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A new application of fuzzy set theory to the Black–Scholes option pricing model ☆

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Abstract

The Black–Scholes Option pricing model (OPM) developed in 1973 has always been taken as the cornerstone of option pricing model. The generic applications of such a model are always restricted by its nature of not being suitable for fuzzy environment since the decision-making problems occurring in the area of option pricing are always with a feature of uncertainty. When an investor faces an option-pricing problem, the outcomes of the primary variables depend on the investor's estimation. It means that a person's deduction and thinking process uses a non-binary logic with fuzziness. Unfortunately, the traditional probabilistic B–S model does not consider fuzziness to deal with the aforementioned problems. The purpose of this study is to adopt the fuzzy decision theory and Bayes' rule as a base for measuring fuzziness in the practice of option analysis. This study also employs 'Fuzzy Decision Space' consisting of four dimensions, i.e. fuzzy state; fuzzy sample information, fuzzy action and evaluation function to describe the decision of investors, which is used to derive a fuzzy B–S OPM under fuzzy environment. Finally, this study finds that the over-estimation exists in the value of risk interest rate, the expected value of variation stock price, and in the value of the call price of in the money and at the money, but under-estimation exists in the value of the call price of out of the money without a consideration of the fuzziness.

Keywords: Black–Scholes; Option pricing model; Fuzzy set theory; Fuzzy decision space

1. Introduction

In this paper, we compare our results with the B–S results (Black & Scholes, 1973). The basic model of the B–S OPM is

$$C = SN(d_1) - K\,\mathrm{e}^{-RT}N(d_2)$$

$$d_1 = [\ln(S/K) + (R + \sigma^2/2)T]/\sigma\sqrt{T}; \quad d_2 = d_1 - \sigma\sqrt{T} \tag{1}$$

where

- C call price;
- S current stock price;
- K striking price;
- R riskless interest rate;
- T time until option expiration;
- σ standard deviation of return on the underlying security;
- $N(d_i)$ cumulative normal distribution function evaluated at d_i.

In decision-making uncertainty is unknown. There are many factors that affect the decision-making, including human psychology state, external information input, which is usually difficult to be derived in terms of probabilistic or stochastic measurement (Cox & Ross, 1976). The well known B–S model, has a number of assumptions such as the riskless interest rate and the volatility are constant, which

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hardly catch human psychology state and external information input. Although, B–S model has been improved, for instant, Cox and Ross (1975) brought out the concept of Constant-Elasticity-of-Variance for volatility. MacBeth and Merville (1979) pointed out that B–S model underprices in-the-money options ($S>K$), and overprices out-of-the-money ($S<K$) options. Cox, Ross, and Rubinstein (1979) used a simplified approach to estimate the volatility. Hull and White (1987) released the assumption that the distribution of price of underlying asset and volatility are constant. Wiggins (1987) and Scott (1987) let go the assumption that the volatility is constant and assumed the volatility follow Stochastic-Volatility. Amin (1993) considered the Jump-Diffusion process of stock price and the volatility were random process. The Bakshi, Cao, and Chen model (1997) derived call price when riskless interest rate and volatility are uncertain. Kenneth (1996) and Rabinovitch (1989) have also used empirical data for verifying the correctness of B–S model. They still did not adequately address the difficulties mentioned above. In this paper, we will use fuzzy concept to address the difficulties mentioned above. Its relevant decision-making is described with decision space $B=\{S,D,P(S_i),C(d_l,S_i)\}$, where $S=\{S_1,S_2,\ldots,S_I\}$ stands for the state set of the environment is element, S_i, $i=1,2,\ldots,I$, stands for a possible state or an actual condition of the state set; $D=\{d_1,d_2,\ldots,d_J\}$ stands for a decision action set; and d_i, $l=1,2,\ldots,L$, stands for an action or alternative available for the investor. $P(S_i)$ is the probability of S_i, and $C(d_j,S_i)$ stands for the premium which is a function on $D\times S$. In the B–S model, $C(d_l,S_i)$ stands for the call price. If the investors know for sure that (S,K,R,σ,T) meet the requirements of a normal distribution, lognormal distribution, or other designated distribution with precise assessment of probabilities, then the optimal alternative (d_l^*) for the investors is such that:

$$C(d_l^*) = \operatorname*{Min}_j \left\{ \sum_{i=1}^{I} C(d_j, S_i)P(S_i) \right\} \quad (2)$$

However, the investor often encounters two difficulties when determining the optimal alternative (d_l^*) with the classical statistical decision model in a B–S model:

(i) An investor usually depends on an expert's judgment to derive the probability distribution of primary variables in a B–S model. However, an investor often subjectively describes the uncertainty he/she faces with implicit fuzziness or impreciseness, which can be expressed as, for example, 'there is a good chance for a riskless interest rate of 3% next year, the riskless interest rate is very unlikely to go below 1%, and it is most probable in the range of 1.5–2.5%.' For another example, 'In a booming economy, there is about a 60% probability that riskless interest rate will grow 10% next year.' The phrases 'booming economy' and 'about 60%' mean implicitly that the probability for the event of '10% riskless interest rate' could be 55, 58, 60, or 65%. In other words, an investor uses both random and fuzzy elements as a base to subjectively assess uncertainty. However, the precondition of the probabilistic and stochastic B–S model assumes that the probability used for the decision analysis is a 'precise' number. In addition, it is calculated and derived from repeated samples and the concept of relative frequency. Thus, it is different from the fuzzy probability calculated and derived in accordance with the 'degree of belief' by experts in the real world. Therefore, it is difficult to use the - traditional probabilistic B–S model under uncertainty for fuzzy decision-making (Bellman and Zadeh, 1970). In this paper, the fuzzy decision theory measures fuzziness and includes the conclusion in the B–S OPM in order to determine an optimal decision (d_l^*).

(ii) While assessing the distribution of a primary variable in a B–S model, an expert should evaluate the influence of sample information. This involves the fuzzy factor of the expert's subjective judgment. That is, the fuzzy factor of the expert's subjective judgment in the call price should not be overlooked. Otherwise, the evaluation will not accurately reflect the problem and will lead to inaccurate decision-making. However, the traditional probability B–S model does not take into consideration on pricing the fact that investors face fuzzy (vague/imprecise/uncertain) factors in B–S analysis. In this paper, the posterior probability will be derived through sample information in accordance with Bayes's rule. The fuzzy sample information will also be included in the B–S OPM to reflect more accurately the situation faced by the investor. An example is illustrated to demonstrate the fuzzy theory to the Black–Scholes call OPM. The results show that the fuzzy B–S OPM to determine an optimal pricing for option is superior to the traditional B–S model in explaining market prices in a fuzzy environment.

The remainder of this paper is organized as follows. The concepts of the probability of fuzzy events are introduced in Section 2. Section 3 describes the B–S model under fuzzy environment, which consist four dimensions: fuzzy state, fuzzy sample information, fuzzy action and evaluation function to describe the decision of investors. Section 4 describes the derivation of fuzzy B–S OPM. Section 5 compares three propositions that is superior to the traditional B–S OPM model in explaining market prices in a fuzzy environment. Section 6 assesses the accuracy of the approximation to the fuzzy B–S with an illustrative example, and conclusions are presented in Section 7.

2. Probability of fuzzy events

The concept of probability is employed in describing fuzzy events and in using sample information to make statistical inferences. An event is an experimental outcome that may or may not occur. Assume the probability of a fuzzy event that measures the chance, or likelihood, the degree of compatibility or degree of truth.

2.1. Prior probability of fuzzy events

In the B–S model under uncertainty, the distributions of the primary variables are assessed subjectively. Therefore, an investor faces the problem of implicit fuzziness. It is difficult to measure the impreciseness with the concept of probability (Zadeh, 1965) because probability is used to measure randomness. Randomness is relevant to the occurrence or non-occurrence of an event, while fuzziness is relevant to the degree of an event (Bellman and Zadeh, 1970). According to the definition given by Zadeh (1965), a fuzzy set is used to describe the set of an event without clear boundaries. The membership function $\mu_{\tilde{A}}: X \to [0,1]$ express the fuzzy set $\tilde{A}$ in set X where if element $x \in X$, then $\mu_{\tilde{A}}(x) \in [0,1]$, $\mu_{\tilde{A}}(x)$ expresses the grade of membership x (also, the degree of compatibility or degree of truth) of X in $\tilde{A}$, which maps X to the membership space. The greater the value of $\mu_{\tilde{A}}(x)$ is, the higher the grade of membership of x belong to $\tilde{A}$. According to the concept of a fuzzy set, when there is no extra sample information, the prior probability $P(\tilde{A})$ of fuzzy event $\tilde{A}$ can be defined as (Zadeh, 1968, 1972):

$$P(\tilde{A}) = \sum_{x_i}^{m} \mu_{\tilde{A}}(x_r)P(x_r) \tag{3}$$

2.2. Posterior probability of fuzzy events

Let $X=\{x_1,x_2,\ldots,x_m\}$ be the sample information space. In S_i state, if the prior probability $P(x_r|S_i)$ of acquiring sample information x_r is known, then the posterior probability of acquiring sample information x_r is:

$$P(S_i|x_r) = \frac{P(x_r|S_i)P(S_i)}{P(x_r)} \tag{4}$$

One could evaluate the sample information acquired through this method. Therefore, it still involves the subjective opinion of experts. For instance, provided that the prior probability R_t of riskless rate in t period is known, and σ_t drops from 60 to 40% due to the change of pricing, experts will then deduce the riskless interest rate as: 'According to the new volatility of the company, the volatility currently drops from 60 to 40% with approximately a 10% riskless rate growth.'

Therefore, the posterior probability of the sample information with fuzziness can be calculated as shown in the following. Let sample information space be $X=\{x_1,x_2,\ldots,x_m\}$, $\{x_r\}$, $r=1,2,\ldots,m$ be an independent event, and let $\tilde{M}=\{\tilde{M}_1,\tilde{M}_2,\ldots,\tilde{M}_J\}$ be the concept of fuzzy sample information. The posterior probability $P(S_i|\tilde{M}_j)$ is calculated in accordance with Bayes' rule after deriving $\tilde{M}_j$

$$P(S_i|\tilde{M}_j) = \frac{P(\tilde{M}_j|S_i)P(S_i)}{P(\tilde{M}_j)}$$

where

$$P(\tilde{M}_j|S_i) = \sum_{r=1}^{m} P(x_r|S_i)\mu_{\tilde{M}_j}(x_r)$$

$$P(\tilde{M}_j) = \sum_{r=1}^{m} P(x_r)\mu_{\tilde{M}_j}(x_r)$$

Therefore

$$P(S_i|\tilde{M}_j) = \frac{\sum_{r=1}^{m} P(x_r|S_i)\mu_{\tilde{M}_j}(x_r)P(S_i)}{\sum_{r=1}^{m} P(x_r)\mu_{\tilde{M}_j}(x_r)} \tag{5}$$

when fuzzy sample information exists, the occurrence probability of S_i state can be described using the above formula. Therefore, there is uncertainty in the future price of option; we want to bring in the concept of 'fuzzy' to describe the B–S model under fuzzy environment.

3. B–S model under fuzzy environment

When dealing with the actual B–S issues, an investor not only faces a fuzzy sample information space, but he/she also stays in a fuzzy state space. For example, industry forecasts its future riskless interest rate in accordance with the classification of 'booming economy', 'fair economy', or 'depression'. The definitions of 'booming economy', 'fair economy', and 'depression', depend on the investor's subjective opinion. Therefore, the state space encountered by the investor also involves implicit fuzziness. Besides, the actions that the investor plans to take will cause the price structure and change accordingly in the analysis. Let us take the pricing change for example, when σ_t drops; R_t is expected to go down. However, due to the investor's environment, timing of the decision-making, and the inability to give it a trial, it is virtually impossible to wait a longtime for the collection of perfect information. Under these circumstances, the alternative adopted by the investor for B–S model under uncertainty contains fuzziness. In summary, the B–S model, which an investor actually deals with, is in a fuzzy state with fuzzy sample information and fuzzy action. As a result, the B–S model can be defined with fuzzy decision space $\tilde{B}=\{\tilde{F},\tilde{A},P(\tilde{F}),C(\tilde{A},\tilde{F})\}$, in which $\tilde{F}=\{\tilde{F}_1,\tilde{F}_2,\ldots,\tilde{F}_K\}$ stands for fuzzy state set $\tilde{F}_k$, $k=1,2,\ldots,K$ stands for a fuzzy set in S; $S=\{S_1,S_2,\ldots,S_I\}$ stands for the state set S_i, $i=1,2,\ldots,I$ stands for a state of the state set. $\tilde{A}=\{\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_N\}$ stands for fuzzy action set $\tilde{A}_n$, $n=1,2,\ldots,N$ stands for a fuzzy action set in D; $D=\{d_1,d_2,\ldots,d_J\}$

stands for action set d_j, $j=1,2,\ldots,J$ stands for an action or alternative available for the investor. $P(\tilde{F}_k)$ is the prior probability of $\tilde{F}_k$. $C(\tilde{A},\tilde{F})$ is the evaluation function of $\tilde{A}\times\tilde{F}$.

3.1. The prior probability of fuzzy state ($\tilde{F}_k$)

The prior probability of fuzzy state $P(\tilde{F}_k)$ is defined in accordance with Eq. (3) as:

$$P(\tilde{F}_k)=\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i) \tag{6}$$

3.2. The posterior probability of fuzzy state ($\tilde{F}_k$)

Let $\tilde{M}=\{\tilde{M}_1,\tilde{M}_2,\ldots,\tilde{M}_J\}$ be the fuzzy sample information space in X. $\tilde{M}_j$, $j=1,2,\ldots,J$, is the fuzzy sample information; $X=\{x_1,x_2,\ldots,x_m\}$ is the sample information space, and $\{x_r\}$, $r=1,2,\ldots,m$ is an independent event.

The posterior probability of fuzzy state $\tilde{F}_k$ is defined in accordance with the posterior probability of fuzzy events after the fuzzy sample information $\tilde{M}_j$ is derived in accordance with Eqs. (3) and (5)

$$\begin{aligned}P(\tilde{F}_k|\tilde{M}_j)&=\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i|\tilde{M}_j)\\&=\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)\frac{\sum_{r=1}^{m}P(x_r|S_i)\mu_{\tilde{M}_j}(x_r)P(S_i)}{P(\tilde{M}_j)}\\&=\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)}\end{aligned} \tag{7}$$

3.3. The expected call price of fuzzy action

An investor must consider the call price after he or she has realized the fuzzy state $\tilde{F}_k$ and the fuzzy sample information $\tilde{M}_j$ of the industry in order to draft an optimal decision and action $\tilde{A}_n$. Let the call price be the value $C(\tilde{A}_n,\tilde{F}_k)$ of evaluation function $C(\tilde{A},\tilde{F})$, then the expected call price $C(\tilde{A}_n|\tilde{M}_j)$ of $\tilde{A}_n$ can be defined as:

$$C(\tilde{A}_n|\tilde{M}_j)=\sum_{k=1}^{K}C(\tilde{A}_n,\tilde{F}_k)P(\tilde{F}_k|\tilde{M}_j) \tag{8}$$

The optimal action $\tilde{A}_n^*$ can be determined by $C(\tilde{A}_n|\tilde{M}_j)$. The call price $C(\tilde{A}_n|\tilde{M}_j)$ of the optimal fuzzy action $\tilde{A}_n^*$ can be defined as

$$C(\tilde{A}_n^*|\tilde{M}_j)=\bigwedge_{n=1}^{N}C(\tilde{A}_n|\tilde{M}_j) \tag{9}$$

where, $\wedge_{n=1}^{N}$ is selecting the minimum value of N values.

The lower the call option prices the better for investor to reduce the loss.

4. The derivation of fuzzy B–S option pricing model

The expected value S of R_t, σ_t, and the fuzzy B–S call option pricing are derived in the following.

4.1. Expected value of R_t

For instance, a company's expert in its sales department has long been performed the riskless interest rate R_t. If this sales expert always forecasts the company's riskless interest rate of the next term in accordance with the economy's condition, which might be classified as a 'booming economy,' a 'fair economy,' and a 'depression'. Let the state set be $S=\{S_1,S_2,\ldots,S_I\}$, where S_i, $i=1,2,\ldots,I$ stands for the riskless interest rate, and the fuzzy stat set be $\tilde{F}=\{\tilde{F}_1,\tilde{F}_2,\ldots,\tilde{F}_K\}$, in which $\tilde{F}_k$, $k=1,2,\ldots,K$ stands for an economy condition. If this company prepares a new lower price plan to respond to the price competition in the market, the sales department expects riskless interest rate. Let the sample information space be $X=\{x_1,x_2,\ldots,x_m\}$, where $\{x_r\}$, $r=1,2,\ldots,m$ stands for rate of riskless interest rates growth under the different price plans and $\{x_r\}$ is an independent event. Also let the fuzzy sample information space $\tilde{M}=\{\tilde{M}_1,\tilde{M}_2,\ldots,\tilde{M}_J\}$, in which $\tilde{M}_j$, $j=1,2,\ldots,J$ stands for a riskless interest growth rate condition that might be classified as 'high riskless interest growth rate' or 'fair riskless interest growth rate'. The expected value of R_t in $\tilde{F}_k$ state, $R_t(\tilde{F}_k)$ can be defined in accordance with Eq. (3) as:

$$R_t(\tilde{F}_k)=\sum_{i=1}^{I}\left[S_i\left(1+\sum_{r=1}^{m}x_rP(x_r|S_i)\right)\right]\mu_{\tilde{F}_k}(S_i)P(S_i) \tag{10}$$

4.2. Expected value of σ_t

Taking a change of the price policy in fuzzy environment, the company estimates its future σ_t, which depends on the investor's subjective judgment. Therefore, the expected value of σ_t in $\tilde{F}_k$ state, $E_{\tilde{A}}(\tilde{\sigma}_t)$ can be defined in accordance with Eq. (3) as:

$$E_{\tilde{A}}(\tilde{\sigma}_t)=\sum_{u=1}^{v}\sigma_{tu}\cdot\mu_{\tilde{A}}(\sigma_{tu})\cdot P(\sigma_{tu}) \tag{11}$$

in which $u=1,2,\ldots,v$ stands for different volatility in σ_{tu}.

4.3. The Fuzzy B–S Option Pricing Model

Combining Eqs. (10) and (11), the Fuzzy B–S Option Pricing Model and the expected call price $C(\tilde{A}_n|\tilde{M}_j)$ can be defined in accordance with Eq. (8) as:

$$C(\tilde{A}_n|\tilde{M}_j) = \sum_{k=1}^{K} C(\tilde{A}_n, \tilde{F}_k)P(\tilde{F}_k|\tilde{M}_j) = \sum_{k=1}^{K} C(\tilde{A}_n, \tilde{F}_k)\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i|\tilde{M}_j) = \sum_{k=1}^{K} C(\tilde{A}_n, \tilde{F}_k) \times \frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)} \quad (12)$$

According to the expected call price $C(\tilde{A}_n|\tilde{M}_j)$, the optimal action $\tilde{A}_n^*$ under the fuzzy B–S model can be defined as:

$$C(\tilde{A}_n|\tilde{M}_j) = \bigwedge_{n=1}^{N} C(\tilde{A}_n|\tilde{M}_j) \quad (13)$$

5. General inference

According to the definition of the fuzzy set given by Zadeh (1965), $\tilde{A}$ is a fuzzy set of X and $\mu_{\tilde{A}}: X \to [0,1]$. $\mu_{\tilde{A}}$ is the membership function of A, that is, when $x \in X$, then $\mu_{\tilde{A}}(x) \in [0,1]$. When the range of the membership function is improved to $\{0,1\}$, then $\tilde{A}$ will be transformed to a crisp set A. The $\mu_{\tilde{A}}$ will be transformed into C_A (characteristic function) . This kind of transformation for a decision maker in the B–S model means that the estimates of the primary variable (S, K, T, R, σ) are without fuzziness, that is, $\mu_{\tilde{A}}(x_r) = 1$ or 0. Under these circumstances, $\mu_{\tilde{A}}(x) = C_A(x) = 1$ if $x \in A$, or $\mu_{\tilde{A}}(x_r) = 0$, if $x \notin A$, therefore, the probability for the occurrence of fuzzy event $\tilde{A}$ can be defined as:

$$P(\tilde{A}) = \sum_{r=1}^{m}\mu_{\tilde{A}}(x_r)\cdot P(x_r) = \sum_{r=1}^{m} C_A(x_r)\cdot P(x_r) = P(A) \quad (14)$$

According to the traditional probabilistic B–S model, the decision space faced by an investor is $B = \{S, D, P(S_i), C(d_l, S_i)\}$, in the B–S model, $C = SN(d_1) - K\,e^{-RT}N(d_2)$, the C value is $C(d_l^*|x_r)$. Therefore, under S_i state and x_r sample information, the expected call price $C(d_l^*|x_r)$ of the optimal alternative is defined as:

$$C(d_l^*|x_r) = \underset{j}{\mathrm{Min}}\left\{\sum_{i=1}^{I} C(d_j, S_i)P(S_i|x_r)\right\} = \bigwedge_{j=1}^{J}\sum_{i=1}^{I} C(d_l, S_i)P(S_i|x_r) \quad (15)$$

The following results can be proved. To avoid distraction, the detailed mathematical proofs are put in the Appendices.

5.1. Proposition 1

Assume that $E_{\tilde{A}}(\tilde{\sigma}_t)$ and $E(\sigma_t)$ stands for the expected value of σ_t of the fuzzy B–S model and the traditional probabilistic B–S model, respectively. When $\tilde{F}_k$ and $\tilde{M}_j$ are existentially, but its fuzziness has been neglected irrationally, then $E(\sigma_t) \geq E_{\tilde{A}}(\tilde{\sigma}_t)$. This means that the expected value of σ_t will be increased falsely and it will lead to false decision-making.

Proof: please see Appendix A.

5.2. Proposition 2

Assume that $R_t(\tilde{F}_k)$ and R_t stands for the expected value of riskless interest rate in the fuzzy B–S model and the traditional probabilistic B–S model, respectively. When $\tilde{M}_j$ and $\tilde{F}_k$ are existentially, but its fuzziness has been overlooked irrationally, then $R_t(\tilde{F}_k) \leq R_t$. This means that the expected value of riskless interest rate will be increased falsely and it will lead to a false decision-making.

Proof: please see Appendix B.

5.3. Proposition 3

(1) When $\tilde{M}_j$ and $\tilde{F}_k$ are existentially and under a fixed σ_t and R_t condition, it is assumed that $\sigma_{tu} = \sigma_{t0}$ and $R_t(\tilde{F}_k) = R_{t0}$. Therefore, the option of in the money ($S > K$) deriving $C(d_l^*|x_r) \geq C(\tilde{A}_n^*|\tilde{M}_j)$. This means that the value of the expected call price of in the money will be overestimated.

Proof: please see Appendix C.

(2) When $\tilde{M}_j$ and $\tilde{F}_k$ are existentially and under a fixed σ_t and R_t condition, it is assumed that $\sigma_{tu} = \sigma_{t0}$ and $R_t(\tilde{F}_k) = R_{t0}$. Therefore, the option of at the money ($S = K$) deriving $C(d_l^*|x_r) \geq C(\tilde{A}_n^*|\tilde{M}_j)$. This means that the value of the expected call price of at the money will be overestimated.

Proof: please see Appendix D.

(3) When $\tilde{M}_j$ and $\tilde{F}_k$ are existentially and under a fixed σ_t and R_t condition, it is assumed that $\sigma_{tu} = \sigma_{t0}$ and $R_t(\tilde{F}_k) = R_{t0}$. Therefore, the option of in the money ($S < K$) deriving $C(d_l^*|x_r) \leq C(\tilde{A}_n^*|\tilde{M}_j)$. This means that the value of the expected call price out of the money will be underestimated.

Proof: please see Appendix E.

From Proposition 3 we know that the option of in the money and at the money will be over-estimated, but the option of out of the money will be under-estimated. If the investor makes a decision in accordance with the estimated call price, then the optimal alternative might not be chosen, because of the target call price or requirement rate of return considerations without a consideration of the fuzziness.

Table 1
$\mu_{\tilde{F}_k}(S_i)$ and $P(S_i)$

$\mu_{\tilde{F}_k}(S_i)$ or $P(S_i)$	$S_i(R_t)$				
	1%	2%	3%	4%	5%
$\mu_{\tilde{F}_1}(S_i)$	0	0	0.8	0.9	1.0
$\mu_{\tilde{F}_2}(S_i)$	0	0.9	1.0	0.8	0
$\mu_{\tilde{F}_3}(S_i)$	1.0	0.9	0.8	0.5	0
$P(S_i)$	0.1	0.2	0.4	0.2	0.1

6. Illustrative example for simulation

This paper takes the call option of stock Y, the target stock of Company Z, as an example to discuss the application of call prices derived by the investor using fuzzy OPM under uncertainty. Company Z's fuzzy-decision space is described below in the following.

6.1. Fuzzy state

It is known to assume that the investor has acquired stock Y, the single target stock from Company Z, of which the estimation of risk interest rate (R_t) has long been carried out by the sales specialists of the company, who have been projecting respective possible risk interest rates in the next term for different future outlooks of 'booming economy', 'fair economy', and 'depression'. Suppose state set $S=\{S_1,S_2,S_3,S_4,S_5\}$, where S_i denotes the risk interest rate of call option, the set represents a collection of fuzzy states (Table 1).

6.2. Fuzzy sample information

In response to recent return fluctuations on the stock market, the investor has readjusted the magnitude of fluctuation for the rate of return on the stock (σ). While S, K, T, and R can be derived directly from observation; σ calculation requires the use of daily return data of the target stock over a past period of time. Based on historical statistics, σ is revised downwards from its current level of 60–40%. It is expected that this change will cause the risk interest rate to drop. Suppose the sample message space $X=\{x_1,x_2,x_3,x_4\}$, where x_r, $r=1$–4, denotes the growth of risk interest rate and where (x_r) is an independent event. The investor produces estimations on two basic assumptions of 'very high risk interest rate growth' and 'relatively flat risk interest rate growth'. Hence, the fuzzy sample message space can be expressed as $\tilde{M}=\{\tilde{M}_1,\tilde{M}_2\}$, where $\tilde{M}_1$ denotes very high-risk interest rate growth and $\tilde{M}_2$, a relatively flat interest risk interest rate growth. It is also known that historically given S_i, Company Z's prior probability of the occurrence of x_r is $P(x_r|S_i)$, as shown in Table 2, and the prior probability of the membership function $\mu_{\tilde{M}_j(x_r)}$ and x_r for fuzzy sample message $\tilde{M}_j$, $j=1$, 2 is $P(x_r)$, as shown in Table 3.

Table 2
$P(x_r|S_i)$

S_i	R_t (%)	x_r			
		10%	15%	20%	25%
S_1	1	0.3	0.3	0.2	0.2
S_2	2	0.5	0.4	0.1	0
S_3	3	0.6	0.3	0.1	0
S_4	4	0.8	0.2	0	0
S_5	5	1.0	0	0	0

Table 3
$\mu_{\tilde{M}_j}(x_r)$ and $P(x_r)$

$\mu_{\tilde{M}_j}(x_r)$ or $P(x_r)$	x_r			
	10%	15%	20%	25%
$\mu_{\tilde{M}_1}(x_r)$	0	0.2	0.8	0.8
$\mu_{\tilde{M}_2}(x_r)$	0.2	0.8	0.6	0
$P(x_r)$	0.2	0.3	0.3	0.2

6.3. Fuzzy action

In response to the growth of risk interest rate in the next term and considering the market status, the investor has decided on his/her action set $D=\{d_1,d_2\}$, where Solution 1 (d_1) is to purchase large volumes of stock options under the expectation of very high stock price fluctuation and Solution 2 (d_2) is when the expected stock price fluctuation is low, hence only small quantities of stock options will be purchased. The evaluation of the respective solutions shows the following results in the call option price:

Solution 1: When the investor purchases large quantities of stock options, the action will either fuel or dampen the market, causing σ to rise creating a larger room for profit. Hence, the call option price will increase.

Solution 2: When the investor purchases only small quantities of stock options, the action has little effect on market fluctuation, while it will limit the level of rising in σ and result in much smaller room for profit. Hence, the call option price will fall.

Suppose the fuzzy action set $\tilde{A}=\{\tilde{A}_1,\tilde{A}_2\}$, where $\tilde{A}_1$ denotes the fuzzy set for d_1 with σ at around 60%, and $\tilde{A}_2$ denotes the fuzzy set for d_2 with σ at around 40%, we then obtain $\mu_{\tilde{A}_n}(\sigma_t)$, as shown in Table 4.

Table 4
$\mu_{\tilde{A}_n}(\sigma_t)$ and $P(\sigma_{tu})$

$\mu_{\tilde{A}_n}(\sigma_t)$	σ_{tu}			
	20%	40%	60%	80%
$\mu_{\tilde{A}_1}(\sigma_t)$	0	0.6	1.0	0.8
$\mu_{\tilde{A}_2}(\sigma_t)$	0.8	1.0	0.5	0
$P(\sigma_{tu})$	0.2	0.3	0.3	0.2

$E_{\tilde{A}}(\tilde{\sigma}_t)=\sum_{u=1}^{v}\sigma_{tu}\cdot\mu_{\tilde{A}}(\sigma_{tu})\cdot P(\sigma_{tu})$, where $E_{\tilde{A}}(\tilde{\sigma}_1)=0.38$, $E_{\tilde{A}}(\tilde{\sigma}_2)=0.242$.

Table 5
$R_t(\tilde{F}_k)$

$\tilde{F}_k$	$\tilde{F}_1$	$\tilde{F}_2$	$\tilde{F}_3$
$R_t(\tilde{F}_k)$	0.020	0.025	0.023

6.4. Evaluation function

With $\tilde{A}_n$ and $\tilde{F}_k$ given, the investor then introduces S, K, and T into the B–S OPM to determine the call option price, which can be derived by calculating the value $C(\tilde{A}_n, \tilde{F}_k)$ of the pricing function $C(\tilde{A}, \tilde{F})$.

6.4.1. Expected value of risk interest rate under $\tilde{F}_k$

After taking into account the risk interest rate growth x_r, $r=1$–4 of Company Z, and with S_i, $i=1$–5 given, the expected value of the new risk interest rate $R_t(S_i)$ is:

$$R_t(S_i) = S_i\left(1 + \sum_{r=1}^{4} x_r P(x_r|S_i)\right)$$

According to Table 2, $R_t(S_1)=0.012$, $R_t(S_2)=0.023$, $R_t(S_3)=0.034$, $R_t(S_4)=0.044$, and $R_t(S_5)=0.055$, we can obtain the expected values of risk interest rates $R_t(\tilde{F}_k)$ under $\tilde{F}_k$, $k=1$–3 as

$$P(\tilde{F}_k) = \sum_{i=1}^{I} \mu_{\tilde{F}_k}(S_i)P(S_i); \qquad R_t(\tilde{F}_k) = P(\tilde{F}_k)R_t(S_i)$$

Using the data provided in Table 1, we can obtain the values of $R_t(\tilde{F}_k)$ as shown in Table 5.

6.4.2. The value of evaluation function

Since the new stock price fluctuation of Company Z is set at 40%, we can bring the previously derived $R_t(\tilde{F}_k)$ values into Eq. (1). We assume that under the current market state of Company Z, where $S=100$ (NT \$), $K=100$ (NT \$), $T=1$ (year) and with Company Z having only one target stock Y, the call option price is then the value $C(\tilde{A}_n, \tilde{F}_k)$ of Company Z's pricing function $C(\tilde{A}, \tilde{F})$. We derive the following: $C(\tilde{A}_1, \tilde{F}_1)= 15.925$; $C(\tilde{A}_2, \tilde{F}_1)= 10.552$; $C(\tilde{A}_1, \tilde{F}_2)= 16.128$; $C(\tilde{A}_2, \tilde{F}_2)= 10.773$; $C(\tilde{A}_1, \tilde{F}_3)= 16.07$; $C(\tilde{A}_2, \tilde{F}_3)= 10.71$.

6.5. Expected value of call option price at optimal actions

To simplify or presentation that assume there are only two alternative under consideration to generalize from two to many alternative can be done similarly. With the investor's derived $C(\tilde{A}_n, \tilde{F}_k)$ and given the fuzzy sample message $\tilde{M}_j$, $j=1$, 2, the expected call option price value $C(\tilde{A}_n^*|\tilde{M}_j)$ for $\tilde{A}_n$ can be defined:

$$C(\tilde{A}_n|\tilde{M}_1, \tilde{M}_2) = \sum_{k=1}^{K} C(\tilde{A}_n, \tilde{F}_k)P(\tilde{F}_k|\tilde{M}_1, \tilde{M}_2)$$

$$C(\tilde{A}_1|\tilde{M}_1, \tilde{M}_2) = 8.385; \qquad C(\tilde{A}_2|\tilde{M}_1, \tilde{M}_2) = 5.587$$

Therefore, the investor should adopt Action A_2.

The investor's expected call option price $C(\tilde{A}_n^*|\tilde{M}_j)$ at optimal action can be as $C(\tilde{A}_n^*|\tilde{M}_j) = \wedge_{n=1}^{2} C(\tilde{A}_n|\tilde{M}_1, \tilde{M}_2) = 5.587$.

6.6. Soundness analysis for fuzzy B–S option pricing model

In pricing options, the fuzzy OPM argues that the investor's estimation of the changes in both correlated variables R and σ that contain hidden fuzzy factors. Therefore, unless the investor possesses complete information on correlated variables and has determined the values of the correlated variables under the constraints of the objective environment, the fuzzy factors cannot be completely excluded.

In the following section, the data from the case of Company Z discussed above will be used as a basis to compare the differences between $C(d_l^*|x_r)$ and $C(\tilde{A}_n^*|\tilde{M}_j)$, in order to better understand the influence of fuzzy factors on the B–S OPM and to examine the soundness of a fuzzy OPM.

6.6.1. Expected value of risk interest rate

Let $R_t(S_i)$ be the expected value of risk interest rate derived from the B–S OPM. According to Eq. (10) and Bayes' theorem, $R_t(S_i)$ can be defined as:

$$R_t(S_i) = \sum_{i=1}^{I} S_i\left[1 + \sum_{r=1}^{m} x_r P(x_r|S_i)P(S_i)\right]$$

Using the data from the previous case of Company Z in Tables 1 and 2, we obtain the expected value of risk interest rate $R_t(S_i)$ as 0.034.

The figures are all higher than the expected values of risk interest rates derived for the same case of Company Z under fuzzy OPM. $R_t(\tilde{F}_1)=0.020$, $R_t(\tilde{F}_2)=0.025$, and $R_t(\tilde{F}_3)=0.023$.

This finding is consistent with the results from Proposition 1 of this study, suggesting that ignoring hidden fuzzy factors in the calculation of the expected values of risk interest rates will result in overestimations and thereby causing mistakes in investment decisions.

6.6.2. Expected value of stock price fluctuation

Let $E(\sigma_t)$ is the expected value of stock price fluctuation derived from the B–S OPM. According to Eq. (12) and Bayes' theorem, $E(\sigma_t)$ can be defined as:

$$E_{\tilde{A}}(\tilde{\sigma}_t) = \sum_{u=1}^{v} \sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})$$

Using the data from the previous case of Company Z in Table 4, we obtain the expected value of stock price fluctuation $E(\sigma_t)$ as 0.5.

The figures are all higher than the expected values of stock price fluctuations derived for the same case of

Company Z under fuzzy OPM. Here, $E_{\tilde{A}}(\tilde{\sigma}_1)=0.38$ and $E_{\tilde{A}}(\tilde{\sigma}_2)=0.242$

This finding is consistent with the results from Proposition 2 of this study, suggesting that ignoring hidden fuzzy factors in the calculation of expected values of stock price fluctuations will result in overestimations and increased investors' motivation for buying and selling of options, thereby fueling or dampening the target stock prices on the market. And this can easily lead to market volatility.

$$E_{\tilde{A}}(\tilde{\sigma}_t)=\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})P(\tilde{F}_k|\tilde{M}_j)=\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)} \quad \text{(A1)}$$

6.6.3. Expected value of call option price

Let $C(d_l^*|x_r)$ and $C(\tilde{A}_n^*|\tilde{M}_j)$ be the expected values of call option prices of the two solutions derived from the B–S OPM and the fuzzy OPM, respectively. Again using the previous case of Company Z with a stock return fluctuation below 40%, and with the values of $R_t(\tilde{F}_k)$ and $E_{\tilde{A}}(\tilde{\sigma}_t)$ derived earlier, we obtain $C(d_l^*|x_r)$ and $C(\tilde{A}_n^*|\tilde{M}_j)$ for the solutions under the two models as 21.09 and 5.587, respectively.

The result shows that the expected values of call option prices derived from the B–S OPM are higher than those obtained from the fuzzy OPM. This finding is again consistent with the results from Proposition 3 ($S=K$) of this study, suggesting that ignoring hidden fuzzy factors in the calculation of the expected values of call option prices will result in overestimations and thereby causing mistakes in relevant investment decisions.

7. Conclusions

The impact of implicit 'Fuzziness' is inevitable due to the subjective assessment made by investors in a B–S OPM. The fuzzy decision theory and Bayes' rule are used to measure the effect of this fuzziness. It is included in the fuzzy B–S OPM to determine the optimal actions for B–S model under uncertainty. The thoughts and controlled behaviors of human involve both fuzziness and non-quantitative quality. Therefore, the fuzzy B–S model would result in a more realistic methodology for a B–S model. Further, corollaries have been made in this paper. It has been proved that if the fuzziness has been neglected irrationally, then the expected values of R, σ and the value of the call price of in the money ($S>K$) and at the money ($S=K$) will be over-estimated, but under-estimation exists in the value of the call price of out of the money ($S<K$) without a consideration of the fuzziness. So, the expected call price will be inaccurately estimated and this will lead to inaccurate decision-making.

Appendix A

When $\tilde{M}_j$ and $\tilde{F}_k$ are existentially, $E(\sigma_t)$ can be defined in accordance with Eq. (11):

Assume $\mu_{\tilde{A}}(\sigma_{tu})=1$, $\mu_{\tilde{F}_k}(S_i)=1$, $\mu_{\tilde{M}_j}(x_r)=1$. It is inputted into Eq. (A1) to derive $E(\sigma_t)$:

$$E(\sigma_t)=\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)} \quad \text{(A2)}$$

$$E(\sigma_t)-E_{\tilde{A}}(\tilde{\sigma}_t)=\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})P(x_r|S_i)P(S_i)[1-\mu_{\tilde{A}}(\sigma_{tu})\mu_{\tilde{F}_k}(S_i)]\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)} \quad \text{(A3)}$$

Due to $\sigma_{tu}>0$, $\mu_{\tilde{A}}(\sigma_{tu})\to[0,1]$, $\mu_{\tilde{F}_k}(S_i)\to[0,1]$, $\mu_{\tilde{M}_j}(x_r)\to[0,1]$, $1-\mu_{\tilde{A}}(\sigma_{tu})\cdot\mu_{\tilde{F}_k}(S_i)\geq 0$.

Therefore, $E(\sigma_t)-E_{\tilde{A}}(\tilde{\sigma}_t)\geq 0$.

Appendix B

When $\tilde{M}_j$ and $\tilde{F}_k$ are existentially, $R_t(\tilde{F}_k)$ can be defined in accordance with Eq. (10) as:

$$R_t(\tilde{F}_k)=\sum_{i=1}^{I}R_t(S_i)\mu_{\tilde{F}_k}(S_i)P(S_i)P(\tilde{F}_k|\tilde{M}_j)=\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}R_t(S_i)(\mu_{\tilde{F}_k}(S_i))^2(P(S_i))^2\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)} \quad \text{(B1)}$$

Assume $\mu_{\tilde{F}_k}(S_i)=1$, $\mu_{\tilde{M}_j}(x_r)=1$. It is inputted into Eq. (B1) to derive the R_t:

$$R_t=\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}R_t(S_i)(P(S_i))^2P(x_r|S_i)}{\sum_{r=1}^{m}P(x_r)} \quad \text{(B2)}$$

$$R_t-R_t(\tilde{F}_k)=\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}R_t(S_i)(P(S_i))^2P(x_r|S_i)[1-(\mu_{\tilde{F}_k}(S_i))^2]\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)} \quad \text{(B3)}$$

Due to $R_t(S_i)=S_i\left[1+\sum_{r=1}^{m}x_rP(x_r|S_i)\right]$, $R_t(S_i)>0$, $\mu_{\tilde{F}_k}(S_i)\to[0,1]$, $\mu_{\tilde{M}_j}(x_r)\to[0,1]$, $1-(\mu_{\tilde{F}_k}(S_i))^2\geq 0$, therefore, $R_t-R_t(\tilde{F}_k)\geq 0$.

Appendix C

Assume option is in the money ($S>K$), let $T=1$ then $C(\tilde{A}_n^*|\tilde{M}_j)$ can be defined in accordance with Eq. (12) as:

$$C(\tilde{A}_n^*|\tilde{M}_j) = SN\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})}\right\} \times\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)} - K\,\mathrm{e}^{-\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i)R_t(S_i)} \times N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})}-\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right\} \times\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)} \tag{C1}$$

Assume $\mu_{\tilde{A}_n}(\sigma_{tu})=\mu_{\tilde{F}_k}(S_i)=\mu_{\tilde{M}_j}(x_r)=1$, when it is inputted into Eq. (C1), the $C(\tilde{A}_n^*|\tilde{M}_j)$ will be transformed into $C(d_l^*|x_r)$:

$$C(d_l^*|x_r) = SN\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}\right\}\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)} - K\,\mathrm{e}^{-\sum_{i=1}^{I}P(S_i)R_t(S_i)} \times N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}-\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right\}\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)} \tag{C2}$$

Let

$$N(d_1) = N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}\right\}$$

$$N(d_2) = N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}-\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right\}$$

$$N(\tilde{d}_1) = N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})}\right\}$$

$$N(\tilde{d}_2) = N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})}-\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right\}$$

then

$$\begin{aligned} C(d_l^*|x_r)-C(\tilde{A}_n^*|\tilde{M}_j) &= SN(d_1)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)} - K\,\mathrm{e}^{-R}\,N(d_2)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)} \\ &\quad - SN(\tilde{d}_1)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)} \\ &\quad + K\,\mathrm{e}^{-\tilde{R}}\,N(\tilde{d}_2)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)} \\ &= \frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\left[SN(d_1)-K\,\mathrm{e}^{-R}\,N(d_2)-SN(\tilde{d}_1)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}\right. \\ &\quad \left.+K\,\mathrm{e}^{-\tilde{R}}\,N(\tilde{d}_2)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}\right] = \frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)} \\ &\quad \times\left\{SN(d_1)-K\,\mathrm{e}^{-R}\,N(d_2)-\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}\left[SN(\tilde{d}_1)-K\,\mathrm{e}^{-\tilde{R}}\,N(\tilde{d}_2)\right]\right\} \end{aligned} \tag{C3}$$

where

$$\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\ge 0,$$

due to, $d_1>d_2$ and $S>K$.

So, $SN(d_1)-K\,\mathrm{e}^{-R}N(d_2)\ge 0$, $SN(\tilde d_1)-K\,\mathrm{e}^{-\tilde R}N(\tilde d_2)\ge 0$, but

$$\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde F_k}(S_i)\mu_{\tilde M_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde M_j}(x_r)}\le 1$$

and

$$SN(d_1)-K\,\mathrm{e}^{-R}N(d_2)-\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde F_k}(S_i)\mu_{\tilde M_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde M_j}(x_r)}[SN(\tilde d_1)-K\,\mathrm{e}^{-\tilde R}N(\tilde d_2)]\ge 0$$

Therefore, $C(d_l^*|x_r)\ge C(\tilde A_n^*|\tilde M_j)$.

Appendix D

Assume option is at the money ($S=K$), let $T=1$ then $C(\tilde A_n^*|\tilde M_j)$ can be defined in accordance with Eq. (12) as:

$$\begin{aligned}C(\tilde A_n^*|\tilde M_j) &= SN\left\{\frac{\sum_{i=1}^{I}\mu_{\tilde F_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde A}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde A}(\sigma_{tu})P(\sigma_{tu})}\right\}\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde F_k}(S_i)\mu_{\tilde M_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde M_j}(x_r)P(x_r)}\\ &\quad -K\,\mathrm{e}^{-\sum_{i=1}^{I}\mu_{\tilde F_k}(S_i)P(S_i)R_t(S_i)}N\left\{\frac{\sum_{i=1}^{I}\mu_{\tilde F_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde A}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde A}(\sigma_{tu})P(\sigma_{tu})}-\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde A}(\sigma_{tu})P(\sigma_{tu})\right\}\\ &\quad\times\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde F_k}(S_i)\mu_{\tilde M_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde M_j}(x_r)P(x_r)}\end{aligned}\tag{D1}$$

Assume $\mu_{\tilde A_n}(\sigma_{tu})=\mu_{\tilde F_k}(S_i)=\mu_{\tilde M_j}(x_r)=1$, when it is inputted into Eq. (D1), the $C(\tilde A_n^*|\tilde M_j)$ will be transformed into $C(d_l^*|x_r)$:

$$\begin{aligned}C(d_l^*|x_r) &= SN\left\{\frac{\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}\right\}\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\\ &\quad -K\,\mathrm{e}^{-\sum_{i=1}^{I}P(S_i)R_t(S_i)}N\left\{\frac{\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}-\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right\}\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\end{aligned}\tag{D2}$$

Let,

$$N(d_1)=N\left\{\frac{\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}\right\}$$

$$N(d_2)=N\left\{\frac{\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}-\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right\}$$

$$N(\tilde d_1)=N\left\{\frac{\sum_{i=1}^{I}\mu_{\tilde F_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde A}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde A}(\sigma_{tu})P(\sigma_{tu})}\right\}$$

$$N(\tilde d_2)=N\left\{\frac{\sum_{i=1}^{I}\mu_{\tilde F_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde A}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde A}(\sigma_{tu})P(\sigma_{tu})}-\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde A}(\sigma_{tu})P(\sigma_{tu})\right\}$$

then

$$\begin{aligned}C(d_l^*|x_r)-C(\tilde{A}_n^*|\tilde{M}_j)&=SN(d_1)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}-K\,\mathrm{e}^{-R}N(d_2)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\\&\quad-SN(\tilde{d}_1)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)}+K\,\mathrm{e}^{-\tilde{R}}N(\tilde{d}_2)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)}\\&=S\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\left[N(d_1)-\mathrm{e}^{-R}N(d_2)-N(\tilde{d}_1)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}\right.\\&\quad\left.+\mathrm{e}^{-\tilde{R}}N(\tilde{d}_2)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}\right]\\&=S\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\left\{N(d_1)-\mathrm{e}^{-R}N(d_2)-\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}[N(\tilde{d}_1)-\mathrm{e}^{-\tilde{R}}N(\tilde{d}_2)]\right\}\end{aligned}\tag{D3}$$

where

$$\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\geq 0,$$

due to $d_1 > d_2$,

So, $N(d_1)-\mathrm{e}^{-R}N(d_2)\geq 0$, $N(\tilde{d}_1)-\mathrm{e}^{-\tilde{R}}N(\tilde{d}_2)\geq 0$, but

$$\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}\leq 1$$

and

$$N(d_1)-\mathrm{e}^{-R}N(d_2)-\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}[N(\tilde{d}_1)-\mathrm{e}^{-\tilde{R}}N(\tilde{d}_2)]\geq 0$$

Therefore, $C(d_l^*|x_r)\geq C(\tilde{A}_n^*|\tilde{M}_j)$.

Appendix E

Assume option is out of the money ($S<K$), let $T=1$ then, $C(\tilde{A}_n^*|\tilde{M}_j)$ can be defined in accordance with Eq. (12) as:

$$\begin{aligned}C(\tilde{A}_n^*|\tilde{M}_j)&=SN\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})}\right\}\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)}\\&\quad-K\,\mathrm{e}^{-\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i)R_t(S_i)}N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})}\right.\\&\quad\left.-\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right\}\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)}\end{aligned}\tag{E1}$$

Assume $\mu_{\tilde{A}_n}(\sigma_{tu})=\mu_{\tilde{F}_k}(S_i)=\mu_{\tilde{M}_j}(x_r)=1$, when it is inputted into Eq. (E1), the $C(\tilde{A}_n^*|\tilde{M}_j)$ will be transformed into $C(d_l^*|x_r)$:

$$\begin{aligned}C(d_l^*|x_r)&=SN\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}\right\}\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\\&\quad-K\,\mathrm{e}^{-\sum_{i=1}^{I}P(S_i)R_t(S_i)}N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}-\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right\}\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\end{aligned}\tag{E2}$$

Let

$$N(d_1)=N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}\right\}$$

$$N(d_2)=N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})}-\sum_{u=1}^{v}\sigma_{tu}P(\sigma_{tu})\right\}$$

$$N(\tilde{d}_1)=N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})}\right\}$$

$$N(\tilde{d}_2)=N\left\{\frac{\ln(S/K)+\sum_{i=1}^{I}\mu_{\tilde{F}_k}(S_i)P(S_i)R_t(S_i)+\left[\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right]^2/2}{\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})}-\sum_{u=1}^{v}\sigma_{tu}\mu_{\tilde{A}}(\sigma_{tu})P(\sigma_{tu})\right\}$$

Then

$$\begin{aligned}
C(d_l^*|x_r)-C(\tilde{A}_n^*|\tilde{M}_j)&=SN(d_1)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}-K\,\mathrm{e}^{-R}N(d_2)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\\
&\quad-SN(\tilde{d}_1)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)}\\
&\quad+K\,\mathrm{e}^{-\tilde{R}}N(\tilde{d}_2)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)P(x_r)}\\
&=\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\left[SN(d_1)-K\,\mathrm{e}^{-R}N(d_2)-SN(\tilde{d}_1)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}\right.\\
&\quad\left.+K\,\mathrm{e}^{-\tilde{R}}N(\tilde{d}_2)\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}\right]=\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\\
&\quad\times\left\{SN(d_1)-K\,\mathrm{e}^{-R}N(d_2)-\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}[SN(\tilde{d}_1)-K\,\mathrm{e}^{-\tilde{R}}N(\tilde{d}_2)]\right\}
\end{aligned} \tag{E3}$$

where

$$\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}P(x_r|S_i)P(S_i)}{\sum_{r=1}^{m}P(x_r)}\geq 0$$

although, $d_1<d_2$, but the degree of $S<K$ is bigger than $d_1<d_2$. So, $S\cdot N(d_1)-\mathrm{K}\,\mathrm{e}^{-\mathrm{R}}\,\mathrm{N}(d_2)\leq 0$, $SN(\tilde{d}_1)-K\,\mathrm{e}^{-\tilde{R}}N(\tilde{d}_2)\leq 0$, but

$$\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}\leq 1$$

and

$$SN(d_1)-K\,\mathrm{e}^{-R}N(d_2)-\frac{\sum_{i=1}^{I}\sum_{r=1}^{m}\mu_{\tilde{F}_k}(S_i)\mu_{\tilde{M}_j}(x_r)}{\sum_{r=1}^{m}\mu_{\tilde{M}_j}(x_r)}[SN(\tilde{d}_1)-K\,\mathrm{e}^{-\tilde{R}}N(\tilde{d}_2)]\leq 0$$

Therefore, $C(d_l^*|x_r)\leq C(\tilde{A}_n^*|\tilde{M}_j)$.

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