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Research Article

Procedure to Solve Network DEA Based on a Virtual Gap Measurement Model

Fuh-hwa Franklin Liu and Yu-cheng Liu

Department of Industrial Engineering & Management, National Chiao Tung University, Hsinchu, Taiwan

Correspondence should be addressed to Yu-cheng Liu; brady.liu@mtigroup.com

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Network DEA models assess production systems that contain a set of network-structured subsystems. Each subsystem has input and output measures from and to the external network and has intermediate measures that link to other subsystems. Most published studies demonstrate how to employ DEA models to establish network DEA models. Neither static nor dynamic network DEA models adjust the links. This paper applies the virtual gap measurement (VGM) model to construct a mixed integer program to solve dynamic network DEA problems. The mixed integer program sets the total numbers of "as-input" and "as-output" equal to the total number of links in the objective function. To obtain the best-practice efficiency, each DMU determines a set of weights for inputs, outputs, and links. The links are played either "as-input" or "as-output." Input and as-input measures reduce slack, whereas output and as-output measures increase slacks to attain their target on the production frontier.

1. Introduction

Data envelopment analysis (DEA) models are used to measure the relative efficiency of each decision-making unit (DMU) relative to its peers regarding multiple input indices versus multiple output indices. Fewer input values and more output values are desired to improve the model's aggregated performance score. The DMU assigns a set of weights to the indices to obtain the best-practice performance score. The drawback of these models is the omission of the internal processes between inputs and outputs.

Seiford and Zhu [1] developed a DEA approach for evaluating US commercial banks in a two-stage process characterized by profitability and marketability. Zhu [2] applied the same two-stage process to the Fortune Global 500 companies. Sexton and Lewis [3] studied the performance of Major League Baseball in a two-stage process. The above two-stage DEA papers are among the first to address formally the links between the two internal stages and the link categories being distinguished and discussed; these are the origin of network DEA. For a two-stage model, the links' role implies a reduction in the first-stage outputs, thereby reducing the efficiency of that stage but increasing the efficiency of the second stage. A number of DEA studies have been developed in an attempt to address this type of conflict. For instance, Lewis and Sexton [4] used the network DEA approaches of Färe and Whittaker [5] and Färe and Grosskopf [6, 7] to compute the efficiency scores of subprocesses. "Link" cannot be adjusted freely in radial models, which adjust the inputs and outputs by the efficiency scores in a two-stage process. For these models, the entire system efficiency cannot be improved by adjusting links; see Kao and Hwang [8] and Lewis and Sexton [4]. "Link" applied in nonradial models (SBM), as has been discussed in recent years. Tone and Tsutsui [9] introduced a network DEA and categorized links into two types of "fixed links" and "free links." "Free links" means that the intermediate items are adjustable or discretionary, not to change direction, and each DMU can be increased or decreased from the observed one and is free to assign each individual link to one of the three characteristics: as-input, as-output, or nondiscretionary so that entire system efficiency could be maximized. "Fixed links" means that the intermediate items are nonadjustable or nondiscretionary. The linking activities are kept unchanged that cannot improve the efficiency of the entire system. In other words, the intermediate products are beyond the control of DMUs.

The recent extension of network DEA development to dynamic DEA does not represent the effect of carry-over activities (links) between two consecutive terms (nodes). Tone and Tsutsui [10] and Kao [11] introduced the dynamic DEA model using the nonradial and radial DEA models, respectively. These researchers express the outputs from term "t" as being carried over to term "$t + 1$," which means that the carry-over can be regarded as one type of link. Tone and Tsutsui [12] categorized carry-over into four types: desirable (good), undesirable (bad), discretionary (free), and nondiscretionary (fixed). The discretionary link is the same as the free link in network DEA. Kao [13] considered general multistage systems in which exogenous inputs are consumed in addition to intermediate products. Cheng and Gao [14] proposed a matrix-type network model which uses data in input-output tables; it is tested and can be feasible in evaluating the relative performance. There are many articles illustrating the application of free links and fixed links. Lozano [15] relaxed the constraints for both the fixed-link and the free-link cases, thus enhancing the discriminating power of the model. This study is to present the performance assessment of the individual processes of an external efficiency model. Avkiran [16] illustrated dynamic network data envelopment analysis (DN-DEA) in commercial banking with emphasis on testing robustness. Huang et al. [17] proposed a two-stage network model with bad outputs and supper efficiency (US-NSBM). Empirical comparisons show that the US-NSBM may be promising and practical for taking the nonperforming loans into account and being able to rank all samples.

Decreasing the volumes of inputs and "as-input" links and increasing the volumes of outputs and "as-output" links would improve the entire system efficiency of DMU_o. Tone and Tsutsui [10] introduced a slack-based measurement (SBM) ex-post approach (adjusted score) and a 0-1 mixed integer fractional program (MIP) to address the discretionary slacks. The MIP model is, in fact, a nonlinear program. Therefore, a 0-1 binary decision variable δ_{it} is assigned to free link i at term t, transforming the process into a SBM model. These links introduced objective function in the MIP model introduced by Tone and Tsutsui [10], which is the ratio of inputs and as-input links to output and as-output links. The symbol $n$free is the total number of free links. We suggest the upper bounds of the furthest right summations in the numeration and denomination should be replaced by $n\text{free}^-$ and $n\text{free}^+$, respectively. The nonradial model SBM has the advantage of determining the slack on each input, output, and free link. However, this model requires the total number of as-input and as-output free links to generate the aggregate efficiency score.

The current paper adopts the virtual gap measurement (VGM) model introduced by Liu and Huang [18]. The prime form of VGM is to seek the minimum virtual gap instead of the maximal efficiency score. The obtained optimal values of dual variables are used to compute the final efficiency score. The prime and dual models are well defined and explainable and the analysis is reliable. The contribution of this paper is to solve the network DEA problem raised in Tone and Tsutsui [10]. We employ a VGM two-phase procedure and linear integer restrictions, which were developed by Cook and Zhu [19]. The researchers adopted linear integer restrictions to capture the nonlinear expression, without actually having to specify it directly in the optimization model. This paper introduces a two-phase approach to solve the problem. Phase-I is a mixed integer program model to partition the links into two sets: as-input and as-output. Phase-II is a linear program model to determine the slack of each input, as-input link, output, and as-output link. The best-practice aggregated efficiency of DMU_o in the entire system was obtained. The slacks of inputs, outputs, and free links are obtainable. The VGM model is presented in Section 2. The proposed two-phase performance evaluation model is presented in Section 3. Because the uniqueness of the optimal solution is important, we present an experiment on this subject in Section 4. Managerial insights are introduced in Section 5. We conclude this paper in the last section.

2. Virtual Gap Measurement Model

A set of DMUs $J = \{1, 2, \ldots, n\}$ uses a production technology that transforms a set of inputs $I = \{1, 2, \ldots, m\}$ into a set of outputs $R = \{1, 2, \ldots, s\}$. Let the notions x_{ij} and y_{rj} be the nonnegative volumes of inputs and outputs of DMU_j. The VGM DEA model is to measure the maximum efficiency score of DMU_o. The multiplier (dual) form of the model depicts the objective function as minimizing the virtual gap (Δ_o^*) between virtual-input ($\sum_{i\in I} x_{io}v_i$) and virtual-output ($\sum_{r\in R} y_{ro}u_r$). Each DMU in set J alternatively acts as DMU_o. Let the symbol "Đ" denote the commensurate virtual unit that is used for virtual gap, virtual-input, and virtual-output. The first set of constraints ensures all DMUs have a non-negative virtual gap. The vectors $v = (v_1, v_2, \ldots, v_m)$ and $u = (u_1, u_2, \ldots, u_s)$ are the decision variables of the weights to be assigned to the inputs and outputs. The second and third constraints define that each weighted input and output have a lower bound τ', a constant value with unit of Đ. These assign dual variables π_i, q_i, and p_r to the three constraints. The envelopment (prime) model of VGM is [M1]:

$$\Delta_o^* = \min \quad \left(\sum_{i\in I} x_{io}v_i' - \sum_{r\in R} y_{ro}u_r'\right), \quad o = 1, 2, \ldots, n; \tag{1}$$

$$\text{s.t.} \quad \sum_{i\in I} x_{ij}v_i' - \sum_{r\in R} y_{rj}u_r' \geq 0, \quad j \in J; \tag{2}$$

$$x_{io}v_i' \geq \tau', \quad i \in I; \tag{3}$$

$$y_{ro}u_r' \geq \tau', \quad r \in R; \tag{4}$$

$$v_i' \geq 0, \quad i \in I; \tag{5}$$

$$u_r' \geq 0, \quad r \in R. \tag{6}$$

The dual to model [M1] can be expressed as [M2]:

$$\delta_o^* = \max \quad \tau' \left(\sum_{i\in I} \frac{q_i}{x_{io}} + \sum_{r\in R} \frac{p_r}{y_{ro}}\right) \tag{7}$$

$$\text{s.t.} \quad \sum_{j\in J} \pi_j x_{ij} = x_{io} - q_i, \quad i \in I; \tag{8}$$

$$\sum_{j\in J}\pi_j y_{rj} = y_{ro} - p_r, \quad r \in R; \tag{9}$$

$$\pi_j \geq 0, \quad j \in J; \tag{10}$$

$$q_i \geq 0, \quad i \in I; \tag{11}$$

$$p_r \geq 0, \quad r \in R. \tag{12}$$

The decision variable π_j denotes the weight of DMU_j. q_i and p_r denote the slacks of ith input and rth output, respectively. The objective function (7) expresses the maximum summation of improvement ratios of inputs and outputs. Set $\tau' = 1$ temporarily for computation convenient, and it will not affect the optimal solutions. The decision variable with a superscript "$*$" denotes its optimal value. The following equation defines the constant value of τ; it is equal to the reciprocal of the maximum virtual-inputs of DMUs:

$$\tau = \frac{1}{\max_{j\in J}\sum_{i\in I} x_{ij} v_i'^*}. \tag{13}$$

Thus, the optimal solutions are normalized as

$$\begin{aligned} v_i^* &= v_i'^* \times \tau, \quad i \in I; \\ u_r^* &= u_r'^* \times \tau, \quad r \in R. \end{aligned} \tag{14}$$

Evaluating different DMU_o's, one may directly compare their weights, virtual gap, virtual-input, and virtual-output vectors. According to (1), the ensuing equation (15) existed. It is obvious that the minimum virtual gap Δ_o^* is equivalent to δ_o^*. Referring to (16), this result ensures the nearest improvement target is found. The maximum total of improvement ratios in (16) ensures the improvement target is located on the envelopment.

$$1 = \frac{\left(\sum_{r\in R} y_{ro} u_r^* + \Delta_o^*\right)}{\sum_{i\in I} x_{io} v_i^*}, \tag{15}$$

$$\theta_o^* = \frac{\sum_{r\in R} y_{ro} u_r^*}{\sum_{i\in I} x_{io} v_i^*}. \tag{16}$$

3. Proposed Network Structure of VGM

3.1. Network Structure. The network contains a set of subprocesses (nodes), H. The nodes are assigned ordinal numbers $1, 2, 3, \ldots, \|H\|$. Let A denote the set of network links. There are n homogeneous DMUs in set J, named $\mathrm{DMU}_1, \mathrm{DMU}_2, \ldots$, and DMU_n, which are randomly processed by the subprocesses in set H.

3.1.1. Inputs and Outputs. At each subprocess h, there is a set of input measures I^h that flow into the network and a set of output measures R^h that flow out of the network. For DMU_j in set J, let $x_{ij}^h \in \mathfrak{R}_+^{I^h}$ and $y_{rj}^h \in \mathfrak{R}_+^{R^h}$ denote the volumes of the ith input measure and the rth output measure at subprocess h, respectively. Let q_i^h and p_r^h be the slack of the ith input and the rth output at subprocess h, respectively.

3.1.2. Links. Each subprocess may have links to other subprocesses. Let (h,k) denote the link between subprocesses h and k, $h > k$. Let $D^{(h,k)}$ denote the set of link measures on link (h,k). $z_{dj}^{(h,k)} \in \mathfrak{R}_+^{D^{(h,k)+}} \cup \mathfrak{R}_+^{D^{(h,k)-}}$ denotes the volume of the dth link in set $D^{(h,k)}$. Each DMU alternatively acts as the DMU_o that is under evaluation. The volume of link d on link (h,k), $z_{do}^{(h,k)}$, could be increased or decreased with a slack $f_d^{(h,k)}$ to improve the efficiency of DMU_o as well.

In Phase-I, we introduce the mixed binary integer virtual gap-based measurement model [M3] to partition the links into two subsets: as-input and as-output.

[M3] is as follows:

$$\delta_o^{(\mathrm{I})*} = \max \quad \tau' \sum_{h\in H}\left(\sum_{i\in I^h}\frac{q_i^h}{x_{io}^h} + \sum_{r\in R^h}\frac{p_r^h}{y_{ro}^h} + \sum_{(h,k)\in A}\sum_{d\in D^{(h,k)}}\frac{f_d^{(h,k)}}{z_{do}^{(h,k)}}\right); \tag{17}$$

$$\text{s.t.} \quad \sum_{j\in J}\pi_j^h x_{ij}^h = x_{io}^h - q_i^h, \quad i \in I^h;\ h \in H; \tag{18}$$

$$\sum_{j\in J}\pi_j^h y_{rj}^h = y_{ro}^h + p_r^h, \quad r \in R^h,\ h \in H; \tag{19}$$

$$\sum_{j\in J}\pi_j^h z_{dj}^{(h,k)} = z_{do}^{(h,k)} - f_d^{(h,k)} + M t_d^{(h,k)}, \quad d \in D^{(h,k)},\ (h,k) \in A;\ h \in H; \tag{20}$$

$$\sum_{j\in J}\pi_j^h z_{dj}^{(h,k)} = z_{do}^{(h,k)} + f_d^{(h,k)} - M\left(1 - t_d^{(h,k)}\right), \quad d \in D^{(h,k)},\ (h,k) \in A;\ h \in H; \tag{21}$$

$$\sum_{j\in J}\left(\pi_j^h - \pi_j^k\right) z_{dj}^{(h,k)} = 0, \quad d \in D^{(h,k)},\ (h,k) \in A;\ h \in H; \tag{22}$$

$$\pi_j^h \geq 0, \quad j \in J;\ h \in H; \tag{23}$$

$$q_j^h \geq 0, \quad j \in J;\ h \in H; \tag{24}$$

$$p_j^h \geq 0, \quad j \in J;\ h \in H; \tag{25}$$

$$t_{do}^{(h,k)} \in \{0,1\}, \quad d \in D^{(h,k)},\ (h,k) \in A,\ h \in H. \tag{26}$$

In (17), the objective function maximizes the total improvement ratios of each input, output, and link of DMU_o. The decision variable π_j^h is the weight of DMU_j at subprocess h. The set of left-side of inequalities (18)~(22) is the efficient frontier with respect to DMU_o. The right-side of (18) is the improved ith input at subprocess h located on the frontier. The right-side of (19) is the improved rth output at subprocess h located on the frontier. M denotes a considerably large constant. When the decision variable $t_{do}^{(h,k)} = 0$, (20) becomes effective and (21) becomes ineffective and this constrain could be ignored, and the dth measurement on link $D^{(h,k)}$ is treated as "as-input" with respect to the network. Conversely, when $t_{do}^{(h,k)} = 1$, (20) becomes ineffective and (21) becomes effective, and the dth measurement on link $D^{(h,k)}$ is treated as "as-output" with respect to the network.

When $t_{do}^{(h,k)} = 0$, (27) and (28) derived from (20) and (22) would ensure the improved dth as-input at link (h,k) projects on the frontier, respectively, to subprocesses h and k with the same value, $z_{do}^{(h,k)} - f_d^{(h,k)}$.

$$\sum_{j\in J} \pi_j^h z_{dj}^{(h,k)} = z_{do}^{(h,k)} - f_d^{(h,k)}, \quad d \in D^{(h,k)},\ (h,k) \in A;\ h \in H; \tag{27}$$

$$\sum_{j\in J} \pi_j^k z_{dj}^{(h,k)} = z_{do}^{(h,k)} - f_d^{(h,k)}, \quad d \in D^{(h,k)},\ (h,k) \in A;\ h \in H. \tag{28}$$

When $t_{do}^{(h,k)} = 1$, the two equations (29) and (30) derived from (21) and (22) would ensure the deteriorated dth as-output at link (h,k) projects on the frontier, respectively, to subprocesses h and k with the same value, $z_{do}^{(h,k)} + f_d^{(h,k)}$.

$$\sum_{j\in J} \pi_j^h z_{dj}^{(h,k)} = z_{do}^{(h,k)} + f_d^{(h,k)}, \quad d \in D^{(h,k)},\ (h,k) \in A;\ h \in H; \tag{29}$$

$$\sum_{j\in J} \pi_j^k z_{dj}^{(h,k)} = z_{do}^{(h,k)} + f_d^{(h,k)}, \quad d \in D^{(h,k)},\ (h,k) \in A;\ h \in H. \tag{30}$$

Therefore, a single constraint (22) would replace the two cases above (Chen et al. [20]).

Proposition 1. *If it is an as-output direction for subprocess h, then it must be an as-input direction for subprocess k.*

Proof. When $t_{do}^{(h,k)} = 0$, equations (20) and (22) are partitioned to (27) and (28) which is explained in prior section. If (28) is multiplied by −1 on each side of the equation, due to the fact that $z_{dj}^{(h,k)} = -z_{dj}^{(k,h)}$ and $f_d^{(h,k)} = f_d^{(k,h)}$, the modified (28) can be converted to (31):

$$\sum_{j\in J} \pi_j^k z_{dj}^{(k,h)} = z_{do}^{(k,h)} + f_d^{(k,h)}, \quad d \in D^{(k,h)},\ (k,h) \in A;\ h \in H. \tag{31}$$

For (27), from π_j^h perspective, the links are as-input. For (31), from π_j^k perspective, the links are as-output. With respect to (27) and (31), the direction is from subprocess k to subprocess h. □

Solve [M3] to obtain the optimal solutions of the binary integer variables. If $t_{do}^{(h,k)*} = 0$, then assign index d to the set "as-input" that regards the entire system. Conversely, if $t_{do}^{(h,k)*} = 1$, then assign index d to the set "as-output" that regards the entire system. On each link (h,k), the set of links on $D^{(h,k)}$ is then partitioned into two subsets, $D_o^{(h,k)-} = \{d \mid t_{do}^{(h,k)*} = 0\}$ and $D_o^{(h,k)+} = \{d \mid t_{do}^{(h,k)*} = 1\}$. The notation $f_d^{(h,k)}$ in [M3] is replaced by $f_d^{(h,k)+}$ and $f_d^{(h,k)-}$ if $d \in D_o^{(h,k)+}$ and $d \in D_o^{(h,k)-}$, respectively. Then, the following linear program [M4] is constructed as Phase-II of the solving procedure:

$$\delta_o^{(\text{II})*} = \max \quad \tau' \sum_{h\in H} \left(\sum_{i\in I^h} \frac{q_i^h}{x_{io}^h} + \sum_{r\in R^h} \frac{p_r^h}{y_{ro}^h} + \sum_{(h,k)\in A} \left(\sum_{d\in D_o^{(h,k)-}} \frac{f_d^{(h,k)-}}{z_{do}^{(h,k)}} + \sum_{d\in D_o^{(h,k)+}} \frac{f_d^{(h,k)+}}{z_{do}^{(h,k)}} \right) \right) \tag{32}$$

$$\text{s.t.} \quad \sum_{j\in J} \pi_j^h x_{ij}^h = x_{io}^h - q_i^h, \quad i \in I^h,\ h \in H; \tag{33}$$

$$\sum_{j\in J}\pi_j^h y_{rj}^h = y_{ro}^h + p_r^h, \quad r \in R^h,\ h \in H; \tag{34}$$

$$\sum_{j\in J}\pi_j^h z_{dj}^{(h,k)} = z_{do}^{(h,k)} - f_d^{(h,k)-}, \quad d \in D_o^{(h,k)-},\ (h,k) \in A;\ h \in H; \tag{35}$$

$$\sum_{j\in J}\pi_j^h z_{dj}^{(h,k)} = z_{do}^{(h,k)} + f_d^{(h,k)+}, \quad d \in D_o^{(h,k)+},\ (h,k) \in A;\ h \in H; \tag{36}$$

$$\sum_{j\in J}\left(\pi_j^h - \pi_j^k\right) z_{dj}^{(h,k)} = 0, \quad d \in D_o^{(h,k)-},\ (h,k) \in A;\ h \in H; \tag{37}$$

$$\sum_{j\in J}\left(\pi_j^h - \pi_j^k\right) z_{dj}^{(h,k)} = 0, \quad d \in D_o^{(h,k)+},\ (h,k) \in A;\ h \in H; \tag{38}$$

$$\pi_j^h \geq 0, \quad j \in J;\ h \in H; \tag{39}$$

$$\begin{aligned} &q_i^h \geq 0, \quad i \in I;\ h \in H;\\ &p_r^h \geq 0, \quad r \in R;\ h \in H; \end{aligned} \tag{40}$$

$$f_d^{(h,k)+} \geq 0, \quad d \in D_o^{(h,k)+},\ (h,k) \in A;\ h \in H; \tag{41}$$

$$f_d^{(h,k)-} \geq 0, \quad d \in D_o^{(h,k)-},\ (h,k) \in A;\ h \in H. \tag{42}$$

The objective function (32) expresses the maximum summation of improvement ratios of inputs, outputs, as-inputs, and as-outputs. Set $\tau' = 1$ for convenient computation and it would be adjusted according to a normalization process. Constraints (33), (34), (35), and (36) ensure the modified values of inputs, outputs, as-inputs, and as-outputs would project on the efficient frontier, respectively. Constraints (37) and (38) ensure the modified as-input and as-output links would project on the efficient frontier (Chen et al. [20]). Assign dual variables $v_i'^h$, $u_r'^h$, $w_d'^{(h,k)-}$, $w_d'^{(h,k)+}$, $\varsigma_d'^{(h,k)-}$, and $\varsigma_d'^{(h,k)+}$ to the functional constraints (33)~(38), respectively. The transformed dual form [M5] is shown below.

$$[\text{M5}]\ \Delta_o'^* = \min \sum_{h\in H}\left[\left(\sum_{i\in I^h} x_{io}^h v_i'^h + \sum_{(h,k)\in A}\sum_{d\in D_o^{(h,k)-}} z_{do}^{(h,k)} w_d'^{(h,k)-}\right) - \left(\sum_{r\in R^h} y_{ro}^h u_r'^h + \sum_{(h,k)\in A}\sum_{d\in D_o^{(h,k)+}} z_{do}^{(h,k)} w_d'^{(h,k)+}\right)\right] \tag{43}$$

$$\begin{aligned} \text{s.t.}\ \sum_{h\in H}\Bigg[&\left(\sum_{i\in I^h} x_{ij}^h v_i'^h + \sum_{(h,k)\in A}\sum_{d\in D_o^{(h,k)-}} z_{dj}^{(h,k)} w_d'^{(h,k)-}\right) - \left(\sum_{r\in R^h} y_{rj}^h u_r'^h + \sum_{(h,k)\in A}\sum_{d\in D_o^{(h,k)+}} z_{dj}^{(h,k)} w_d'^{(h,k)+}\right)\\ &+\left(\sum_{(h,k)\in A}\sum_{d\in D_o^{(h,k)-}} z_{dj}^{(h,k)} \varsigma_d'^{(h,k)-} + \sum_{(h,k)\in A}\sum_{d\in D_o^{(h,k)+}} z_{dj}^{(h,k)} \varsigma_d'^{(h,k)+}\right)\\ &-\left(\sum_{(k,h)\in A}\sum_{d\in D_o^{(k,h)-}} z_{dj}^{(k,h)} \varsigma_d'^{(k,h)-} + \sum_{(k,h)\in A}\sum_{d\in D_o^{(k,h)+}} z_{dj}^{(k,h)} \varsigma_d'^{(k,h)+}\right)\Bigg] \geq 0; \end{aligned} \tag{44}$$

$$w_d'^{(h,k)+} z_{do}^{(h,k)} = \tau', \quad d \in D_o^{(h,k)+},\ (h,k) \in A;\ h \in H; \tag{45}$$

$$w_d'^{(h,k)-} z_{do}^{(h,k)} = \tau', \quad d \in D_o^{(h,k)-},\ (h,k) \in A;\ h \in H; \tag{46}$$

$$x_{io}^h v_i'^h \geq \tau', \quad i \in I^h,\ h \in H; \tag{47}$$

$$y_{ro}^h u_r'^h \geq \tau', \quad r \in R^h,\ h \in H; \tag{48}$$

$$v_i'^h \geq 0, \quad i \in I^h,\ h \in H; \tag{49}$$

$$u_r'^h \geq 0, \quad r \in R^h, \ h \in H; \tag{50}$$

$$w_d'^{(h,k)-} \geq 0, \quad d \in D_o^{(h,k)-}, \ (h,k) \in A; \ h \in H; \tag{51}$$

$$w_d'^{(h,k)+} \geq 0, \quad d \in D_o^{(h,k)+}, \ (h,k) \in A; \ h \in H; \tag{52}$$

$$\varsigma_d'^{(h,k)-} \text{ free in sign}, \quad d \in D_o^{(h,k)-}, \ (h,k) \in A; \ h \in H; \tag{53}$$

$$\varsigma_d'^{(h,k)+} \text{ free in sign}, \quad d \in D_o^{(h,k)+}, \ (h,k) \in A; \ h \in H; \tag{54}$$

$$\varsigma_d'^{(k,h)-} \text{ free in sign}, \quad d \in D_o^{(k,h)-}, \ (k,h) \in A; \ h \in H; \tag{55}$$

$$\varsigma_d'^{(k,h)+} \text{ free in sign}, \quad d \in D_o^{(k,h)+}, \ (k,h) \in A; \ h \in H. \tag{56}$$

Replace the coefficient τ' in VGM models by τ, which is expressed as (57). Therefore, all the upper bound of normalized values of virtual-input plus virtual-as-input of DMUs is 1. Furthermore, all of the upper bound of normalized values of virtual-output plus virtual-as-output of DMUs is 1, as well.

$$\tau = \frac{1}{\max_{j \in J} \left[\sum_{h \in H} \left(\sum_{i \in I^h} x_{ij}^h v_i'^h + \sum_{(h,k) \in A} \sum_{d \in D_o^{(h,k)-}} z_{dj}^{(h,k)} w_d'^{(h,k)-} \right) \right]}. \tag{57}$$

Then, the optimal solutions are normalized as

$$\Delta_o^* = \Delta_o'^* \times \tau, \quad h \in H; \tag{58}$$

$$v_i^{h*} = v_i'^{h*} \times \tau, \quad i \in I^h, \ h \in H; \tag{59}$$

$$u_r^{h*} = u_r'^{h*} \times \tau, \quad r \in R^h, \ h \in H; \tag{60}$$

$$w_d^{(h,k)-*} = w_d'^{(h,k)-*} \times \tau, \quad d \in D_o^{(h,k)-}, \ (h,k) \in A; \tag{61}$$

$$w_d^{(h,k)+*} = w_d'^{(h,k)+*} \times \tau, \quad d \in D_o^{(h,k)+}, \ (h,k) \in A; \tag{62}$$

$$\varsigma_d^{(h,k)-*} = \varsigma_d'^{(h,k)-*} \times \tau, \quad d \in D_o^{(h,k)-}, \ (h,k) \in A; \tag{63}$$

$$\varsigma_d^{(h,k)+*} = \varsigma_d'^{(h,k)+*} \times \tau, \quad d \in D_o^{(h,k)+}, \ (h,k) \in A. \tag{64}$$

Evaluating different DMU_o's, one may directly compare their weights, virtual gap, virtual-input, virtual-as-input, virtual-as-output, and virtual-output vectors. According to (43), the ensuing equation (65) existed. It is obvious that the minimum virtual gap "Δ_o^*" is equivalent to $\delta_o^{(\text{II})*}$. The maximum efficiency score of the entire network could be computed as (66).

$$1 = \frac{\sum_{h \in H} \left(\sum_{r \in R^h} y_{ro}^h u_r^{h*} + \sum_{(h,k) \in A} \sum_{d \in D_o^{(h,k)+}} z_{do}^{(h,k)} w_d^{(h,k)+*} \right) + \Delta_o^*}{\sum_{h \in H} \left(\sum_{i \in I^h} x_{io}^h v_i^{h*} + \sum_{(h,k) \in A} \sum_{d \in D_o^{(h,k)-}} z_{do}^{(h,k)} w_d^{(h,k)-*} \right)}; \tag{65}$$

$$E_o^* = \frac{\sum_{h \in H} \left(\sum_{r \in R^h} y_{ro}^h u_r^{h*} + \sum_{(h,k) \in A} \sum_{d \in D_o^{(h,k)+}} z_{do}^{(h,k)} w_d^{(h,k)+*} \right)}{\sum_{h \in H} \left(\sum_{i \in I^h} x_{io}^h v_i^{h*} + \sum_{(h,k) \in A} \sum_{d \in D_o^{(h,k)-}} z_{do}^{(h,k)} w_d^{(h,k)-*} \right)}. \tag{66}$$

3.2. Subprocess Efficiencies. (33)~(36) depict the production technology of the network. The equations' right-side is the target on each index. At the improvement target, reducing the slacks q_i^{h*}, $i \in I^h$, $h \in H$; $f_d^{(h,k)-*}$, $d \in D_o^{(h,k)-}$, $(h,k) \in A$ to the associated indices and adding the slacks p_r^{h*}, $r \in R^h$, $h \in H$; $f_d^{(h,k)+*}$, $d \in D_o^{(h,k)+}$, $(h,k) \in A$ from the associated indices will improve the efficiency score to 1. The efficient reference set of DMUs with respect to DMU_o is defined as $\text{ER}_o = \{j \mid \pi_j^{h*} > 0, \ h \in H, \ j \in J\}$. The efficiency score of subprocess h is computed as follows:

$$E_o^{h*} = \frac{\sum_{r \in R^h} y_{ro}^h u_r^{h*} + \sum_{(h,k) \in A} \sum_{d \in D_o^{(h,k)+}} z_{do}^{(h,k)} w_d^{(h,k)+*}}{\sum_{i \in I^h} x_{io}^h v_i^{h*} + \sum_{(h,k) \in A} \sum_{d \in D_o^{(h,k)-}} z_{do}^{(h,k)} w_d^{(h,k)-*}}. \tag{67}$$

3.3. Virtual Gap Diagram. For the virtual gap diagram, this paper defines the summation of input and as-input as x-axis ($\sum_{i \in I^h} x_{io}^h v_i^{h*} + \sum_{(h,k) \in A} \sum_{d \in D_o^{(h,k)-}} z_{do}^{(h,k)} w_d^{(h,k)-*}$) and the summation of output and as-output as y-axis ($\sum_{r \in R^h} y_{ro}^h u_r^{h*} + \sum_{(h,k) \in A} \sum_{d \in D_o^{(h,k)+}} z_{do}^{(h,k)} w_d^{(h,k)+*}$). For (66), it represents not only the maximum efficient score of the entire system but also the slope of the line from DMU_o to origin. From (65), if the virtual gap is zero, the optimal efficient score is 1. We define the line with a slope equal to 1 to be the frontier. A larger virtual gap will entail a smaller slop and lower efficiency. Figure 1 depicts the DMU_j performance on the virtual gap diagram. The slope of DMU_1 is 1; it is located on the efficiency frontier, indicating high efficiency. DMU_2 slope is 0.6, indicating lower efficiency; its location on the virtual

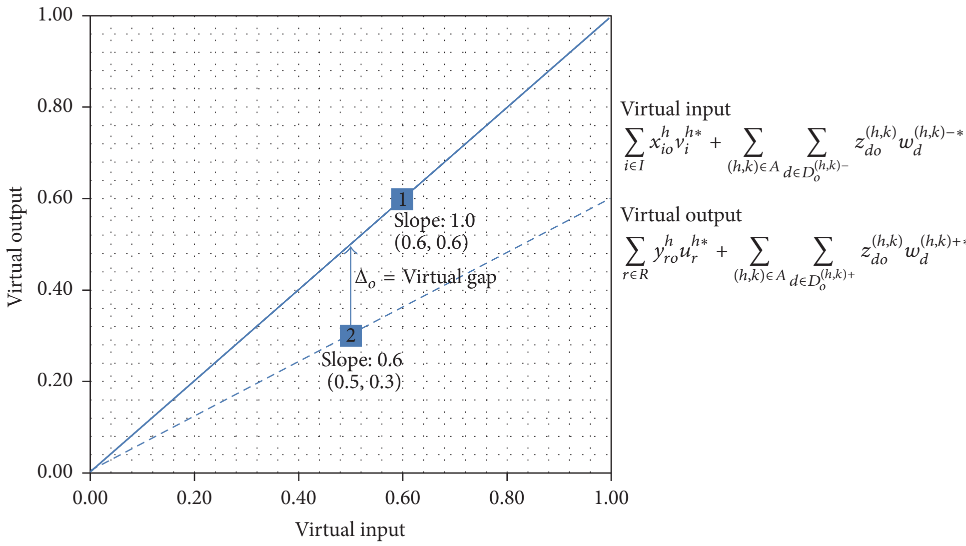

FIGURE 1: Virtual gap diagram.

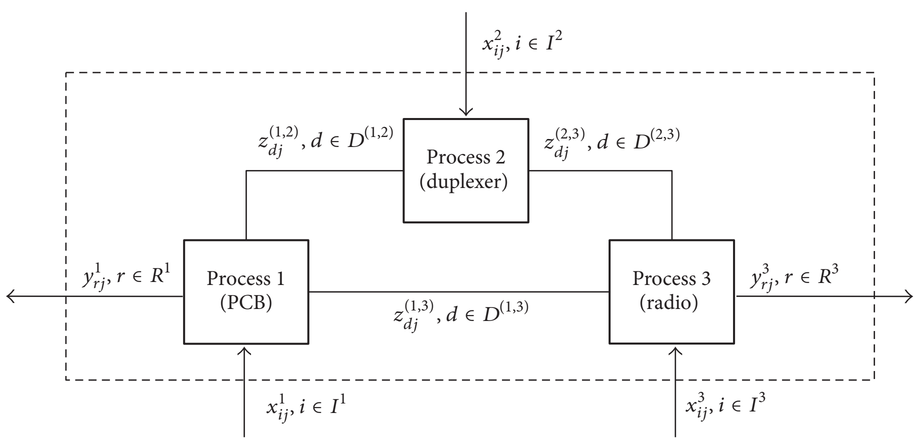

FIGURE 2: Example of network DEA.

gap diagram is (0.5, 0.3). For this DMU$_o$, the virtual gap is 0.2 (0.5 minus 0.3). In order to improve the DMU$_o$ efficiency, the virtual gap needs to be decreased.

4. Illustrative Examples

4.1. Data. This paper introduces a -realworld application of the network VGM model. One manufactory company produces microwave radio; this company owns three factories to produce printed circuit board (PCB), duplexer, and microwave radio. The relationship between the three factories is depicted in Figure 2; process 1 is PCB factory, process 2 is duplexer factory, and process 3 is microwave radio factory. Process 1 (PCB factory) purchases raw material from outside suppliers (x_{ij}^1) and provides PCB to processes 2 and 3 ($z_{dj}^{(1,2)}$ and $z_{dj}^{(1,3)}$). It also sells PCB to other customers (y_{rj}^1). Process 2 (duplexer factory) purchases mechanical housing from other suppliers (x_{ij}^2) and PCB from process 1 ($z_{dj}^{(1,2)}$) to build duplexer and it sells duplexer to process 3 ($z_{dj}^{(2,3)}$). Process 3 (microwave radio factory) purchases PCB from process 1 ($z_{dj}^{(1,3)}$), duplexer from process 2 ($z_{dj}^{(2,3)}$), and other materials (x_{ij}^3) from other suppliers outside of this company to build microwave radio which it sells to end-users (y_{rj}^3). Figure 2 depicts the entire network DEA system that is drawn as the rectangular dash box. The system contains a set of processes, $H = \{1, 2, 3\}$. The inputs, outputs, and links are shown in the figure. The set of links is $A = \{(1, 2), (1, 3), (2, 3)\}$. For instance, the set of indices are $I^1 = \{1\}$, $I^2 = \{1\}$, $I^3 = \{1\}$, $R^1 = \{1\}$, $R^2 = \{\}$, $R^3 = \{1\}$, $D^{(1,2)} = \{1, 2\}$, $D^{(1,3)} = \{1\}$, and $D^{(2,3)} = \{1, 2\}$. The set of DMUs is $J = \{1, 2, \ldots, 11\}$. The arrows of inputs and outputs express the directions of their flows with respect to the system as well as to the processes. As regards the free link, the linking activities are freely determined while maintaining continuity between input and output. It demonstrates whether the current link flow is appropriate or needs to be increased or decreased.

Table 1 lists the hypothetical data of the indices of all DMUs.

Table 1: The dataset of indices.

DMU_j	x^1_{1j}	x^2_{1j}	x^3_{1j}	$z^{(1,2)}_{1j}$	$z^{(1,2)}_{2j}$	$z^{(1,3)}_{1j}$	$z^{(2,3)}_{1j}$	$z^{(2,3)}_{2j}$	y^1_{1j}	y^3_{1j}
1	5	10	714	20	43	538	74	6256	78	234
2	6	32	786	20	65	773	84	6814	83	254
3	5	25	851	25	60	612	70	9567	99	123
4	5	30	544	24	60	591	64	7205	80	161
5	9	13	700	30	70	547	98	4404	85	226
6	9	41	678	27	58	592	72	6529	96	206
7	5	33	948	12	59	587	85	6201	97	144
8	6	10	658	13	42	540	70	8898	83	175
9	5	41	641	10	46	676	73	4927	69	181
10	9	20	885	28	60	659	79	6793	84	174
11	6	25	727	22	41	655	84	4346	70	117

Table 2: Partitions of links, as-input and as-output.

DMU_o	Binary decision variables for the five measures on links					Sets of as-input and as-output on the three links					
	$t^{(1,2)*}_{1o}$	$t^{(1,2)*}_{2o}$	$t^{(1,3)*}_{1o}$	$t^{(2,3)*}_{1o}$	$t^{(2,3)*}_{2o}$	$D^{(1,2)+}_o$	$D^{(1,2)-}_o$	$D^{(1,3)+}_o$	$D^{(1,3)-}_o$	$D^{(2,3)+}_o$	$D^{(2,3)-}_o$
1	0	1	0	1	1	{2}	{1}	{}	{1}	{1, 2}	{}
2	0	1	0	1	1	{2}	{1}	{}	{1}	{1, 2}	{}
3	1	0	0	1	0	{1}	{2}	{}	{1}	{1}	{2}
4	0	0	0	0	0	{}	{1, 2}	{}	{1}	{}	{1, 2}
5	0	0	1	0	1	{}	{1, 2}	{1}	{}	{2}	{1}
6	0	0	1	1	1	{}	{1, 2}	{1}	{}	{1, 2}	{}
7	1	0	1	0	0	{1}	{2}	{1}	{}	{}	{1, 2}
8	1	1	1	1	0	{1, 2}	{}	{1}	{}	{1}	{2}
9	1	1	0	1	1	{1, 2}	{}	{}	{1}	{1, 2}	{}
10	0	1	1	1	1	{2}	{1}	{1}	{}	{1, 2}	{}
11	0	1	0	1	1	{2}	{1}	{}	{1}	{1, 2}	{}

4.2. Results of VGM Case. Employ [M3] of this paper to solve the optimal solutions. Each row in Table 2 lists the optimal solutions of the binary variables. When $\text{DMU}_o = \text{DMU}_1$, according to the first row, optimal values of five binary decision variables can be read as 0, 1, 0, 1, and 1, and the six sets of as-input and as-output are {2}, {1}, {}, {1}, {1, 2}, and {}. The decision variable $t^{(1,2)*}_{1o} = 0$ means that link 1 between processes 1 and 2 is treated as an as-input measure and $t^{(1,2)*}_{1o} = 1$ means that link 2 between processes 1 and 2 is treated as an as-output measure.

With the partitions of the links depicted in Table 2, one is ready to employ [M4]. The optimal values of the indices' slacks are listed in Table 3. When DMU_1 is being evaluated, $\text{DMU}_o = \text{DMU}_1$ in the first column, all the slacks are zero. DMU_1 is not dominated by the other DMUs, and it does not modify any values of indices. When $\text{DMU}_o = \text{DMU}_3$, in the third column, the optimal solution of [M4] decreases input values x^1_2 and x^2_2 by 3.286 and 25.411, respectively. The values of links $z^{(1,2)}_1$, $z^{(1,2)}_2$, $z^{(1,3)}_1$, $z^{(2,3)}_1$, and $z^{(2,3)}_2$ are modified by the values −8.326, −2.078, −48.096, +14.863, and −4651.133, respectively. Increase output values y^3_1 by 3.169.

Solving model [M4], one would also obtain the optimal values of dual variables as each DMU is played as DMU_o. The optimal values of dual variables are listed and summarized in Table 4.

This paper adopts the coefficient τ (57) to limit the upper bound and normalize the optimal solution; τ is equal to the reciprocal of the maximum virtual-inputs of DMUs listed in Table 5.

According to (58)~(64), the normalized weights of the indices are listed in Table 6.

The bottom rows of Tables 7 and 8 are the virtual-input and virtual-output of the DMUs, respectively. This paper defines a virtual gap diagram; the summation of input and as-input is the x-axis, and the summation of output and as-output is the y-axis. Figure 3 depicts the locations of DMUs on this virtual gap diagram. DMU_1 is located on the diagonal line because its efficiency score equals 1. The other DMUs are located below the diagonal line because there is a virtual gap for each DMU.

Use (58) to calculate the solution of the virtual gap, Δ^*_o. Use (66) to calculate the overall efficiency score of DMU_o, E^*_o. Use (67) to calculate the subprocess efficiencies at process h, E^{h*}_o, $h = 1, 2, 3$. Table 9 lists their values.

Table 3: Slacks of the indices of every DMU_o.

Slacks	DMU_o										
	1	2	3	4	5	6	7	8	9	10	11
q_1^{1*}	0	0	3.286	0	0	3.644	0.822	0	0	0	2.618
q_1^{2*}	0	11.289	25.411	21.713	18.914	0.822	19.009	0	21.267	11.035	3.242
q_1^{3*}	0	101.510	0	0	0	0	0	0	197.015	0	0
$f_1^{(1,2)-*}$	0	0	8.326	4.877	13.241	14.168	—	—	—	3.846	6.215
$f_1^{(1,2)+*}$	—	—	—	—	—	—	3.111	10.077	12.672	—	—
$f_2^{(1,2)-*}$	0	0	2.078	15.380	13.937	18.851	10.436	—	—	—	—
$f_2^{(1,2)+*}$	—	—	—	—	—	—	—	11.846	0.270	17.651	10.383
$f_1^{(1,3)-*}$	0	0	48.096	50.473	—	—	—	—	21.746	—	122.544
$f_1^{(1,3)+*}$	—	—	—	—	85.253	63.823	171.004	143.581	—	31.949	—
$f_1^{(2,3)-*}$	0	—	—	1.532	5.011	—	6.939	—	—	—	—
$f_1^{(2,3)+*}$	—	12.374	14.836	—	—	12.752	—	5.385	2.689	13.752	38.305
$f_2^{(2,3)-*}$	0	—	4651.133	4397.778	—	—	4745.688	5510.308	—	—	—
$f_2^{(2,3)+*}$	—	5389.739	—	—	5477.336	6432.22	—	—	2412.840	8079.703	8118.121
p_1^{1*}	0	0	0	0	12.075	0	7.369	9.794	0	21.783	15.000
p_1^{3*}	0	80.803	3.169	4.781	9.085	0	0	26.768	64.298	112.870	80.000

"—" means the decision variable is not in model [M4].

Table 4: Optimal weights of indices of model [M5].

Weight	DMU_o										
	1	2	3	4	5	6	7	8	9	10	11
$u_1^{\prime*}$	0.053	0.012	0.271	0.075	0.025	0.043	0.053	0.012	0.014	0.004	0.014
$u_3^{\prime*}$	0.022	0.005	0.008	0.006	0.028	0.005	0.007	0.006	0.006	0.000	0.009
$v_1^{\prime*}$	0.200	0.167	4.953	0.288	0.111	0.111	1.190	0.199	0.789	0.111	0.236
$v_2^{\prime*}$	0.100	0.031	0.040	0.033	0.077	0.024	0.030	0.442	0.024	0.050	0.040
$v_3^{\prime*}$	0.001	0.002	0.001	0.003	0.012	0.009	0.001	0.003	0.006	0.005	0.007
$w_1^{\prime(1,2)*+}$	—	—	—	—	—	—	0.083	0.077	0.100	—	—
$w_2^{\prime(1,2)*+}$	—	—	—	—	—	—	—	0.024	0.022	0.017	0.024
$w_1^{\prime(1,3)*+}$	—	—	—	—	0.002	0.002	0.002	0.002	—	0.002	—
$w_1^{\prime(2,3)*+}$	—	0.012	0.014	—	—	0.014	—	0.014	0.014	0.013	0.012
$w_2^{\prime(2,3)*+}$	—	0.000	—	—	0.000	0.000	—	—	0.000	0.000	0.000
$w_1^{\prime(1,2)*-}$	0.050	0.050	0.040	0.042	0.033	0.037	—	—	—	0.036	0.045
$w_2^{\prime(1,2)*-}$	0.051	0.015	0.017	0.017	0.014	0.017	0.017	—	—	—	—
$w_1^{\prime(1,3)*-}$	0.002	0.001	0.002	0.002	—	—	—	—	0.001	—	0.002
$w_1^{\prime(2,3)*-}$	0.014	—	—	0.016	0.010	—	0.012	—	—	—	—
$w_2^{\prime(2,3)*-}$	0.000	—	0.000	0.000	—	—	0.000	0.000	—	—	—
$\varsigma_1^{\prime(1,2)*+}$	0.112	0.069	0.095	0.050	0.024	0.035	0.086	—	—	0.044	0.082
$\varsigma_2^{\prime(1,2)*+}$	—	—	0.121	0.063	0.015	0.038	0.092	0.044	—	0.016	0.016
$\varsigma_1^{\prime(1,3)*+}$	—	0.002	0.016	0.000	—	—	0.006	—	0.002	—	0.002
$\varsigma_1^{\prime(2,3)*+}$	0.090	—	—	—	—	—	—	—	—	—	—
$\varsigma_2^{\prime(2,3)*+}$	—	—	—	—	—	—	—	—	—	—	—
$\varsigma_1^{\prime(1,2)*-}$	—	—	—	—	—	—	—	0.068	0.068	—	—
$\varsigma_2^{\prime(1,2)*-}$	0.029	0.009	—	—	—	—	—	—	0.033	—	—
$\varsigma_1^{\prime(1,3)*-}$	0.002	—	—	—	0.000	0.001	—	0.000	—	0.001	—
$\varsigma_1^{\prime(2,3)*-}$	—	0.008	0.062	0.004	0.012	0.046	0.016	0.003	0.029	0.026	0.030
$\varsigma_2^{\prime(2,3)*-}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 5: The coefficient τ for normalization.

	DMU_o										
	1	2	3	4	5	6	7	8	9	10	11
τ	0.069	0.144	0.020	0.090	0.060	0.084	0.064	0.045	0.073	0.140	0.089

Table 6: Optimal weights of indices after normalization.

Weight	DMU$_o$										
	1	2	3	4	5	6	7	8	9	10	11
u_1^*	0.004	0.002	0.005	0.007	0.001	0.004	0.003	0.001	0.001	0.000	0.001
u_3^*	0.002	0.001	0.000	0.001	0.002	0.000	0.000	0.000	0.000	0.000	0.001
v_1^*	0.014	0.024	0.098	0.026	0.007	0.009	0.077	0.009	0.058	0.016	0.021
v_2^*	0.007	0.004	0.001	0.003	0.005	0.002	0.002	0.020	0.002	0.007	0.004
v_3^*	0.000	0.000	0.000	0.000	0.001	0.001	0.000	0.000	0.000	0.001	0.001
$w_1^{(1,2)*+}$	—	—	—	—	—	—	0.005	0.003	0.007	—	—
$w_2^{(1,2)*+}$	—	—	—	—	—	—	—	0.001	0.002	0.002	0.002
$w_1^{(1,3)*+}$	—	—	—	—	0.000	0.000	0.000	0.000	—	0.000	—
$w_1^{(2,3)*+}$	—	0.002	0.000	—	—	0.001	—	0.001	0.001	0.002	0.001
$w_2^{(2,3)*+}$	—	0.000	—	—	0.000	0.000	—	—	0.000	0.000	0.000
$w_1^{(1,2)*-}$	0.003	0.007	0.001	0.004	0.002	0.003	—	—	—	0.005	0.004
$w_2^{(1,2)*-}$	0.004	0.002	0.000	0.002	0.001	0.001	0.001	—	—	—	—
$w_1^{(1,3)*-}$	0.000	0.000	0.000	0.000	—	—	—	—	0.000	—	0.000
$w_1^{(2,3)*-}$	0.001	—	—	0.001	0.001	—	0.001	—	—	—	—
$w_2^{(2,3)*-}$	0.000	—	0.000	0.000	—	—	0.000	0.000	—	—	—
$\varsigma_1^{(1,2)*+}$	0.008	0.010	0.002	0.004	0.001	0.003	0.006	—	—	0.006	0.007
$\varsigma_2^{(1,2)*+}$	—	—	0.002	0.006	0.001	0.003	0.006	0.002	—	0.002	0.001
$\varsigma_1^{(1,3)*+}$	—	0.000	0.000	0.000	—	—	0.000	—	0.000	—	0.000
$\varsigma_1^{(2,3)*+}$	0.006	—	—	—	—	—	—	—	—	—	—
$\varsigma_2^{(2,3)*+}$	—	—	—	—	—	—	—	—	—	—	—
$\varsigma_1^{(1,2)*-}$	—	—	—	—	—	—	—	0.003	0.005	—	—
$\varsigma_2^{(1,2)*-}$	0.002	0.001	—	—	—	—	—	—	0.002	—	—
$\varsigma_1^{(1,3)*-}$	0.000	—	—	—	0.000	0.000	—	0.000	—	0.000	—
$\varsigma_1^{(2,3)*-}$	—	0.001	0.001	0.000	0.001	0.004	0.001	0.000	0.002	0.004	0.003
$\varsigma_2^{(2,3)*-}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 7: Input and as-input weighted data of the VGM model.

Weight data	DMU$_o$										
	1	2	3	4	5	6	7	8	9	10	11
$x_1 v_1^*$	0.069	0.144	0.488	0.130	0.060	0.084	0.383	0.053	0.289	0.140	0.126
$x_2 v_2^*$	0.069	0.144	0.020	0.090	0.060	0.084	0.064	0.198	0.073	0.140	0.089
$x_3 v_3^*$	0.069	0.194	0.020	0.132	0.506	0.538	0.064	0.075	0.269	0.570	0.442
$z_1^{(1,2)} w_1^{(1,2)-*}$	0.069	0.144	0.020	0.090	0.060	0.084	—	—	—	0.140	0.089
$z_2^{(1,2)} w_2^{(1,2)-*}$	0.151	0.144	0.020	0.090	0.060	0.084	0.064	—	—	—	—
$z_1^{(1,3)} w_1^{(1,3)-*}$	0.069	0.144	0.020	0.090	—	—	—	—	0.073	—	0.089
$z_1^{(2,3)} w_1^{(2,3)-*}$	0.069	—	—	0.090	0.060	—	0.064	—	—	—	—
$z_2^{(2,3)} w_2^{(2,3)-*}$	0.083	—	0.020	0.090	—	—	0.064	0.045	—	—	—
Summation = virtual input	0.648	0.913	0.607	0.802	0.808	0.873	0.704	0.370	0.705	0.991	0.834

5. Managerial Insights

This paper adopts the VGM model to evaluate the indices of input items, output items, and links and identify the major efficiency improvement. As shown in Table 9, it indicates the efficiency scores of each subprocess. For DMU$_2$, the actions needed to improve its low efficiency are subprocesses 1 and 2. The efficiency scores are 0.25 and 0.667, respectively. For DMU$_{11}$, the actions needed to improve the efficiency are to improve subprocesses 1 and 3. The efficiency scores are 0.585 and 0.502, respectively. If their efficiency scores are too low, even if subprocess 2 has higher performance, that will affect overall efficiency. The manager should review and consider the slacks of improvement, weight of each index, and virtual gap to define a plan and improve the efficiency of entire system.

The resulting VGM DEA scores provide complete information on how to improve the efficiency of DMUs for a

Table 8: Output and as-output weighted data of the VGM model.

Weight data	DMU$_o$										
	1	2	3	4	5	6	7	8	9	10	11
$y_1u_1^*$	0.288	0.144	0.528	0.538	0.126	0.342	0.330	0.045	0.073	0.042	0.089
$y_3u_3^*$	0.360	0.174	0.020	0.090	0.385	0.084	0.064	0.045	0.073	0.000	0.089
$z_1^{(1,2)}w_1^{(1,2)+*}$	—	—	—	—	—	—	0.064	0.045	0.073	—	—
$z_2^{(1,2)}w_2^{(1,2)+*}$	—	—	—	—	—	—	—	0.045	0.073	0.140	0.089
$z_1^{(1,3)}w_1^{(1,3)+*}$	—	—	—	—	0.060	0.084	0.064	0.045	—	0.140	—
$z_1^{(2,3)}w_1^{(2,3)+*}$	—	0.144	0.020	—	—	0.084	—	0.045	0.073	0.140	0.089
$z_2^{(2,3)}w_2^{(2,3)+*}$	—	0.144	—	—	0.060	0.084	—	—	0.073	0.140	0.089
Summation = virtual output	0.648	0.606	0.568	0.628	0.632	0.677	0.523	0.268	0.440	0.603	0.444

Table 9: Efficiency and virtual gaps solved by VGM model.

DMU$_j$	1	2	3	4	5	6	7	8	9	10	11
Virtual input	0.648	0.913	0.607	0.802	0.808	0.873	0.704	0.370	0.705	0.991	0.834
Virtual output	0.648	0.606	0.568	0.628	0.632	0.677	0.523	0.268	0.440	0.603	0.444
Virtual gap	0	0.307	0.039	0.174	0.176	0.196	0.181	0.102	0.265	0.470	0.390
E_o	1.000	0.663	0.936	0.783	0.782	0.775	0.743	0.724	0.624	0.608	0.532
E_o^1	0.804	0.250	0.965	1.344	1.029	1.697	1.026	3.357	0.606	1.148	0.585
E_o^2	0	0.667	0.250	0	0.250	0.667	0.250	0.553	4.000	1.500	1.500
E_o^3	1.242	1.368	0.667	0.224	0.893	0.622	0.667	1.121	0.642	0.738	0.502

specific network process. The VGM impartially measures items with considerable measuring unit difference and is unit-invariant. The VGM can be applied in supply chain management which takes the perspective of organization mechanism to deal with the complex interactions in supply chain. The broadcasting company in Tone and Tsutsui [9] includes two departments: one is program department which produces programs and the other is transmission department which broadcasts programs. The links between the two departments are the program broadcasting plan which is generated from program producer. The producer would apply VGM and consider time, advertisement revenue, and customer preference to increase or decrease the program transmission.

6. Discussion and Conclusions

The contributions and innovative progress of this paper are that (1) it solves the MIP unsolvable nonlinear program model through a two-phase procedure by using a mixed integer program and (2) it creates a mathematical model and converts multiefficiency frontiers for the separation processes to an aggregation efficiency frontier for the entire production system, eventually obtaining the best-practice performance. The objective of efficiency assessment is to identify weaknesses such that the appropriate steps to improve the entire system performance can be taken. This paper introduces a two-phase procedure to evaluate the network DEA with "free" links. This new procedure employs VGM and considers not only the input and output slacks in the objective function but also the slacks of intermediate measures. The adjustment in the slacks of input, links, and output items defines the best-practice performance. The resulting DEA scores provide complete information on how to project inefficient DMUs onto the DEA frontier for a specific network DEA. The VGM impartially measures items with considerable measuring unit differences and is unit-invariant. Instead of the two conflicting roles that each link plays in existing models, each link plays a single role in the proposed network system in that it is either desirable or undesirable. We derived the dual method of the envelopment form, the multiplier form, to express how to obtain the weights of the as-input and as-output items. Each link is assigned a single weight. This procedure is similar to the legacy radial DEA models that determine the virtual weights of the inputs (as-input) and outputs (as-output) of each DMU$_o$. In computing the performance score, the signs of the as-input and as-output items are always opposite. The single assignment of weights for all of the performance indices, inputs, outputs, and links is crucial for performance analysis. The quantity of all process links may be considerably large. The current two-phase procedure is capable of solving the problem in nonpolynomial complexity. The new procedure will also be applied in series multistage, shared resource (Chen et al. [21] and Liang et al. [22]), dynamic network DEA (Tone and Tsutsui [10] and Kao [11]), assurance region (Thompson Jr. et al. [23]), cone ratio model (Charnes et al. [24]), and virtual weight analysis models (Sarrico and Dyson [25]) in future research.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

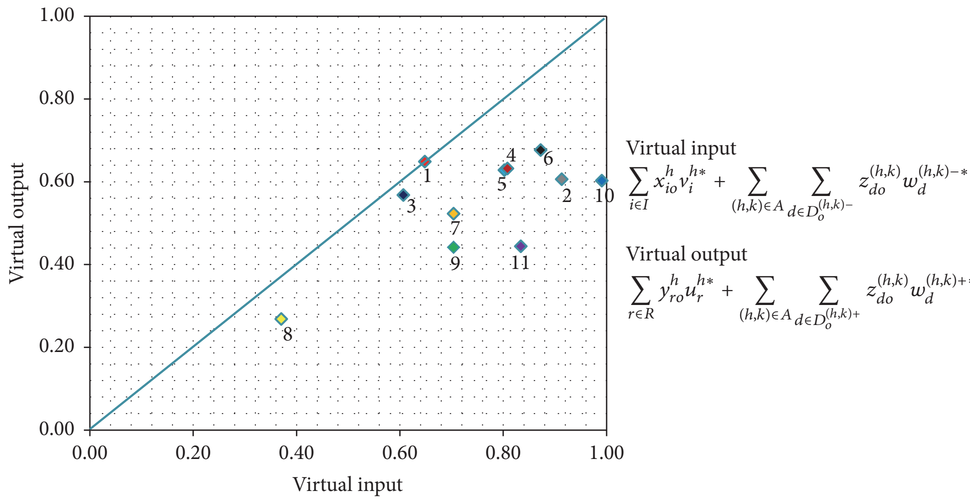

Figure 3: Locations of the 11 DMUs on the virtual gap diagram.

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