidence structure $(\mathscr{X}, \mathscr{B})$ is a QSSD$(q^6, q^3, [q+1], [q])$ with nexus $\alpha = q^2 + q$. Its point graph, denoted by Alt$(4, q)$, is a strongly regular graph with parameters $n = q^6$, $b_0 = (q^2+1)(q^3-1)$, $a = q^4 + q^3 - q^2 - 2$, and $c = (q^2+1)q^2$.

Remark. The half dual polar graph $D_{4,4}(q)$ is defined on one family of maximal totally isotropic subspaces of $\mathscr{S}$ (in Example (iii)). Two vertices x and y are adjacent if and only if the codimension of $x \cap y$ is 2. It is well known that $D_{4,4}(q)$ is isomorphic to the point graph of $(\mathscr{X}, \mathscr{B})$ (this is implicit in Wells, 1984, p. 384). It is also known that $\mathrm{Alt}(4,q)$ is the induced subgraph of $D_{4,4}(q)$ over the distance 2 neighborhood of some vertex $x \in D_{4,4}(q)$ (see Brouwer et al., 1989, Proposition 9.5.11).

Note that the Examples (iii) and (iv) are QSSDs with $\mu = \lambda$ and $\mu = \lambda - 1$, respectively. It is worth mentioning here that $D_{4,4}(q)$ and $\mathrm{Alt}(4,q)$ are Zara graphs (Zara, 1984) with maximal cliques of size $(q^4 - 1)/(q - 1)$ and q^3, respectively. For the rest of this section, we study the existence of QSSDs with $\mu = \lambda - 1 \geqslant 2$.

Lemma 1.6. *If there exists a QSSD*$(v, k, [\mu+1], [\mu])$, $\mu \geqslant 2$, *with nexus* $\alpha = \beta(\mu+1)/\mu$, *then* $k - 1 = m(\mu - 1)$ *for some integer* m, *and*

$$\beta - 1 \geqslant F(m,\mu) = \frac{(m\mu - m - 1)(\mu^2 - 1)}{m\mu - m - 1 + \mu^2 - 2\mu}.$$

Proof. A result of Neumaier (1981, Lemma 1.6) gives a lower bound for β:

$$\beta - 1 \geqslant \frac{(k-2)(\mu^2 - 1)}{k - 2 + \mu^2 - 2\mu}.$$

Lemma 1.2(iii) shows that $r - 1 = \mu(k - 1)/(\mu - 1)$, hence $\mu - 1$ divides $k - 1$, and so $k - 1 = m(\mu - 1)$ for some integer m. The expression for $F(m,\mu)$ is obtained by substituting $m(\mu - 1)$ for $k - 1$ in the above inequality. □

Corollary 1.7. *If there exists a QSSD*$(v, k, [\mu + 1], [\mu])$, $\mu \geqslant 2$, *with nexus* $\alpha = \beta(\mu + 1)/\mu$ *and* $k > \beta + 1$, *then* $\beta \geqslant f(\mu)$, *where* $f(2) = 4$, $f(3) = 7$, $f(4) = 10$, $f(5) = 13$, $f(\mu) = 3\mu - 4$ *if* $6 \leqslant \mu \leqslant 13$ *and* $f(\mu) = 3\mu - 5$ *if* $\mu \geqslant 14$.

Proof. From Lemma 1.6,

$$\frac{F(m,\mu)}{\mu^2 - 1} = 1 - \frac{\mu(\mu - 2)}{(\mu - 1)m + \mu^2 - 2\mu - 1}.$$

Thus $F(m,\mu)$ is a nondecreasing function of m for each fixed $\mu \geqslant 2$. If $\mu = 2$, then $\beta \geqslant 1 + F(m,2) = 4$. If $\mu = 3, 4$ or 5, then $2 \leqslant k - \beta \leqslant m(\mu - 1) - F(m,\mu)$ implies $m \geqslant 4$, so $\beta \geqslant 1 + F(4,\mu)$. If $\mu \geqslant 6$, then $\beta \geqslant 1 + F(3,\mu) = 3\mu - 6 + ((16\mu - 24)/(\mu^2 + \mu - 4))$. In particular, $\beta \geqslant 3\mu - 4$ if $6 \leqslant \mu \leqslant 13$, and $\beta \geqslant 3\mu - 5$ if $\mu \geqslant 14$. The corollary follows immediately. □

Proposition 1.8. *For each pair* (μ, β) *with* $\mu \geqslant 2$, *there are only finitely many QSSD* $(v, k, [\mu + 1], [\mu])$ *with nexus* $\alpha = \beta(\mu + 1)/\mu$.

Proof. Substituting $\lambda = \mu + 1$ and $r - 1 = (k - 1)\mu/(\mu - 1)$ in Proposition 1.5(ii) shows that

$$\frac{\mu k(k-1)(k-\mu)(k\mu - 1)}{\alpha(\mu - 1)[\mu(k\mu - 1) + (\mu^3 - \mu)(k - \beta - 1)]}$$

is an integer. Let $f(k) = \mu(k\mu - 1) + (\mu^3 - \mu)(k - \beta - 1)$. Then $f(k)$ divides $\mu k(k-1)(k-\mu)(k\mu - 1)$. Use of the Euclidean algorithm shows that GCD$(f(k), \mu)$, the greatest commom divisor of $f(k)$ and μ, is μ; GCD$(f(k), k)$ divides $\mu(\beta\mu^2 + \mu^2 - \beta)$; GCD$(f(k), k-1)$ divides $\mu(\mu-1)(\beta\mu + \beta - 1)$; GCD$(f(k), k-\mu)$ divides $\mu(\mu^2-1)(\beta - \mu)$; and GCD$(f(k), k\mu - 1)$ divides $(\mu^2 - 1)(\beta\mu + \mu - 1)$. It follows that $f(k)$ divides $N = \mu^4(\mu-1)^3(\mu+1)^2(\beta\mu^2 + \mu^2 - \beta)(\beta\mu + \beta - 1)(\beta - \mu)(\beta\mu + \mu - 1)$, and there are only finitely many possible k's for a given pair (μ, β) such that $f(k)$ divides N. Hence, by Theorem 1.4, there are only finitely many possible v's. □

2. Some extremal conditions

In this section, we introduce two extremal conditions that provide an upper bound and a lower bound, respectively, for α. The following two equivalent conditions, called the $(*)$-*conditions*, were studied for $(s, r; \mu)$-nets in Huang and Laurent (1993) and for partial λ-geometries in Cameron and Drake (1980). As mentioned in the previous section, for a nonflag (x, B), $|x^\perp \cap B|$ is a constant β, where $\beta\lambda = \alpha\mu$, and we let $\Pi_{x,B}$ be the incidence structure defined over $x^\perp \cap B$. The structure of $\Pi_{x,B}$, together with the $(*)$-condition, gives a sharp lower bound for β (and hence for α).

Lemma 2.1. *Let $\Pi = (\mathscr{X}, \mathscr{B})$ be a QSSD$(v, k, [\lambda], [\mu])$, $\mu \geqslant 2$, with nexus α. The following two conditions are equivalent:*

(i) *if B_1, B_2 are two distinct blocks, with $B_1 \cap B_2 \neq \emptyset$, then there exist $B_3, \ldots, B_\lambda \in \mathscr{B}$ such that $\cap_{1 \leqslant i \leqslant \lambda} B_i = B_1 \cap B_2$, which consists of μ points.*

(ii) *if B_1, B_2, B_3 are three distinct blocks with $|B_1 \cap B_2 \cap B_3| \geqslant 2$, then $|B_1 \cap B_2 \cap B_3| = \mu$.*

Proof. First we assume (i), and let B_1, B_2, and B_3 be three distinct blocks such that $|B_1 \cap B_2 \cap B_3| \geqslant 2$. Let $x, y \in B_1 \cap B_2 \cap B_3$ be distinct; then by (i) there are blocks $B_4, \ldots, B_\lambda$ containing x and y such that $|\cap_{1 \leqslant i \leqslant \lambda} B_i| = |B_1 \cap B_2| = \mu$, hence $|B_1 \cap B_2 \cap B_3| = \mu$. Conversely, we assume (ii), and let B_1, $B_2 \in \mathscr{B}$ be distinct with $B_1 \cap B_2 \neq \emptyset$, then $|B_1 \cap B_2| = \mu$. If $x, y \in B_1 \cap B_2$ are distinct, then there are another $B_3, \ldots, B_\lambda \in \mathscr{B}$ containing x and y. Since x, $y \in B_1 \cap B_2 \cap B_i$, $|B_1 \cap B_2 \cap B_i| = \mu$ by (ii) for $i = 3, \ldots, \lambda$. Hence $|\cap_{1 \leqslant i \leqslant \lambda} B_i| = |B_1 \cap B_2| = \mu$. □

Corollary 2.2. *Let (x, B) be a nonflag of a QSSD satisfying the $(*)$-condition, and let A_1 and A_2 be two distinct blocks of x intersecting B. Then $|A_1 \cap A_2 \cap B| \leqslant 1$.*

Proof. If $|A_1 \cap A_2 \cap B| \geqslant 2$, then, by Lemma 2.1(ii), $|A_1 \cap A_2 \cap B| = \mu = |A_1 \cap A_2|$, and hence $x \in A_1 \cap A_2 \cap B \subseteq B$, a contradiction. □

Lemma 2.3. *Let $\Pi = (\mathscr{X}, \mathscr{B})$ be a QSSD$(v, k, [\lambda], [\mu])$, $\mu \geqslant 2$, satisfying the $(*)$-condition with nexus α, and let (x, B) be a nonflag. Then*

(i) *$\beta \geqslant \lambda(\mu - 1) + 1$, and hence $\alpha \geqslant (\lambda^2(\mu-1) + \lambda)/\mu$,*

(ii) *equality holds if and only if the structure $\Pi_{x,B}$ is a 2-$(\lambda(\mu-1)+1, \mu, 1)$ design.*

Proof. Let $y \in x^{\perp} \cap B$, and let $A_1, \ldots, A_\lambda$ be the blocks containing x and y. Then each A_i intersects B in μ points. By the $(*)$-condition and Corollary 2.2, $(A_i \cap B) \cap (A_j \cap B) = y$ $(i \neq j)$, so $\beta = |x^{\perp} \cap B| \geqslant |\cup_{1 \leqslant i \leqslant \lambda} A_i \cap B| = \sum_{1 \leqslant i \leqslant \lambda} |A_i \cap B \backslash \{y\}| + 1 = \lambda(\mu - 1) + 1$, and the lower bound for α $(= \beta\lambda/\mu)$ follows immediately.

If equality holds then any point $z \in x^{\perp} \cap B \backslash \{y\}$ is in exactly one $A_i \cap B$ for some i $(1 \leqslant i \leqslant \lambda)$. Since y is arbitrary, every two distinct points are in exactly one 'block' of $\Pi_{x,B}$, and hence $\Pi_{x,B}$ is a 2-$(\lambda(\mu - 1) + 1, \mu, 1)$ design. □

Substituting $\mu = \lambda$ $(= q + 1)$ and $\mu = \lambda - 1$ $(= q)$ in the previous lemma, we have $\alpha \geqslant q^2 + q + 1$ and $\alpha \geqslant q^2 + q$, respectively. Examples (iii) and (iv) in the previous section show that both bounds are sharp. Moreover, the 2-designs mentioned above in the QSSDs of Examples (iii) and (iv) are projective planes and affine planes of order q, respectively.

An upper bound for β (and hence for α) is obtained by the following extremal condition, called the (Δ)-*condition*,

Any three distinct pairwise collinear points are in at least one common block.

Lemma 2.4. *Let $\Pi = (\mathscr{X}, \mathscr{B})$ be a QSSD$(v, k, [\lambda], [\mu]), \mu \geqslant 2$, satisfying the (Δ)-condition with nexus α. Then $\beta \leqslant \lambda(\mu - 1) + 1$, and hence $\alpha \leqslant (\lambda^2(\mu - 1) + \lambda)/\mu$.*

Proof. Let (x, B) be a nonflag, $y \in x^{\perp} \cap B$, and $A_1, \ldots, A_\lambda$ be the blocks containing x and y. By the (Δ)-condition, every point of $x^{\perp} \cap y^{\perp}$ is in at least one block of $A_1, \ldots, A_\lambda$, and hence every point $z \in x^{\perp} \cap B \backslash \{y\}$ $(\subseteq x^{\perp} \cap y^{\perp})$ is in at least one $A_i \cap B$ for some i $(1 \leqslant i \leqslant \lambda)$, so $\beta = |x^{\perp} \cap B| = |\cup_{1 \leqslant i \leqslant \lambda} A_i \cap B| \leqslant \sum_{1 \leqslant i \leqslant \lambda} |A_i \cap B \backslash \{y\}| + 1 = \lambda(\mu - 1) + 1$. □

Corollary 2.5. *Let $\Pi = (\mathscr{X}, \mathscr{B})$ be a QSSD$(v, k, [\lambda], [\mu])$ satisfying the $(*)$-condition with nexus $\alpha = (\lambda^2(\mu - 1) + \lambda)/\mu$. Then the (Δ)-condition holds.*

Proof. Let x, y, and z be three distinct pairwise collinear points, and let B be a block containing y and z. If $x \in B$ then we are done; otherwise $x \notin B$. Let $A_1, \ldots, A_\lambda$ be the blocks containing x and y. Since the $(*)$-condition holds and $\alpha = (\lambda^2(\mu - 1) + \lambda)/\mu$, the structure $\Pi_{x,B}$ is a 2-design by Lemma 2.3. Then z $(\in x^{\perp} \cap B \backslash \{y\})$ lies in one $A_i \cap B$ for some i $(1 \leqslant i \leqslant \lambda)$, and hence $x, y, z \in A_i$, as required. □

For a nonflag (x, B), the incidence structure $\Pi_{x,B}$ is determined under the $(*)$- and (Δ)-conditions.

Corollary 2.6. *Let $\Pi = (\mathscr{X}, \mathscr{B})$ be a QSSD$(v, k, [\lambda], [\mu])$, $\mu \geqslant 2$, with nexus α. The following are equivalent:*

(i) *Π satisfies the $(*)$- and (Δ)-conditions,*
(ii) *$\Pi_{x,B}$ is a 2-$(\lambda(\mu - 1) + 1, \mu, 1)$ design.*

Proof. By Lemmas 2.3 and 2.4, we have (i) implies (ii). Conversely, let x, y, and z be three distinct pairwise collinear points, and let B be a block containing y and z

but $x \notin B$. Since $\Pi_{x,B}$ is a 2-design, z ($\in x^{\perp} \cap B \setminus \{y\}$) lies in one 'block' of $\Pi_{x,B}$, and hence x,y, and z are in a common block of $\mathscr{B}$. So the (Δ)-condition holds. Let A, B, and C be three distinct blocks with $|A \cap B \cap C| \geqslant 2$, say u, $v \in A \cap B \cap C$. We want to show that $|A \cap B \cap C| = \mu$. Suppose, to the contrary, $|A \cap B \cap C| < \mu$; then $C \cap A \neq B \cap A$. Choose any point $w \in (C \cap A) \setminus (B \cap A)$. Then u and v are in two 'blocks' $A \cap B$ and $C \cap B$ of $\Pi_{w,B}$. This contradicts the assumption that $\Pi_{w,B}$ is a 2-design with index 1. So $|A \cap B \cap C| = \mu$, and hence the (*)-condition holds. □

3. A characterization of Alt$(4, q)$

Cameron and Drake (1980) showed that a QSSD$(v, k, [\lambda], [\lambda])$ satisfying the (*)-condition with nexus $\alpha = \lambda^2 - \lambda + 1$ is obtained from a polar space of type $D_4(q)$ with one family of maximal totally isotropic subspaces as the block set. As a result, its point graph is isomorphic to $D_{4,4}(q)$. In this section, we shall prove a similar result for a QSSD$(v, k, [\lambda], [\lambda - 1])$ with nexus $\alpha = \lambda^2 - \lambda$.

Let $\Pi = (\mathscr{X}, \mathscr{B})$ be a QSSD$(v, k, [\lambda], [\mu])$ satisfying the (*)-condition with nexus $\alpha = (\lambda^2(\mu - 1) + \lambda)/\mu$ (i.e., $\beta = \lambda(\mu - 1) + 1$). Associate Π with an incidence structure $\Pi' = (\mathscr{X}, \mathscr{L})$ with a collection $\mathscr{P}$ of substructures, where $\mathscr{L} = \{A \cap B \mid A, B \in \mathscr{B}$ are distinct with $A \cap B \neq \emptyset\}$ and let $\mathscr{P} = \{x^{\perp} \cap B \mid x \in \mathscr{X}, B \in \mathscr{B}, x \notin B\}$. Members of $\mathscr{L}$ and $\mathscr{P}$ are called *lines* and *planes*, respectively. Clearly, the point graphs of Π and Π' are identical. For any two collinear points x and y, let $A_1, \ldots, A_\lambda$ be the blocks containing x and y and denote by xy the line $A_1 \cap A_2 = \cap_{1 \leqslant i \leqslant \lambda} A_i$ (by the (*)-condition). Since α reaches the lower bound, the (Δ)-condition also holds, by Corollary 2.5. Thus $x^{\perp} \cap y^{\perp} = \cup_{1 \leqslant i \leqslant \lambda} A_i$, and $\{A_i \setminus xy \mid 1 \leqslant i \leqslant \lambda\}$ forms a partition of $x^{\perp} \cap y^{\perp} \setminus xy$. Hence the incidence structure $\Pi' = (\mathscr{X}, \mathscr{L})$ is a gamma space, and each block of Π induces a maximal singular subspace in Π' (refer to Brouwer et al., 1989, for the definitions of gamma spaces and singular subspaces). Note also that each plane in $\mathscr{P}$ is a singular subspace too. A triple of points is called a *triangle* if they are pairwise collinear but not contained in a common line. The main theorem in this section is as follows:

Theorem 3.1. *Let $\Pi = (\mathscr{X}, \mathscr{B})$ be a QSSD$(v, k, [\lambda], [\mu])$ satisfying the (*)-condition with nexus $\alpha = (\lambda^2(\mu - 1) + \lambda)/\mu$ (i.e., $\beta = \lambda(\mu - 1) + 1$). Then*

(i) if $\mu = \lambda$ ($= q + 1 \geqslant 3$), then $\Pi' = (\mathscr{X}, \mathscr{L})$ is the polar space of type $D_4(q)$ and the point graph of Π is isomorphic to $D_{4,4}(q)$.

(ii) if $\mu = \lambda - 1$ ($= q \geqslant 4$), then either $\Pi' = (\mathscr{X}, \mathscr{L})$ is the affine polar space of type $D_4(q) \setminus \infty^{\perp}$ and the point graph of Π is isomorphic to Alt$(4, q)$, *or $k = 5^5$, 11^5.*

We refer to Cohen and Shult (1990) for the notion of affine polar spaces and hyperplanes of the form $\infty^{\perp}$ for some point ∞ of a polar space. Assertion (i) of Theorem 3.1 is proved in Cameron and Drake (1980, Section 3) together with the fact that $D_{4,4}(q)$ is isomorphic to the point graph of $D_4(q)$ (see Wells, 1983). For the rest of this section, we assume that $\Pi = (\mathscr{X}, \mathscr{B})$ is a QSSD as mentioned in Theorem 3.1 with $\mu = \lambda - 1 = q \geqslant 4$. It follows that $\alpha = q^2 + q$ and $\beta = q^2$.

Lemma 3.2. *Every plane $\pi \in \mathscr{P}$ together with the lines it contains is an affine plane of order q.*

Proof. Every plane $\pi \in \mathscr{P}$ is $x^{\perp} \cap B$ for some nonflag (x, B). By Lemma 2.3 (ii) with $\mu = \lambda - 1 = q$, π is a 2-$(q^2, q, 1)$ design, and hence is an affine plane of order q. □

Lemma 3.3. *Every triangle is in a unique block and hence in a unique plane.*

Proof. Let $\{x, y, z\}$ be a triangle, and let $A_1, \ldots, A_{q+1}$ be the blocks containing x and y. Then $\{A_i \backslash xy \mid 1 \leqslant i \leqslant q+1\}$ forms a partition of $x^{\perp} \cap y^{\perp} \backslash xy$. Since $z \in x^{\perp} \cap y^{\perp} \backslash xy$, z is in a unique block A_i, $1 \leqslant i \leqslant q+1$. Moreover, since $z \notin A_j$ for $j \neq i$, there exists a point $w \in (z^{\perp} \cap A_j) \backslash xy$ such that x, y, and z are in the plane $w^{\perp} \cap A_i$. Since Π' is a gamma space, $w^{\perp} \cap A_i$ is the unique plane containing the triangle $\{x, y, z\}$. □

The following corollary follows from a classical result of Buekenhout (1969).

Corollary 3.4. *q is a prime power and every block together with the lines it contains is an affine space of dimension $d \geqslant 3$ over $GF(q)$.*

We may now assume $k = q^d$ for some integer $d \geqslant 3$, and hence, by Lemma 1.2(iii), $r = (q^{d+1} - 1)/(q - 1)$. Since $r\beta/\lambda$ is an integer (Theorem 1.4(ii)), we have $q^2 - 1$ divides $q^2(q^{d+1} - 1)$, and hence the following holds.

Lemma 3.5. *d is an odd integer.*

By Proposition 1.5(i), $\alpha\mu(k + \mu(r - \alpha - 1))$ is a divisor of $kr(k-1)(r-\lambda)$. Substituting the values of μ, α, k, and r, we have

$$\frac{q^{d+1}(q^{d+1} - 1)(q^d - 1)(q^{d-1} - 1)}{(q^2 - 1)(q^{d+3} + q^{d+2} - q^{d+1} - q^5)}$$

is an integer. Let $f(q) = q^{d+3} + q^{d+2} - q^{d+1} - q^5$. Then $f(q)$ divides $q^{d+1}(q^{d+1} - 1)(q^d - 1)(q^{d-1} - 1)$. From the facts that $\mathrm{GCD}(f(q), q^{d+1})$ divides q^5; $\mathrm{GCD}(f(q), q^{d+1} - 1)$ divides $q^5 - q^2 - q + 1$; $\mathrm{GCD}(f(q), q^d - 1)$ divides $q^5 - q^3 - q^2 + q$; and $\mathrm{GCD}(f(q), q^{d-1} - 1)$ divides $q^5 - q^4 - q^3 + q^2$, we have $f(q)$ divides $q^5(q^5 - q^2 - q + 1)(q^5 - q^3 - q^2 + q)(q^5 - q^4 - q^3 + q^2)$ ($< q^{20}$ if $q \geqslant 2$). Hence $d + 3 < 20$, i.e., $d \leqslant 15$.

For odd $d \geqslant 5$, $f(q) = q^5(q^2 - 1)(q^{d-4} + q^{d-5} + q^{d-7} + \cdots + 1)$. Let $D(d, q) = q^{d-4} + q^{d-5} + q^{d-7} + \cdots + 1$. From above, we conclude that $D(d, q)$ divides $q^3(q+1)(q-1)^3(q^3 + q - 1)(q^3 + q^2 - 1)$. If $d = 15$, we denote $D(15, q) = q^{11} + q^{10} + q^8 + q^6 + q^4 + q^2 + 1$ by D. The Euclidean algorithm shows that $\mathrm{GCD}(q, D) = 1$, $\mathrm{GCD}(q+1, D)$ divides 5, $\mathrm{GCD}(q-1, D)$ divides 7, $\mathrm{GCD}(q^3 + q - 1, D)$ divides 51, and $\mathrm{GCD}(q^3 + q^2 - 1, D)$ divides 61. Therefore, D divides $5 \cdot 7^3 \cdot 51 \cdot 61$ (denoted by M).

But $D > M$ if $q \geqslant 5$, a contradiction. One can eliminate the remaining possibilities for $q \leqslant 4$ by computing $D(15,q)$. Therefore $d \leqslant 13$. One can eliminate the possibilities for $7 \leqslant d \leqslant 13$ by the same arguments. For the case of $d = 5$,

$$\frac{q^6(q^6-1)(q^5-1)(q^4-1)}{(q^2-1)(q^8+q^7-q^6-q^5)}$$

is an integer. After being simplified, $q(q^4+q^2+1)(q^5-1)(q^2+1)/(q+1)$ is an integer. The same argument as above shows that $\mathrm{GCD}(q+1,q) = 1$, $\mathrm{GCD}(q+1,q^4+q^2+1)$ divides 3, $\mathrm{GCD}(q+1,q^5-1)$ divides 2, and $\mathrm{GCD}(q+1,q^2+1)$ divides 2. Hence $q+1$ divides $3 \cdot 2 \cdot 2 = 12$. For $q \geqslant 4$, only $q = 5$ and $q = 11$ remain. This completes the proof of the following lemma.

Lemma 3.6. *Either $d = 3$, or $(d,q) = (5,5)$, $(5,11)$.*

Note that every block of $\mathscr{B}$ carries the structure of an affine space of dimension 3 if $d = 3$. The following proposition shows that the associated gamma space Π' of Π is obtained from an affine polar space.

Proposition 3.7. *If $d = 3$, then $(\mathscr{X},\mathscr{L})$ is an affine polar space of type $D_4(q)\backslash\infty^{\perp}$.*

Proof. By Lemma 1.2 (i), $v = q^6$. We shall verify that $(\mathscr{X},\mathscr{L})$ together with the collection $\mathscr{P}$ of affine planes satisfies the axioms for affine polar spaces in Cohen and Shult (1990). It is clear that $x^{\perp} \subseteq y^{\perp}$ implies $x = y$ for any two points, and that $(\mathscr{X},\mathscr{L})$ is a connected gamma space. Note that every block carries the structure of an affine space of dimension 3. Any three pairwise collinear points not on a line lie in a unique plane of $\mathscr{P}$ (Lemma 3.3). Let $\pi \in \mathscr{P}$ and $z \in \mathscr{X}$ with $z \notin \pi$, and let B be a block containing π. If $z \in B$, then $\pi \subseteq z^{\perp}$; otherwise $z \notin B$, and $z^{\perp} \cap B$ is an affine plane. If the two planes π and $z^{\perp} \cap B$ are identical, then $\pi \subseteq z^{\perp}$; otherwise, $\pi \cap (z^{\perp} \cap B)$ is either an empty set or a line. Hence $z^{\perp} \cap \pi$ either is empty, is the set of points on a line, or coincides with the set of all points in π. It follows that, by Cohen and Shult (1990, Corollary 4.2), $(\mathscr{X},\mathscr{L})$ is an affine polar space (consisting of q^6 points) of rank 4 derived from a polar space of rank 4 by removing a hyperplane. According to the classification given in Cohen and Shult (1990, Proposition 5.2 and Theorem 5.12), $(\mathscr{X},\mathscr{L})$ is obtained from the polar space of type $D_4(q)$ by deleting a hyperplane $\infty^{\perp}$ for some point ∞ (see also Cooperstein and Shult, 1991). □

Since $D_{4,4}(q)$ is isomorphic to the point graph of the polar space of type $D_4(q)$ and $\mathrm{Alt}(4,q)$ is isomorphic to the subgraph of $D_{4,4}(q)$ induced over $D_{4,4}(q)\backslash x^{\perp}$ for some vertex x, the proof of assertion (ii) of Theorem 3.1. is completed.

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