A Unifying Approach to Determine the Necessary and Sufficient Conditions for Nonblocking Multicast 3-Stage Clos Networks

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***Abstract*—The 3-stage Clos network is the most-studied switching network. However, exact conditions on the strictly nonblocking multicast three-stage Clos network under various models are difficult to get, due to the messy detail and because each case employs a different argument. Hwang and Liaw made the latest attempt and pointed out errors in previous attempts. However, they made errors, too. In this paper, we propose a unifying approach to study those models systematically (which also applies to some wide-sense nonblocking (WSNB) networks). We also propose a new routing algorithm, and use the unifying approach to derive a necessary and sufficient condition for WSNB.**

***Index Terms*—3-stage Clos network, strictly nonblocking switching network, wide-sense nonblocking (WSNB) switching network, window algorithm.**

I. Introduction

A 3-stage Clos network $C(n_1, r_1, n_2, r_2, m)$ [1] has r_1 switches of size $n_1 \times m$ in the input stage (stage 1), m switches of size $r_1 \times r_2$ in the middle stage (stage 2), and r_2 switches of size $m \times n_2$ in the output stage (stage 3). Every middle switch has exactly one link to each input switch and each output switch. Thus, the network has $N_1 = n_1 r_1$ inputs and $N_2 = n_2 r_2$ outputs (see Fig. 1).

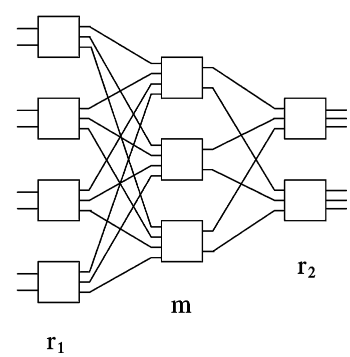

Fig. 1. $C(2,4,3,2,3)$ with eight inputs and six outputs.

Each switch is assumed to be nonblocking, in the sense that any one-to-one mapping between its inputs and outputs can be connected simultaneously. A switch is said to have fan-out capability if any one-to-many mapping between its inputs and outputs can be connected. Under multicast traffic, each input can request to connect to many idle outputs. A request is called an f-request if the number of outputs involved is f. In an f_2-cast traffic, no f-request is allowed for $f > f_2$. A network is f_2-cast strictly nonblocking (SNB) if it can always route an f-request, $f \le f_2$, regardless of how existing connections are routed. It is f_2-cast wide-sense nonblocking (WSNB) if routing is possible under the condition that all requests are routed according to a given routing strategy.

Giacomazzi and Tricordi [3] first studied necessary and sufficient conditions for $C(n_1, r_1, n_2, r_2, m)$ to be SNB for f_2-cast traffic. (Although Masson and Jordan [8] wrote a paper some 20 years earlier, giving a sufficient condition for SNB in a multicast 3-stage Clos network, this network was later [2] determined to be WSNB). Let m^o denote the number of middle switches required for SNB. Giacomazzi and Tricordi gave the following necessary and sufficient condition:

$$m \ge \max_{f, f_c, r} \{f + f_c + \min\{N_2 - f - f_c, f_2 r\}\} \quad (1)$$

where f indicates that the current request c is f-cast, f_c is the number of outputs connected by other inputs on the same input switch as c, and

$$r = \min\{N_2 - f - f_c, (n_2 - 1)f\}$$

is the maximum number of connections over all requests going to at least one of the output switches involved in the current request.

Note that (1) is complicated and implicit, since one has to maximize over three variables. No closed-form solution was given in [3]. Recently, Pattavina and Tesei [9] gave a counterexample against (1).

Giacomazzi and Tricordi also gave a similar equation for the model where the input stage has no fan-out capability.

Hwang and Liaw [6] used a different approach to obtain a necessary and sufficient condition for SNB in a f_2*-cast* 3-stage Clos network

$$m^o = \min\{(n_1 - 1)f_2 + n_2, (N_1 - 1)f_2 + 1, N_2\} \quad (2)$$

where the second term reflects the boundary condition from the input size, and the third term reflects the boundary condition from the output side. Compared with (1), (2) is explicit and

Paper approved by D. K. Hunter, the Editor for Optical Communication of the IEEE Communications Society. Manuscript received February 24, 2003; revised April 14, 2004. This work was supported in part by the Republic of China National Science Council under Grant NSC-91-2115-M-009-010, and in part by the LI-MCI Center.

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Digital Object Identifier 10.1109/TCOMM.2005.852839

easily computed. Pattavina and Tesei extended (2) to the case that a multicast call must connect to at least f_1 outputs.

Call this model, which assumes fan-out capability for every switch, model 0. Hwang and Liaw also gave similar equations to models 1, 2, and 3, where model i assumes that switches in stage i have no fan-out capability. Note that such a model can also be treated as a WSNB [4], [5] network by interpreting that no fan-out in stage i is not due to the lack of capability of the switch, but due to the rule of a routing strategy.

Hwang and Liaw also studied a genuine WSNB model, in which an f-request can have at most one path going to an output switch, regardless of how many outputs on that switch are involved in the request. This routing strategy is called "no-split," and was the one used in [8].

Actually, Hwang and Liaw considered two cases for each of the above five models according to whether a busy input can generate another request by adding extra outputs to existing ones. If this is not allowed, the traffic is closed end; otherwise, it is open end. So, in total, they gave ten sets of necessary and sufficient conditions.

In this paper, we revise these ten sets of necessary and sufficient conditions. Our method is similar to Hwang and Liaw, but some intricate points are more carefully analyzed. In particular, we develop a unifying approach which works for all ten cases, instead of the case-by-case arguments given in [6]. This systematic approach discovers some errors in [6], and allows an easy check on the correctness of the current results. We also propose a new WSNB algorithm using the window algorithm (to be defined in Section IV) first proposed by Tscha and Lee [10], and also studied by Kabacinski and Danilewitz [7] for the multi-$\log_2 N$ network. We use the unifying approach to derive necessary and sufficient conditions for $C(n_1, r_1, n_2, r_2, m)$ to be WSNB under the window algorithm. Interestingly, models 1 and 2 becomes two special cases where the window size is maximum or minimum.

II. A Unifying Approach

Our approach gives a more systematic and orderly counting to the number of blocked middle switches. Let the current request be from input i on switch I to f outputs on a set O of switches. Then the order is blocking by paths from I, blocking by paths from other input switches, blocking by paths from O. Such an ordered counting allows us to eliminate duplicated counting. We also highlight the notion of the minimum number of middle switches required to connect the current requests when blocking from other paths is ignored. This notion helps to crystalize the counting. While previous approaches are not devoid of the above elements, they are usually less explicit and exact.

Let $b(I)$ denote the potential maximum number of middle switches occupied by paths from I which c cannot use, while ignoring the boundary effect from the output side, i.e., there are only N_2 outputs. Then the actual maximum number, after considering the boundary effect, is

$$\min\{b(I), N_2 - f\} \tag{3}$$

since each path through these middle switches must end at a distinct output, but there are only $N_2 - f$ of them available.

Suppose c involves outputs on several output switches. In general, it suffices to consider just one of them, since they do not compete for links to the middle stage, and the condition derived for one is the same for another. An exception is when those output switches involved in c are required to be routed through the same middle switch, as in model 1, then we must consider the output switches together, because a failure to reach one output switch is a failure for all.

First consider the case where connections to the switches in O are independent. Let $O_j \in O$ be an output switch containing k outputs in c. Then each of the other $n_2 - k$ outputs can occupy a distinct middle switch, as long as there are enough inputs to generate requests involving these $n_2 - k$ outputs. Since paths generated by inputs of I are already counted in (3), there are $N_1 - n_1$ inputs available. Let $b(N_1)$ denote the potential maximum number of middle switches the paths from these $N_1 - n_1$ inputs can occupy while ignoring the output side.

Another point to consider is that some of the $n_2 - k$ outlets may already be used in (3). There are $N_2 - f - (n_2 - k)$ outlets not in the f-request and not from O_j. So if

$$N_2 - f - n_2 + k < b(I) < N_2 - f$$

then

$$b(I) - (N_2 - f - n_2 + k)$$

among the $n_2 - k$ outputs from O_j are already consumed in (3). Hence, only $(N_2 - f - b(I))^+$ outputs from O_j can generate new paths, where $(x)^+ = \max\{x, 0\}$. Summarizing, the maximum number of new paths generated by available outputs from O_j is

$$\min\left\{n_2 - k, b(N_1), (N_2 - f - b(I))^+\right\}. \tag{4}$$

Let $b(c)$ denote the minimum number of middle switches required to guarantee the routing of c when there is no other connection. Then to route c among other connections, we must have $b(c)$ additional switches not counted in (3) and (4). In the worst-case scenario, where the switches in (3) and (4) are all distinct, then a sufficient condition to route c is

$$\begin{aligned} m \geq\ & \min\{b(I), N_2 - f\} \\ & + \min\left\{n_2 - k, b(N_1), (N_2 - f - b(I))^+\right\} + b(c) \\ =\ & \min\{b(I) + n_2 - k + b(c), \\ & \quad b(I) + b(N_1) + b(c), N_2 - f + b(c)\} \end{aligned} \tag{5}$$

obtained by pairing $b(I)$ with each term in the second min. Note that if the first min is $N_2 - f$, then the second min must be $(N_2 - f - b(I))^+ = 0$, and the condition becomes $N_2 - f + 0 + b(c)$, which is same as the last term in (5). A careful examination of the above arguments reveals that the worst-case scenario generating (3) and (4) can happen, hence, (5) is also a necessary condition. For $f = 1$, then $b(I) = n_1 - 1$, $b(N_1) = N_1 - n_1$, and $b(c) = 1$, (5) is then reduced to $\min\{n_1 + n_2 - 1, N_1, N_2\}$, the famous Clos result.

To save writing, let m^o denote the number such that $m \geq m^o$ is necessary and sufficient for $C(n_1, r_1, n_2, r_2, m)$ to be f_2-cast SNB (WSNB for the no-split model).

Lemma 1: Suppose the routings of c to the output switches in O are independent. Then

$$m^o = \max_{f,k} \min \{b(I) + n_2 - k + b(c), \\ b(I) + b(N_1) + b(c), N_2 - f + b(c)\}.$$

Corollary 2: If $b(c)$ is independent of k, then

$$m^o = \max_{f} \min \{b(I) + n_2 - 1 + b(c), \\ b(I) + b(N_1) + b(c), N_2 - f + b(c)\}.$$

Lemma 1 applies to models 0, 2, 3, and no-split, while *Corollary 2* applies to any model whose output stage has fan-out capability.

A traffic model is called consistent if every pair of paths from the same input, if legitimate separately, is a legitimate pair under the model. None of models 1, 2, or 3 is consistent, since two legitimate paths can go through the same stage-i switch and force a fan-out at that switch. Nor is the no-split model consistent, since the two paths going to the same output may go through different middle switches.

Lemma 3: m^o for open-end traffic is same as the closed-end case for a consistent model.

Proof: It was proved in [6] that if m^o is sufficient for open-end traffic, then it suffices for closed-end traffic. Therefore, we only need to prove the converse.

Consider an f-cast request from input i in open-end traffic such that i is already connected to f' outputs, $f' \leq f_2 - f$. We modify the traffic by deleting the requests (and connections) from i to these f' outputs, and add them to the f-cast request. Further, any set of existing requests from an input $i' \neq i$ are interpreted as one request with the set of connecting paths intact. Then the modified traffic is closed end, hence, the current $(f + f')$-cast request from i is routable.

Back to the original open-end traffic. We route the f-cast request using the same paths as in the modified traffic. Note that all existing paths, other than those from i, are same as in the modified traffic. So any overlapping of these paths with existing ones must be with paths from i. Since the model is consistent, all paths from the same input can co-exist. ■

Although the notion of consistency has limited applicability in *Lemma 3*, we will see in the next section that with modifications, it can apply to various models.

III. Necessary and Sufficient Conditions for Ten Models

We apply the results obtained in Section II to the ten models studied in [6].

Theorem 4: For model 0 under closed-end traffic

$$m^o = \min \{(n_1 - 1)f_2 + n_2, (N_1 - 1)f_2 + 1, N_2\}.$$

Proof: $b(I) = (n_1 - 1)f_2$, $b(N_1) = (N_1 - n_1)f_2$, and $b(c) = 1$. By *Corollary 2*

$$m = \max_{f} \min \{(n_1 - 1)f_2 + n_2, (N_1 - 1)f_2 + 1, N_2 - f + 1\} \\ = \min \{(n_1 - 1)f_2 + n_2, (N_1 - 1)f_2 + 1, N_2\} \text{ at } f = 1. \quad ■$$

Corollary 5: Open-end traffic has the same m^o.

Proof: Follows from *Lemma 3*. ■

Theorem 6: For model 2 under closed-end traffic

$$m^o = \min \{(n_1 - 1)f_2 + n_2 - 1 + \min\{f_2, r_2\}, \\ (N_1 - 1)f_2 + \min\{f_2, r_2\}, N_2\}.$$

Proof: $b(I) = (n_1 - 1)f_2$, $b(N_1) = (N_1 - n_1)f_2$, and $b(c) = \min\{f, r_2\}$, since if $f > r_2$, r_2 additional middle switches still suffice by routing each such middle switch to a distinct output switch, and then using the fan-out capability of the output switches to reach multiple outputs. The reason that f_2, but not $\min\{f_2, r_2\}$, is in $b(I)$ and $b(N_1)$ is because under the SNB rule, each connection can be routed arbitrarily, as long as the paths are available. Hence, a connection can use two different paths to reach two inputs on the same output switch, even if it is a waste. By *Corollary 2*

$$m^o = \max_{f} \min\{(n_1 - 1)f_2 + n_2 - 1 + \min\{f, r_2\}, (N_1 - 1)f_2 \\ + \min\{f, r_2\}, N_2 - f + \min\{f, r_2\}\} \\ = \min \{(n_1 - 1)f_2 + n_2 - 1 + \min\{f_2, r_2\}, (N_1 - 1)f_2 \\ + \min\{f_2, r_2\}, N_2\} \text{ at } f = \min\{f_2, r_2\}. \quad ■$$

Theorem 7: For model 2 under open-end traffic

$$m^o = \min\{n_1 f_2 + n_2 - 1, N_1 f_2, N_2\}.$$

Proof: Unlike the case in *Corollary 5*, the middle switches cannot fan out. Hence, i cannot use a middle switch already carrying a path from i to route c. Since $I \setminus \{i\}$ can connect to a maximum of $(n_1 - 1)f_2$ outputs, while i can already have connected to a maximum of $f_2 - f$ outputs, $b(I) = n_1 f_2 - f$. $b(N_1)$, and $b(c)$ are same as the open-end case. By *Corollary 2*

$$m^o = \max_{f} \min \{n_1 f_2 - f + n_2 - 1 + \min\{f, r_2\}, N_1 f_2 - f \\ + \min\{f, r_2\}, N_2 - f + \min\{f, r_2\}\} \\ = \min\{n_1 f_2 + n_2 - 1, N_1 f_2, N_2\} \text{ at any } f \leq r_2. \quad ■$$

Note that the conditions for closed-end traffic and open-end traffic are different, contrary to the conclusion in [4]. We give an example that m^o in *Theorem 6* is not sufficient for open-end traffic.

Example: Consider $C(2, 2, 4, 3, 10)$ with $f_2 = 4$. Then I_1 has inlets $\{i_1, i_2\}$, I_2 has inlets $\{i_3, i_4\}$, and O_j has outlets $\{O_{4(j-1)+1}, O_{4(j-1)+2}, O_{4(j-1)+3}, O_{4(j-1)+4}\}$ for $j = 1, 2, 3$. Then $m^o = 10$ in *Theorem 6* and $m^o = 11$ in *Theorem 7*. Let $M_1, \ldots, M_k$ be the ten middle switches. The current request is (i_i, o_1), and the existing paths are (i_1, M_1, o_5), (i_1, M_2, o_6), (i_1, M_3, o_7), (i_2, M_4, o_8),

(i_2, M_5, o_9), (i_2, M_6, o_{10}), (i_2, M_7, o_{11}), (i_3, M_8, o_2), (i_3, M_9, o_3), and (i_3, M_{10}, o_4). Then none of the M_i can carry (i_1, o_1).

Theorem 8: For model 3 under closed-end traffic

$$m^o = \min\{(n_1 - 1)f_2 + n_2, N_1 f_2, N_2\}.$$

Proof: It is easily argued that $b(I) = (n_1 - 1)f_2$ and $b(N_1) = (N_1 - n_1)f_2$. To derive $b(c)$, note that if c involves outputs on the same output switch, then these outputs must each be routed through a distinct middle switch. On the other hand, outputs of different output switches can be routed through the same middle switch. Let k denote the maximum number of outputs on the same output switch. Then $b(c) = k$, where $k \le \min\{f, n_2\}$. By *Lemma 1*

$$\begin{aligned} m^o &= \max_{f,k} \min\{(n_1 - 1)f_2 + n_2 - k + k, \\ &\qquad (N_1 - 1)f_2 + k, N2 - f + k\} \\ &= \max_f \min\{(n_1-1)f_2+n_2, (N_1-1)f_2+\min\{f, n_2\}, \\ &\qquad N_2 - f + \min\{f, n_2\}\} \text{ at } k = \min\{f, n_2\} \\ &= \min\{(n_1-1)f_2+n_2, (N_1-1)f_2+\min\{f_2, n_2\}, N_2\} \\ &\qquad \text{at } f = \min\{f_2, n_2\} \\ &= \min\{(n_1 - 1)f_2 + n_2, N_1 f_2, N_2\} \end{aligned}$$

since if $\min\{f_2, n_2\} = n_2$, then

$$(n_1 - 1)f_2 + n_2 \le (N_1 - 1)f_2 + \min\{f_2, n_2\}. \quad \blacksquare$$

Theorem 9: Open-end traffic has the same m^o.

Proof: We use the same method as in the proof of *Lemma 3*, but with some modification. Note that with the modified traffic, the paths from i to outputs in the same output switch must go through different middle switches. Further, these paths can be interchanged without affecting the routability. We will make necessary interchanges, such that no path from i to $o \in O_j$ goes through a middle switch which routes i to $o' \in O_j$ in an existing path in open-end traffic. Then we can route the current request under open-end traffic by using the paths in the modified traffic. ■

Theorem 10: For the no-split algorithm under closed-end traffic

$$\begin{aligned} m^o = \min\{&(n_1 - 1)\min\{f_2, r_2\} + n_2, \\ &(N_1 - 1)\min\{f_2, r_2\} + 1, N_2\}. \end{aligned}$$

Proof: An f-request can use at most $\min\{f, r_2\}$ paths to reach the output stage. Hence, $b(I) = (n_1 - 1)\min\{f_2, r_2\}$ and $b(N_1) = (N_1 - n_1)\min\{f_2, r_2\}$, while $b(c) = 1$ (since there is no other connection, any middle switch will do). By *Corollary 2*

$$\begin{aligned} m^o &= \max_{f,|O|} \min\{(n_1 - 1)\min\{f_2, r_2\} + n_2, \\ &\qquad (N_1 - 1)\min\{f_2, r_2\} + 1, N_2 - f + 1\} \\ &= \min\{(n_1 - 1)\min\{f_2, r_2\} + n_2, \\ &\qquad (N_1 - 1)\min\{f_2, r_2\} + 1, N_2\} \text{ at } f = 1. \quad \blacksquare \end{aligned}$$

Theorem 11: Open-end traffic has the same m^0.

Proof: If the current request from i contains an output o from the same output switch O_j, such that there exists a path from i to O_j, then use the same path to route (i, o). So we may assume that the current request contains only outputs whose output switches are not connected to i. Use the same method as given in the proof of *Lemma 3* to route them, since the path from i to these outputs can coexist with existing paths from i. ■

Finally, we deal with model 1. In this model, we need to find one middle switch which can route to all output switches in O. Therefore, we must replace $n_2 - k$ in (4) by $|O|n_2 - f$. The corresponding change in m^o from *Lemma 1* is

$$\begin{aligned} m^o = \max_{f,|O|} \min\{&b(I) + |O|n_2 - f + b(c), \\ &b(I) + b(N_1) + b(c), N_2 - f + b(c)\}. \quad (6) \end{aligned}$$

Theorem 12: For model 1 under closed-end traffic

$$\begin{aligned} m^o = \max\{&\min\{n_1 + (n_2 - 1)\lfloor f^o \rfloor, N_1\}, \\ &\min\{N_2 - \lceil f^o \rceil + 1, N_1\}\} \\ \text{where } f^o = &\frac{(N_2 + 1 - n_1)}{n_2}. \end{aligned}$$

Proof: Since each f-request must go to a single middle switch, $b(I) = n_1 - 1$, $b(N_1) = N_1 - n_1$, and $b(c) = 1$. Further, outputs in C which are on the same O_i must be connected through the same path, and then reached through the output switch fan-out. Thus we may assume $f \le f_2 \le r_2$. By (6)

$$\begin{aligned} m^o &= \max_{f,|O|} \min\{n_1 + |O|n_2 - f, N_1, N_2 - f + 1\} \\ &= \max_f \min\{n_1 + (n_2 - 1)f, N_1, N_2 - f + 1\} \\ &\qquad \text{at } |O| = f \text{ (the maximum value of } |O|). \end{aligned}$$

Since the first term is increasing in f, and the third decreasing, the maximum should occur at

$$n_1 + (n_2 - 1)f = N_2 - f + 1$$

which is solved by f^o if the integrability of f is ignored. It is easily verified that the maximum occurs either at $\lfloor f^o \rfloor$ or $\lceil f^o \rceil$. Further

$$\begin{aligned} N_2 - \lfloor f^o \rfloor + 1 &\ge n_1 + (n_2 - 1)\lfloor f^o \rfloor \\ N_2 - \lceil f^o \rceil + 1 &\le n_1 + (n_2 - 1)\lceil f^o \rceil. \quad \blacksquare \end{aligned}$$

For open-end traffic, it was shown in [4] that no m^o is large enough to guarantee SNB.

IV. Wide-Sense Nonblocking Under the Window Algorithm

We first describe the window algorithm. Partition the output switches into sets of t (assuming t divides r_2), and call each set a window. An f-request involving outputs in w windows will be treated as w multicast requests, where the ith subrequest involves only the f_i outputs in window i. Each subrequest must be routed through one middle switch only, while two subrequests from the same input are treated as different requests, and hence, cannot be routed through the same middle switch.

Let f denote the sum of f_i. Note that the window algorithm implies the no-split rule, since two outputs on the same output switch must be in the same window, and their routes go through the same middle switch. Clearly, their routes must also go through the same link between that middle switch and their output switch. Therefore, in deriving the tightest nonblocking condition, we may assume that each f-request involves at most one output from an output switch. Hence, f_i is also the number of output switches in window i involved in the f-request. Clearly, $w \leq \min\{f, r_2/t\}$, $f_i \leq \min\{t, f-w+1\}$, and $f \leq \min\{f_2, r_2\}$.

Theorem 13: For the t-window algorithm under closed-end traffic

$$m^o = \max\left[\min\left\{n_1 \min\left\{f_2, \frac{r_2}{t}\right\} + (\lfloor v^o \rfloor + 1)(n_2-1), N_1 \min\left\{f_2, \frac{r_2}{t}\right\}\right\}, \min\left\{N_2 - \lceil v^o \rceil, N_1 \min\left\{f_2, \frac{r_2}{t}\right\}\right\}\right]$$

where

$$v^o = \min\left\{\frac{[N_2 - (n_1-1)\min\{f_2, \frac{r_2}{t}\}(n_2-1) - w]}{n_2}, t-1, f_2 - w\right\}.$$

Further, if $\lfloor v^o \rfloor = \lceil v^o \rceil$, then

$$m^o = \min\left\{(n_1-1)\min\left\{f_2, \frac{r_2}{t}\right\} + (v^o+1)(n_2-1) + w, (N_1-1)\min\left\{f_2, \frac{r_2}{t}\right\} + w, N_2 - v^o\right\}.$$

Proof: Suppose the request c consists of w subrequests. Note that each subrequest involves a distinct set of outputs. We compute the maximum number of middle switches needed to connect subrequest i. Since each subrequest is connected through a single middle switch, the connections of the other $w-1$ subrequests consume $w-1$ middle switches, regardless of the numbers of outputs in these subrequests, as long as they are positive. Therefore, the worst case for subrequest i occurs when f_i is maximum, with f and w fixed, i.e., $f_i = \min\{t, f-w+1\}$. Since the subrequests are interchangeable, the worst-case number of middle switches that suffices for one subrequest suffices for all.

Since each subrequest is routed through a single middle switch, (6) for model 1 can be used to compute the number of middle switches required to connect subrequest i, except replacing $|O| = f = f_i$. To compute m^o for connecting c, we also have to maximize over f and w, and add the $w-1$ middle switches required to connect the other $w-1$ subrequests. Since c consists of at most $\min\{f_2, r_2/t\}$ subrequests, we have $b(I) = (n_1-1)\min\{f_2, r_2/t\}$,

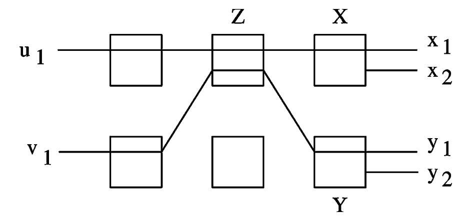

Fig. 2. Request (u_1, y_2) is unroutable.

$b(N_1) = (N_1 - n_1)\min\{f_2, r_2/t\}$, $b(c) = 1$, and $w \leq \min\{f_2, \lceil r_2/t \rceil\}$

$$\begin{aligned} m^o &= \max_{f,w,f_i}\left[\min\left\{(n_1-1)\min\left\{f_2, \frac{r_2}{t}\right\} + f_i(n_2-1)+1, (N_1-1)\min\left\{f_2, \frac{r_2}{t}\right\}+1, N_2-f+1\right\}\right] + w - 1 \\ &= \max_{f,w}\left[\min\left\{(n_1-1)\min\left\{f_2, \frac{r_2}{t}\right\} + \min\{t, f-w+1\}(n_2-1) + w, (N_1-1)\min\left\{f_2, \frac{r_2}{t}\right\} + w, N_2 - f + w\right\}\right] \\ &= \max_{\substack{0 \leq v \leq t-1 \\ 1 \leq w \leq \min\left\{f_2, \frac{r_2}{t}\right\}}}\left[\min\left\{(n_1-1)\min\left\{f_2, \frac{r_2}{t}\right\} + (v+1)(n_2-1) + w, (N_1-1)\min\left\{f_2, \frac{r_2}{t}\right\} + w, N_2 - v\right\}\right] \end{aligned} \quad (7)$$

by changing the variable from $f-w$ to v. Setting equal the two terms in (7) containing v, we obtain the first term of v^o. The second and third terms represent some boundary conditions imposed by the other two terms in (7).

Clearly, the maximum is obtained at $v =$ either $\lfloor v^o \rfloor$ or $\lceil v^o \rceil$, and at $w = \max w = \min\{f_2, r_2/t\}$. At $v = \lfloor v^o \rfloor$, we can drop the third term of (6), as it is larger than the first term. At $v = \lceil v^o \rceil$, we can drop the first term.

Thus, *Theorem 13* follows. □

Corollary 14: For $t = r_2$, m^o in *Theorem 13* is the same as m^o in *Theorem 12*.

Proof: For $t = r_2$, $\min\{f_2, r_2/t\} = 1$ and $v^o = [N_2 - (n_1-1) - (n_2-1) - 1]/n_2 = f^o - 1$, where f^o was given in *Theorem 12*. *Corollary 14* is easily verified. □

Corollary 14 is not surprising, since if all output switches are in the same window, then every f-cast request must be routed through the same middle switch, the same constraint as in model 1.

Corollary 15: For $t = 1$, $m^o = \min\{n_1 \min\{f_2, r_2\} + n_2 - 1, N_1 \min\{f_2, r_2\}, N_2\}$.

Proof: For $t = 1$, $v^o = 0$. *Corollary 15* easily follows from *Theorem 13* with $\lfloor v_o \rfloor = \lceil v^o \rceil$. □

The m^o in *Corollary 15* is same as in model 2, except f_2 is replaced by $\min\{f_2, r_2\}$. This difference is due to the fact that no-split is forced under the window algorithm, but not in model 2.

Fig. 2 shows that for open-end traffic under the window algorithm, $C(n_1, r_1, n_2, r_2, m)$ cannot guarantee WSNB, no matter how large m is. Let X and Y be in the same window. Then,

TABLE I
RESULTS OF THE FIVE MODELS (FIRST ROW REPRESENTS CLOSED-END, SECOND ROW, OPEN-END, AND $*$ INDICATES EITHER A NEW RESULT OR A CORRECT PROOF IS NOW PROVIDED)

model	main condition	input boundary	output boundary
0	$(n_1-1)f_2+n_2$	$(N_1-1)f_2+1$	N_2
	same		
1	$n_1+(n_2-1)\lfloor(N_2+1-n_1)/n_2{}^*\rfloor$	N_1	$N_2-\lceil(N_2+1-n_1)/n_2\rceil+1$
	blocking		
2	$(n_1-1)f_2+n_2-1+\min\{f_2,r_2\}$	$(N_1-1)f_2+\min\{f_2,r_2\}$	N_2
	$n_1f_2+n_2-1^*$	$N_1f_2{}^*$	N_2
3	$(n_1-1)f_2+n_2$	N_1f_2	N_2
	same*		
no-split	$(n_1-1)\min\{f_2,r_2\}+n_2$	$(N_1-1)\min\{f_2,r_2\}+1$	N_2
	same*		

request (u_1, y_2) is unroutable, since it must be routed through Z under the window algorithm, but the link from Z to Y is occupied. Contrasting this example with *Theorem 13*, we see a qualitative difference between the closed-end and the open-end traffic for the window algorithm (just as for model 1).

The window algorithm offers a continuum of choices between models 1 and 2. It also compares favorably with models in Section II. As we said, model 1 corresponds to choosing $t = r_2$, and model 2 is close to, but dominated by, the choice $t = 1$. For f_2 large (for example, the unconstrained broadcast case, where $f_2 = r_2$), models 0, 2, 3, and the no-split algorithm essentially require $n_1 r_2$ middle switches, while the window algorithm requires $n_1\sqrt{r_2}$ by choosing $t = \sqrt{r_2}$. On the other hand, for f_2 small, we can choose $t = 1$, such that the window algorithm essentially requires the same number of middle switches as models 0, 2, 3, and the no-split algorithm.

V. CONCLUSIONS

We summarize our results in Table I for easy comparisons. There are several interesting findings.

1) The network in model 0 is more powerful (and more costly) than the networks in models 1, 2, 3. Yet m^o in model 0 is only slightly better than those in models 2 and 3, and not comparable to m^o in model 1. This surprising phenomena, that a more powerful network may not have better performance, is due to the fact that in the SNB setup, if you give the routing freedom to do stupid things, it will.
2) m^o of the no-split is same as model 0 except for replacing f_2 by $\min\{f_2, r_2\}$, an improvement if $f_2 > r_2$. The improvement is not surprising, since the no-split model guarantees WSNB, while model 0 guarantees SNB.
3) The boundary effect from the output side is uniformly N_2, except in model 1. This is because only in model 1, the three terms in m^o contain one term increasing in f, and another (the output boundary) decreasing in f, resulting in an f-value which is not an extreme value.

The window algorithm not only provides a continuum of choices, but also some interesting connections between the models in Section II. For example, models 1 and 2 correspond to the two extreme cases of the window algorithm. Also, we have known that model 1 implies the no-split rule, and model 3 contradicts the no-split rule; now we further know the consequence of combining model 2 and the no-split rule. We have also shown that the performance of the window algorithm is about the same as others for f_2 small, but can be much better for f_2 large.

ACKNOWLEDGMENT

The author wishes to thank the reviewers for many helpful comments, the Associate Editor for an efficient handling of the review process, and Ms. B.C. Lin for a careful proofreading.

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F. K. Hwang, photograph and biography not available at time of publication.