Transition to intermittent chaotic synchronization

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Coupled chaotic oscillators can exhibit intermittent synchronization in the weakly coupling regime, as characterized by the entrainment of their dynamical variables in random time intervals of finite duration. We find that the transition to intermittent synchronization can be characteristically distinct for geometrically different chaotic attractors. In particular, for coupled phase-coherent chaotic attractors such as those from the Rössler system, the transition occurs immediately as the coupling is increased from zero. For phase-incoherent chaotic attractors such as those in the Lorenz system, the transition occurs only when the coupling is sufficiently strong. A theory based on the behavior of the Lyapunov exponents and unstable periodic orbits is developed to understand these distinct transitions.

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I. INTRODUCTION

Synchronization in coupled chaotic oscillators has been an area of tremendous interest in nonlinear science. The phenomenon was described by Fujisaka and Yamada [1], and later independently reported [2,3]. Pecora and Carroll [3] triggered much interest in this topic [4–6]. A common setting in which chaotic synchronization is investigated consists of a number of chaotic oscillators linearly coupled in a simple manner. For instance, consider two identical oscillators described by $d\mathbf{x}/dt=\mathbf{f}(\mathbf{x})$ and $d\mathbf{y}/dt=\mathbf{f}(\mathbf{y})$, where $\mathbf{x}$ and $\mathbf{y}$ are the dynamical variables of the two oscillators, respectively, and $\mathbf{f}$ is the nonlinear vector field that generates a chaotic attractor. Coupling can simply be modeled by a term added to each vector field, which is proportional to a coupling parameter K and to the difference between the two variables ($K=0$ thus indicates no coupling). For sufficiently large values of K, say, $K>K_s$, synchronization between the chaotic oscillators can occur in the sense that the distance $|\mathbf{x}(t)-\mathbf{y}(t)|$ decreases exponentially to zero with time, while both variables remain chaotic by themselves. Mathematically, in the simple setting described, the identity $\mathbf{x}(t)=\mathbf{y}(t)$, which characterizes the synchronization state, is always a solution of the system, but the solution is unstable with respect to small deviations away from the synchronization state for $K<K_s$, and it becomes stable for $K>K_s$. In principle, chaotic synchronization is numerically or experimentally observable for $K>K_s$.

It is also known that, for K near K_s, under noise, chaotic synchronization can occur in an intermittent fashion [7]. That is, the difference $|\mathbf{x}(t)-\mathbf{y}(t)|$ can become small and remain so, but only in finite time intervals. These intervals of temporal synchronization are random and interspersed by desynchronization events characterized by large differences between $\mathbf{x}(t)$ and $\mathbf{y}(t)$. The question that motivates this work is how intermittent synchronization occurs as the coupling parameter is increased from zero. As we will show in this paper, our study leads to quite unexpected results that are fundamental to chaotic synchronization.

Our results can be stated in terms of the probability of synchronization $\Phi(K)$, which numerically can be approximated by the fraction of time during which $|\mathbf{x}(t)-\mathbf{y}(t)|<\epsilon$ occurs, where ϵ is a small but arbitrary threshold. This probability depends on the coupling parameter. Our finding is that the probability exhibits characteristically distinct behavior for chaotic attractors of different geometry. In particular, we focus on commonly studied attractors that possess single-scroll and multiple-scroll geometry in the phase space. For the former, which are *phase coherent* and are typically represented by Rössler-type attractors [8], $\Phi(K)$ increases from zero immediately as coupling is turned on. For the latter, which are *phase-incoherent* chaotic attractors such as the Lorenz attractors [9], this probability increases only when K exceeds a critical value K_c, $(0\leq K_c<K_s)$, which is the point for which one of the originally null Lyapunov exponents becomes negative. That is, for coupled phase-coherent attractors, the transition to intermittent chaotic synchronization is *immediate*, but for systems of coupled phase-incoherent attractors, the transition is *delayed* in the sense that it occurs only when the coupling is sufficiently strong. The general observation is that the transition route to intermittent chaotic synchronization depends on whether the coupled chaotic attractors are phase coherent or phase incoherent.

To develop a theoretical understanding of these phenomena, we first study the behavior of the Lyapunov exponent, which is zero in the absence of coupling, and examine how it changes as the coupling is increased from zero. (For convenience, in this paper, we call this exponent the *null* Lyapunov exponent, keeping in mind that it is actually zero only when there is no coupling.) We argue that for phase-coherent chaotic attractors, the null exponent becomes negative as soon as the coupling is turned on, giving rise to the observed immediate transition to intermittent synchronization. However, for phase-incoherent chaotic attractors, as the coupling is increased from zero, the null exponent first becomes positive, reaches a maximum, then decreases from the maximum, and eventually becomes negative. That is, it requires a finite amount of coupling for the exponent to become negative, rendering delayed the transition to intermittent chaotic synchronization.

To understand the two distinct transition scenarios at a more fundamental level, we focus on unstable periodic orbits, which are the building blocks of any chaotic set [10], and examine how they can be synchronized under coupling. For $K=0$, all unstable periodic orbits are transversely unstable, i.e., they are unstable with respect to perturbations away from the synchronization state. Intermittent synchronization sets in when an orbit becomes *transversely stable*. Our analysis of the transverse stabilities of unstable periodic orbits yields the surprising finding that for phase-coherent chaotic attractors, there exist unstable periodic orbits that can be made transversely stable by arbitrarily small coupling. For phase-incoherent chaotic attractors, no periodic orbit has this property, i.e., finite coupling is required for any periodic orbit to become transversely stable. Note that here, the bifurcation leading to intermittent chaotic synchronization is opposite to the previously studied riddling [11] and bubbling bifurcations [12,13] that are triggered when an unstable periodic orbit first becomes *transversely unstable*. Although a periodic-orbit theory can be conveniently formulated for coupled identical oscillators, we find that similar types of transition occur for coupled systems of slightly nonidentical chaotic oscillators.

The rest of the paper is organized as follows: In Sec. II, we present numerical evidence for the two distinct transition scenarios in coupled phase-coherent and phase-incoherent chaotic attractors. In Sec. III, we analyze the behavior of the null Lyapunov exponent to explain the transitions. In Sec. IV, we develop an unstable periodic-orbit theory to further understand the transitions. A discussion is presented in Sec. V.

II. NUMERICAL EVIDENCE FOR DISTINCT ROUTES TO INTERMITTENT CHAOTIC SYNCHRONIZATION

A typical setting for studying synchronization in coupled phase-coherent chaotic attractors [6,14] is the following coupled Rössler system:

$$\dot{x}_{1,2} = -y_{1,2} - z_{1,2} + K(x_{2,1} - x_{1,2}),$$

$$\dot{y}_{1,2} = x_{1,2} + 0.15y_{1,2},$$

$$\dot{z}_{1,2} = 0.2 + z_{1,2}(x_{1,2} - 10), \tag{1}$$

where K is the coupling parameter. For this system, transition to complete synchronization occurs for $K_s \approx 0.105$. On the other hand, the following coupled Lorenz system can be used as a prototype model for studying synchronization in coupled phase-incoherent chaotic attractors:

$$\dot{x}_{1,2} = 10(y_{1,2} - x_{1,2}) + K(x_{2,1} - x_{1,2}),$$

$$\dot{y}_{1,2} = 28x_{1,2} - y_{1,2} - x_{1,2}z_{1,2},$$

$$\dot{z}_{1,2} = -(8/3)z_{1,2} + x_{1,2}y_{1,2}, \tag{2}$$

for which the transition to complete synchronization occurs for $K_s \approx 3.9$. For both systems, transition to intermittent synchronization occurs for values of K much smaller than the values of K_s in the respective systems.

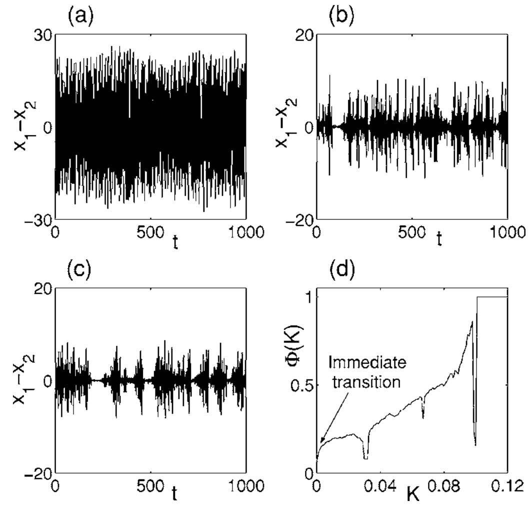

FIG. 1. For the coupled Rössler system, (a)–(c) show the time series $x_1(t)-x_2(t)$ for $K=0$, $K=0.01$, and $K=0.05$, respectively. (d) Probability of synchronization $\Phi(K)$. The transition to intermittent chaotic synchronization is immediate, as predicted by Fig. 7(a).

Figures 1(a)–1(c) show, for the coupled Rössler system, the time series $x_1(t)-x_2(t)$ for $K=0$, $K=0.01$, and $K=0.05$, representing zero, small, and moderate coupling, respectively. There is no synchronization for $K=0$, but as K is increased from zero, intermittent synchronization appears, as characterized by the intermittent time intervals during which $x_1(t)-x_2(t)$ is close to zero [Figs. 1(b) and 1(c)]. In fact, as the coupling becomes stronger, the fraction of the synchronized time intervals, or the probability of synchronization $\Phi(K)$, increases and reaches one for $K \geqslant K_s$ (complete synchronization). For a given value of K, this probability can be approximated by the fraction of time for which $|x_1(t) - x_2(t)| < \epsilon$, where ϵ is a small threshold [we choose it to be 1% of the range of $x_1(t)-x_2(t)$]. Figure 1(d) shows $\Phi(K)$ for the coupled Rössler system. We see that $\Phi(K)$ increases from zero as soon as K is increased from zero [15], suggesting an immediate transition to intermittent chaotic synchronization.

For the coupled Lorenz system, we find a quite different transition scenario. Figures 2(a)–2(c) show the time series $x_1(t)-x_2(t)$ for $K=0$, $K=0.8 < K_c \approx 1.1$, and $K=1.5 > K_c$, respectively. There is no indication of intermittent synchronization when $K < K_c$ [Fig. 2(b)], in contrast to the coupled Rössler system. Intermittent synchronization occurs when $K > K_c$, as shown in Fig. 2(c). Figure 2(d) shows the prob-

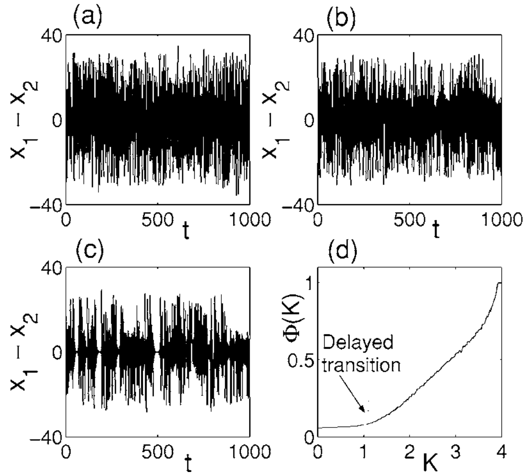

FIG. 2. For the coupled Lorenz system, (a)–(e) show the time series $x_1(t) - x_2(t)$ for $K=0$, $K=0.8 < K_c$, and $K=1.5 > K_c$, respectively. (d) Probability of synchronization $\Phi(K)$. The transition to intermittent chaotic synchronization is delayed, as predicted by Fig. 7(b).

ability of synchronization $\Phi(K)$ vs K, where the numerical threshold ϵ for synchronization is still set to be 1% of the range of $x_1(t) - x_2(t)$. We observe that $\Phi(K)$ remains close to zero for $K \leq 1.1$, indicating that the transition to intermittent synchronization is delayed because it requires a relatively large coupling. This behavior is strikingly different from that in the coupled Rössler system.

In the following two sections, we will present a systematic theoretical description of these distinct routes to intermittent chaotic synchronization.

III. LYAPUNOV-EXPONENT THEORY FOR THE TRANSITION TO INTERMITTENT CHAOTIC SYNCHRONIZATION

First, we consider coupled phase-coherent chaotic attractors. Qualitatively, the phase dynamics of such an attractor can be described by [14,16]

$$\dot{\phi} = \omega + g[r(t)], \tag{3}$$

where ω is the average frequency of the chaotic oscillations, $r(t)$ is the chaotic amplitude, and the function g describes the influence of the $r(t)$ on the phase dynamics. For a pair of coupled oscillators, we have

$$\dot{\phi}_{1,2} = \omega + g_{1,2}[r_{1,2}(t)] + Kh(\phi_{2,1}, \phi_{1,2}), \tag{4}$$

where K is the coupling parameter, and h is a 2π-periodic function in each of its arguments. To be able to analytically calculate the Lyapunov exponent, we assume the simplest case for the function $h(\phi_{2,1}, \phi_{1,2})$

$$h(\phi_{2,1}, \phi_{1,2}) = \sin(\phi_2 - \phi_1). \tag{5}$$

In fact, for the system of coupled phase-coherent Rössler attractors, the leading term in $h(\phi_{2,1}, \phi_{1,2})$ takes the form [17] in Eq. (5). The equation for the phase difference $\Delta\phi = \phi_2 - \phi_1$ can thus be written as

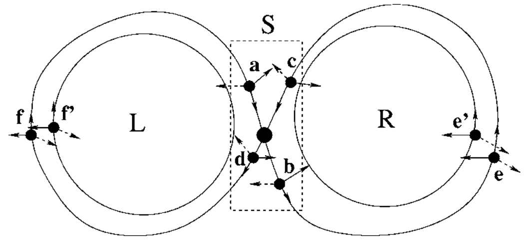

FIG. 3. Schematic illustration of a phase-incoherent chaotic attractor and the associated dynamical consistency, e.g., point $a(c)$ can only go to $b(d)$.

$$\Delta\dot{\phi} = -2K \sin \Delta\phi + g_2[r_2(t)] - g_1[r_1(t)]. \tag{6}$$

For a phase-coherent chaotic attractor, the dependence of the frequency on the amplitude is typically weak [14], so the term $g_2[r_2(t)] - g_1[r_1(t)]$ in Eq. (6) can be neglected. We thus obtain

$$\Delta\dot{\phi} \approx -2K \sin \Delta\phi, \tag{7}$$

and the solution

$$\Delta\phi(t) \approx 2 \tan^{-1}\{\tan[\Delta\phi(0)/2] e^{-2Kt}\}. \tag{8}$$

For infinitesimal initial phase difference $\Delta\phi(0)$ and large time, we have $\Delta\phi(t) \approx \Delta\phi(0) e^{-2Kt}$, which gives

$$\lambda_0 = -2K. \tag{9}$$

We see that a null Lyapunov exponent becomes negative immediately as K is increased from zero, resulting in phase coherence between the two coupled chaotic oscillators. In terms of the dynamical variables in the phase space, this gives rise to intermittent chaotic synchronization.

Now consider a phase-incoherent chaotic attractor, as shown schematically by a double-scroll geometry in Fig. 3. The classical Lorenz chaotic attractor [9], for example, belongs to this type. The left- and right-hand scrolls are denoted by L and R, respectively. A typical trajectory visits both scrolls in time, and it tends to stay in one scroll, executing chaotic motion for a time, then switch to the other scroll and wander chaotically for some time there, then switch back, and so on. Switchings occur in the region denoted by S, in which there is typically an unstable steady state [18]. In the deterministic case, the way that switchings occur must be consistent with the natural dynamics. For instance, a trajectory moving to point a near the switching region must go to point b after the switching. It cannot go to point d. Similarly, under the dynamics, point c can only move to point d. Another aspect of the dynamical consistency is that the relative frequencies with which a trajectory visits L and R appear to be constant, as can be easily verified numerically.

Coupling can, however, disturb the dynamical consistency and, consequently, causes a null Lyapunov exponent to become positive. To see how this can happen, we focus on the dynamics of an infinitesimal vector along the neutral eigendirection that corresponds to the null Lyapunov exponent. The coupling from one chaotic attractor to another can effectively be regarded as random perturbation. Depending on the location of a trajectory, the effect of the random perturbation can be quite different. When the trajectory is not in the switching region, the perturbation can perturb its position, say from point $e(f)$ to point $e'(f')$, or vice versa. As shown in Fig. 3, perturbations at such locations will have little effect on the local eigenspace. Taking a pair of original and perturbed points (e,e') as an example, we see that the original eigenvector in the neutral direction (direction of the flow) at e remains to be a neutral direction at the perturbed point e'. There can, of course, be small deviations from the neutral direction, but they will be averaged out as the trajectory moves in region R. When a trajectory is in the switching region S, random perturbation of arbitrarily small amplitude can alter the local eigenspace significantly. For instance, when the trajectory is at point a, random perturbation can kick it to point c. Such a perturbation has two effects. First, since the local eigenspaces at the two points are distinct, the neutral eigenvector at a, when carried over by the trajectory perturbed to c, will not be in the neutral direction at c. The vector typically will have a component in the unstable direction at c, and its length will consequently be stretched exponentially. Thus the length of the neutral vector on the attractor, when it is perturbed in the switching region as described, will generally increase exponentially, causing the null Lyapunov exponent to become positive. Second, the random perturbation that moves the trajectory from a to c, is in fact inconsistent with the deterministic dynamics because, in the absence of the perturbation, the trajectory would move past point b.

Let $f^L(D)$ and $f^R(D)$ be the frequencies of visits of a typical trajectory to the L and R scrolls, respectively, under random perturbation of amplitude K, and let $f^S(K)$ be the probability that the trajectory experiences inconsistent perturbations in the switching region S, where $f^L(K)+f^R(K)+f^S(K)=1$. (If the trajectory simply passes through the switching region in a way consistent with the deterministic flow, we regard it as either in L or in R.) In the absence of coupling, we have $f^S(0)=0$, so the null Lyapunov exponent of the chaotic attractor can be written as $\lambda_0=f^L(0)\lambda_0^L+f^R(0)\lambda_0^R$, where λ_0^L and λ_0^R are the average rates of change of infinitesimal vectors along the corresponding eigendirections when the trajectory is in the left scroll and in the right scroll, respectively. The null exponent can then be trivially written as $\lambda_0(0)=f^L(0)\lambda_0^L+f^R(0)\lambda_0^R=0$. Under coupling, when the trajectory is perturbed inconsistently in the switching region, the neutral vector is stretched exponentially there. In the typical case where there is a dominant unstable steady state in the switching region, the rate is mainly determined by the largest eigenvalue of the steady state. Let $\bar{\lambda}>0$ be the Lyapunov exponent associated with this eigenvalue. We have

$$\lambda_0(K) \approx f^L(K)\lambda_0^L + f^R(K)\lambda_0^R + f^S(K)\bar{\lambda} = f^S(K)\bar{\lambda}. \quad (10)$$

For $K \gtrsim 0$, $f^S(K)$ is proportional to the probability that a trajectory falls in the switching region, which is proportional to the probability that a trajectory crosses the stable manifold of the dominant unstable steady state. In the three-dimensional phase space, a perturbed trajectory near the unstable steady state can be found in a sphere centered at the steady state. For small coupling, the radius of the sphere is proportional to

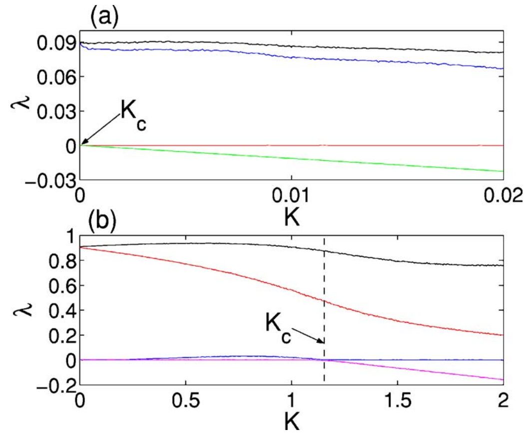

FIG. 4. (Color online) For the coupled Rössler system [Eq. (1)] and coupled Lorenz system [Eq. (2)] [(a) and (b), respectively], the four largest Lyapunov exponents are shown vs the coupling parameter K.

the amplitude of the "chaotic disturbance" from the coupled oscillator, and hence, as an approximation, the radius is proportional to the coupling parameter K. Since the dimension of the unstable steady state is 2, we have, for three-dimensional flows [19], $f^S(K) \sim K^2$ for $K \gtrsim 0$. The relevant observation is that as soon as the coupling is increased from zero, a null Lyapunov exponent becomes positive, destroying any phase coherence between the two coupled oscillators. Insofar as this exponent remains positive, intermittent chaotic synchronization cannot occur.

The increase of the null Lyapunov exponent cannot continue indefinitely. In fact, as the coupling parameter K is increased further, the exponent reaches a maximum and then decreases. We can imagine that, eventually, for $K > K_c > 0$, the exponent will become negative. This marks the onset of phase coherence and intermittent chaotic synchronization. Our analysis suggests that K_c is finite, which means that the transition to intermittent synchronization is delayed.

Figures 4(a) and 4(b) show the numerically calculated behavior of the Lyapunov exponents for the coupled Rössler and coupled Lorenz system, respectively. We see that in Fig. 4(a), a null Lyapunov exponent becomes negative immediately as K is increased from zero, while in Fig. 4(b), for $K < K_c \approx 1.1$, a null exponent is actually positive. This exponent becomes negative for $K > K_c$. The numerically estimated value of K_c is consistent with the transition point to intermittent chaotic synchronization, as in Fig. 2.

IV. UNSTABLE PERIODIC-ORBIT THEORY

To make feasible the formulation of a periodic-orbit theory for intermittent chaotic synchronization, we consider the following system of two coupled, identical chaotic oscillators:

$$\dot{\mathbf{x}}_{1,2} = \mathbf{f}(\mathbf{x}_{1,2}) + \mathbf{K} \cdot (\mathbf{x}_{2,1} - \mathbf{x}_{1,2}), \quad (11)$$

where $\mathbf{K}$ is the linear coupling matrix. The synchronization manifold $\mathcal{M}$ of the system is given by $\mathbf{x}_1(t)=\mathbf{x}_2(t)\equiv\mathbf{x}(t)$, which is a solution of Eq. (11). Using the change of variables

$$\mathbf{u} \equiv (\mathbf{x}_1 + \mathbf{x}_2)/2 \quad \text{and } \mathbf{v} \equiv (\mathbf{x}_1 - \mathbf{x}_2)/2, \qquad (12)$$

we obtain the following equations for the motions in $\mathcal{M}$ and in the neighborhood of $\mathcal{M}$ in the transverse direction, respectively:

$$\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u}), \qquad (13)$$

$$\dot{\mathbf{v}} = [\mathbf{Df}(\mathbf{x}) - 2\mathbf{K}] \cdot \mathbf{v}, \qquad (14)$$

where the dynamics in $\mathcal{M}$ is determined by the velocity field $\mathbf{f}(\mathbf{u})$ that generates a chaotic attractor, and the matrix $\mathbf{Df}(\mathbf{x})$ is evaluated with respect to the chaotic synchronization solution $\mathbf{x}(t)$. Because of the existence of the well-defined synchronization manifold, it is possible to examine the infinite set of unstable periodic orbits embedded in the chaotic attractor in terms of their abilities to synchronize with each other. In particular, given an unstable periodic orbit $\mathbf{x}^p(t)$, whether it can be synchronized with its replica is determined by its transverse Lyapunov spectrum, which can be computed by diagonalizing the matrix $\mathbf{Df}[\mathbf{x}^p(t)] - 2\mathbf{K}$. If the largest transverse Lyapunov exponent $\lambda_\perp^p$ is negative, this orbit embedded in the chaotic attractor of one oscillator can be synchronized with its replica in the attractor of the other coupled oscillator.

A chaotic trajectory can be regarded as consisting of sequential, intermittent visits to the infinite set of unstable periodic orbits embedded in the attractor. Thus, if some periodic orbits synchronize with their respective replicas, it is possible for chaotic trajectories from different oscillators to synchronize during the time intervals when they are in the vicinities of these periodic orbits. Because of the intermittent nature of visits to any subset of periodic orbits, synchronization between the trajectories will appear intermittent as well. In the absence of coupling, this is not possible, because the transverse stability of a periodic orbit is determined by the matrix $\mathbf{Df}[\mathbf{x}^p(t)]$, which gives unstable dynamics, as all periodic orbits embedded in a chaotic attractor are unstable. As the coupling is increased from zero, because of the $-2\mathbf{K}$ adjustment to $\mathbf{Df}[\mathbf{x}^p(t)]$, it is possible that the largest transverse Lyapunov exponents of some periodic orbits become negative, so as to make them synchronized. For a given coupling strength, let $\min\{\lambda_\perp\}$ be the minimum value of the largest transverse Lyapunov exponents of all unstable periodic orbits. Intermittent chaotic synchronization is possible if $\min\{\lambda_\perp\} < 0$.

Depending on the value of $\min\{\lambda_\perp\}$ for $K \gtrsim 0$, there can be two distinct types of transition to intermittent chaotic synchronization.

(i) $\min\{\lambda_\perp\} \approx 0$ for $K \gtrsim 0$. In this case, a small increase in K can cause $\min\{\lambda_\perp\}$ to become negative, so it is likely for intermittent synchronization to set in as soon as K is increased from zero. This is an immediate transition to intermittent chaotic synchronization which, practically, means that the synchronization is observable even in the weakly coupling regime.

(ii) $\min\{\lambda_\perp\}$ is not close to zero and relatively large. In this case, a large value of K is necessary to make $\min\{\lambda_\perp\}$ negative. This is a delayed transition to intermittent chaotic synchronization, which means that the synchronization can be observed only when the coupling is sufficiently strong.

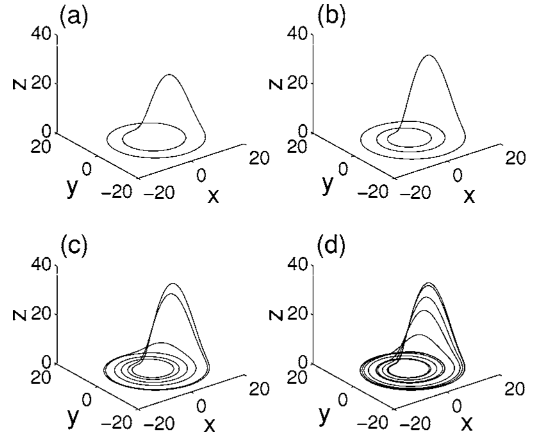

FIG. 5. Some representative unstable periodic orbits for the chaotic Rössler attractor in Eq. (1)—(a) period 2, (b) period 3, (c) period 6, and (d) period 12.

A practical issue is that only a limited number of unstable periodic orbits can be numerically computed. It is thus necessary to be able to infer whether the value of $\min\{\lambda_\perp\}$ for $K \approx 0$ is close to zero, or not, from a finite data set. We developed the following empirical procedure to address this problem. For a given chaotic oscillator, we first compute as many unstable periodic orbits, of as high periods as possible, to within the limit of our computational resource. Let p_{max} be the maximum period. For any $p \leqslant p_{max}$, we compute $\lambda_\perp^{min}(p)$, the mimimum value of the largest transverse Lyapunov exponents for all the available periodic orbits of period p, and compare all values of $\lambda_\perp^{min}(p)$. If $\min\{\lambda_\perp^{min}(p)\}$, the minimum value of all $\lambda_\perp^{min}(p)$, occurs for $p \approx p_{max}$, it is conceivable that $\lambda_\perp^{min}(p)$ can further decrease, should one be able to compute periodic orbits of higher periods. We can thus infer that

$$\min\{\lambda_\perp\} = \lim_{p\to\infty} \min\{\lambda_\perp^{min}(p)\} \approx 0,$$

which implies an immediate transition to intermittent chaotic synchronization. However, if $\min\{\lambda_\perp^{min}(p)\}$ occurs for $p < p_{max}$, it is possible that $\lambda_\perp^{min}(p)$ will not decrease significantly even if periodic orbits of higher periods can be computed. This implies that the value of $\min\{\lambda_\perp\}$ for $K \approx 0$ is not close to zero, giving rise to a delayed transition to intermittent chaotic synchronization.

A convenient numerical criterion is to examine the minimum value of the largest transverse Lyapunov exponent for all periodic orbits of period up to p. Let $\min\{\lambda_\perp(\bar{p})\}$ denote this value. Clearly $\min\{\lambda_\perp(\bar{p})\}$ is a nonincreasing function of p. For immediate transition to intermittent chaotic synchronization, we expect $\min\{\lambda_\perp(\bar{p})\}$ to show a continuous tendency to decrease as p is increased. However, for delayed transition to intermittent synchronization, $\min\{\lambda_\perp(\bar{p})\}$ will decrease initially, but it tends to plateau for some $p = p_c$, where $p_c < p_{max}$.

To compute unstable periodic orbits for the Rössler and Lorenz systems, we use the algorithm in Ref. [20] and adopt

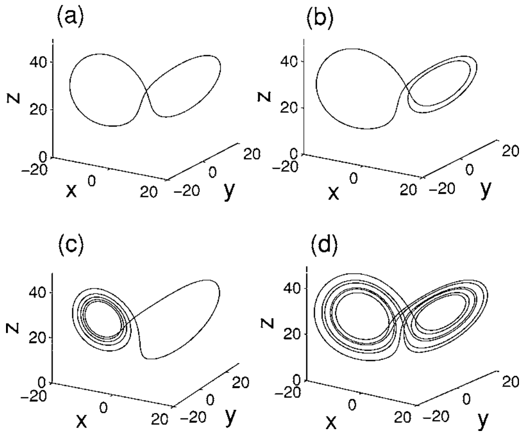

FIG. 6. Some representative unstable periodic orbits for the chaotic Lorenz attractor in Eq. (2)—(a) period 2, (b) period 3, (c) period 6, and (d) period 12.

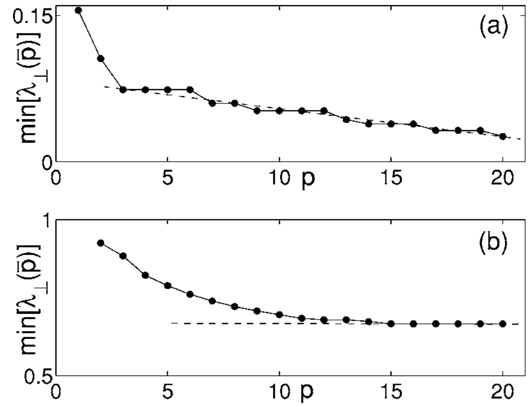

FIG. 7. The minimum value of the largest transverse Lyapunov exponents for all computed periodic orbits of periods up to 20 vs the periods for (a) the coupled Rössler system and (b) the coupled Lorenz system. For both systems, the coupling parameter is set to be about 10% of the value required for complete synchronization.

it to continuous-time systems, paying particular attention to distinguishing different orbits of the same period. For the Rössler and Lorenz attractors, we obtained about 1400 and 1100 distinct periodic orbits, respectively, of periods up to 20. Some representative orbits in the phase space are shown in Figs. 5 and 6. To see the transition to intermittent chaotic synchronization, we compute $\min\{\lambda_{\perp}(\bar{p})\}$ for both systems for the coupling value of about 10% of K_s, i.e., $K=0.01$ for the coupled Rössler system and $K=0.4$ for the coupled Lorenz system, as shown in Figs. 7(a) and 7(b). We see that for the coupled Rössler system, the minimum value of the transverse exponent occurs for $p=20=p_{max}$, and there is a clear tendency for $\min\{\lambda_{\perp}(\bar{p})\}$ to approach zero as p is increased. However, for the coupled Lorenz system, the minimum value of the largest transverse exponent occurs for $p=15<20=p_{max}$, and $\min\{\lambda_{\perp}(\bar{p})\}$ plateaus as p is increased from 15. The periodic-orbit analysis thus suggests that the transition to intermittent chaotic synchronization is immediate for the coupled Rössler system while it is delayed for the coupled Lorenz system.

V. DISCUSSION

In summary, we have addressed the phenomenon of intermittent synchronization in coupled chaotic oscillators and uncovered two types of transition, one immediate and another delayed. For the immediate transition, intermittent chaotic synchronization sets in as soon as the coupling is increased from zero, whereas the delayed transition requires a relatively large amount of coupling for the synchronization. We have presented evidence that coupled phase-coherent chaotic attractors exhibit immediate transition, while coupled phase-incoherent chaotic attractors exhibit the delayed transition. These can be understood by the behavior of a null Lyapunov exponent as a function of the coupling parameter and, at a more fundamental level, by the synchronization dynamics of unstable periodic orbits embedded in the chaotic attractor. We emphasize that these results hold for phase-coherent and phase-incoherent chaotic attractors in general, regardless of their dynamical details. The phenomenon of intermittent chaotic synchronization can be expected to occur commonly in coupled oscillator systems, which are important for many physical, chemical, and biological systems.

We have observed that the characteristic difference in the transition persists even if the coupled oscillators are slightly nonidentical or under small noise. This can be understood by noting that noise or small parameter mismatch typically makes complete synchronization intermittent. It is thus intuitively apparent that intermittent synchronization is robust under noise or small parameter mismatch. At a fundamental level, since the transition to intermittent synchronization is determined by the unstable periodic orbits, and since small noise or mismatch cannot destroy these orbits, the distinct transitions reported in this paper are robust.

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[1] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); **75**, 1087 (1986); T. Yamada and H. Fujisaka, *ibid.* **70**, 1240 (1983); **72**, 885 (1984).

[2] V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, Izv. Vyssh. Uchebn. Zaved., Radiofiz. **29**, 1050 (1986); A. R. Volkovskii and N. F. Rul'kov, Sov. Tech. Phys. Lett. **15**, 249 (1989).

[3] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).

[4] W. L. Ditto and K. Showalter, Chaos **7**, 509 (1997).

[5] J. Kurths, S. Boccaletti, C. Grebogi, and Y.-C. Lai, Chaos **13**, 126 (2003).

[6] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization—A Universal Concept in Nonlinear Sciences* (Cambridge University Press, Cambridge, England, 2001).
[7] J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. E **50**, 1874 (1994); **52**, R1253 (1995).
[8] J. Rössler, Phys. Lett. **57A**, 397 (1976).
[9] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[10] D. Auerbach, P. Cvitanović, J.-P. Eckmann, G. Gunaratne, and I. Procaccia, Phys. Rev. Lett. **58**, 2387 (1987); G. H. Gunaratne and I. Procaccia, *ibid.* **59**, 1377 (1987); D. Auerbach, B. O'Shaughnessy, and I. Procaccia, Phys. Rev. A **37**, 2234 (1988).
[11] Y.-C. Lai, C. Grebogi, J. A. Yorke, and S. C. Venkataramani, Phys. Rev. Lett. **77**, 55 (1996).
[12] P. Ashwin, J. Biescu, and I. Stewart, Phys. Lett. A **193**, 126 (1994).
[13] S. C. Venkataramani, B. R. Hunt, E. Ott, D. J. Gauthier, and J. C. Bienfang,Phys. Rev. Lett. **77**, 5361 (1996); A. V. Zimin, B. R. Hunt, and E. Ott, Phys. Rev. E **67**, 016204 (2003).
[14] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).
[15] The dips in the plot of $\Phi(K)$ are due to periodic windows where time-lagged synchronization between periodic attractors occurs.
[16] J. D. Farmer, Phys. Rev. Lett. **47**, 179 (1981).
[17] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).
[18] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos and Strange Attractors* (Springer-Verlag, New York, 1982).
[19] Z. Liu, Y.-C. Lai, and M. A. Matias, Phys. Rev. E **67**, 045203(R) (2003); Y.-C. Lai and Z. Liu, Chaos **14**, 189 (2004).
[20] R. L. Davidchack and Y.-C. Lai, Phys. Rev. E **60**, 6172 (1999).