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Calculation of Zero-Norm States and Reduction of Stringy Scattering Amplitudes

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We give a simplified method to generate two types of zero-norm states in the old covariant first quantized (OCFQ) spectrum of open bosonic string. Zero-norm states up to the fourth massive level and general formulas of some zero-norm tensor states at arbitrary mass levels are calculated. On-shell Ward identities generated by zero-norm states and the *factorization* property of stringy vertex operators can then be used to argue that the string-tree scattering amplitudes of the degenerate lower spin propagating states are fixed by those of higher spin propagating states at each fixed mass level. This decoupling phenomenon is, in contrast to Gross's high-energy symmetries, valid to *all* energies. As examples, we explicitly demonstrate this stringy phenomenon up to the fourth massive level (spin-five), which justifies the calculation of two other previous approaches based on the massive worldsheet sigma-model and Witten's string field theory (WSFT).

§1. Introduction

The theory of string, as a consistent quantum theory, has no free parameter and an infinite number of states. It is thus conceivable that there exists huge hidden symmetry group which is responsible for the ultraviolet finiteness of the theory. In fact, it was conjectured by Gross[1)] more than a decade ago that an infinite broken gauge symmetries get restored at energy much higher than the Planck energy. Moreover, he conjectured that, for the closed string, there existed an infinite number of linear relations among the scattering amplitudes of different string states that are valid order by order and are of the *identical form* in string perturbation theory as α' goes to infinity. As a result, the scattering amplitudes of all string states can be expressed in terms of, say, the dilaton amplitudes. A similar result was presented in Ref. 2) for the open string case.

Soon after, it was discovered that[3)] the equations of motion for massive background fields of the degenerate positive-norm propagating states can be expressed in terms of those of higher spin propagating states at each fixed mass level. This decoupling phenomenon was argued to be arisen from the existence of two types of zero-norm states with the same Young representations as those of the degenerate positive-norm states in the OCFQ spectrum. This was demonstrated by using massive worldsheet sigma-model approach in the lowest order weak field approximation but valid to all orders in α', and thus was, in contrast to Gross's result, valid to *all* energies. To compare with the usual sigma-model loop (α') approximation, this result was argued to be a sigma-model $n+1$ loop result for the n-th massive level

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(spin-n+1).[3)–5)] This calculation applies to both open and closed string cases. In a recent paper,[6)] the same decoupling phenomenon was demonstrated by using WSFT for the open string case up to the spin-five level. It was shown that the background fields of these degenerate positive-norm states can be gauged to the higher *rank* fields at the same mass level.

In this paper, we will derive this interesting stringy decoupling phenomenon from the third and a more direct method, namely, the S-matrix approach. The key was to explicitly calculate both types of zero-norm states[7)] in the OCFQ spectrum. An infinite number of *nonlinear* relations between string scattering amplitudes of different string states with the same momenta at each fixed mass level can then be written down.[8)] By nonlinearity, one means that the coefficients among scattering amplitudes of different string states depend on the center of mass scattering angle $\phi_{\rm CM}$ through the dependence of momentum k.[9)] These relations, or stringy on-shell Ward identities are, as in Gross's case, valid order by order and are of the identical form in string perturbation theory since zero-norm states should be decoupled from the string amplitudes at each order of string perturbation theory. These Ward identities, together with the *factorization* property of stringy vertex operators, will be used in this paper to express the scattering amplitudes of the degenerate lower spin propagating states in terms of those of higher spin propagating states, and thus reduce the number of independent scattering amplitudes at each fixed mass level. These Ward identities and the resulting decoupling phenomenon are, in contrast to Gross's high-energy symmetries, valid to *all* energies. However, these nonlinear Ward identities, which are valid to all energies, are difficult to solve. The high-energy limit of these stringy Ward identities are recently[9)] used to explicitly prove Gross's conjecture on *linear* relations among high-energy scattering amplitudes of different string states with the same momenta. It was shown that these stringy Ward identities get simplied as $\alpha' \to \infty$, and the number of independent scattering amplitudes reduces further. As a result, there is only one independent component of high energy scattering amplitude at each fixed mass level. All other components of high energy scattering amplitudes are proportional to it. Moreover, the proportionality constants between scattering amplitudes of different string states are calculated. These proportionality constants were found to be independent of the scattering angle $\phi_{\rm CM}$ and the loop order χ of string perturbation theory as conjectured by Gross.[1),2)] For the case of string-tree amplitudes, a general formula can even be given[9)] to determine all high energy stringy scattering amplitudes for arbitrary mass levels in terms of those of tachyons – another conjecture by Gross.[1)]

It is now clear that zero-norm states are of crucial importance to uncover the fundamental symmetries of string theory.[9)] The power of zero-norm states and their direct relation to spacetime w_∞ symmetry and Ward identities[10)] of toy 2D string model were stressed in Ref. 11). A general formula of 2D zero-norm states at an arbitrary mass levels with Polyakov's momentum was given in terms of Schur Polynomials. These zero-norm states were shown to carry the charges of w_∞ symmetry, which was used to determine the tachyon scattering amplitudes *without* any integration. In §2 of this paper, with the help of a simplified method to construct D=26 stringy *positive*-norm vertex operators,[12)] we will first tabulate Young diagrams of

D=26 *zero*-norm states at each mass level given Young diagrams of positive-norm states at the same mass level. A consistent check of counting of number of zero-norm states by using the background ghost fields in WSFT was given in Ref. 6). Here we go one step further and invent a simplified method to explicitly construct D=26 stringy *zero*-norm states. As examples, we calculate all relevant zero-norm states up to the spin-five level. General formulas of some zero-norm tensor states at an arbitrary mass levels will also be given. In §3, we then use these zero-norm states and their corresponding stringy Ward identities, together with the *factorization* property of stringy vertex operators, to explicitly show the reduction of string-tree scattering amplitudes of degenerate positive-norm propagating states up to the spin-five level. This calculation justifies two previous independent calculations based on the massive worldsheet sigma-model approach[3)] and WSFT approach.[6)]

§2. Calculation of zero-norm states

The vertex operator of a physical state of open bosonic string

$$|\Psi\rangle = \sum C_{\mu_1\cdots\mu_m}\alpha_{-n_1}^{\mu_1}...\alpha_{-n_m}^{\mu_m}\,|0;k\rangle\,, [\alpha_m^{\mu},\alpha_n^{\nu}] = m\eta^{\mu\nu}\delta_{m+n} \quad (1)$$

is given by[13)]

$$\Psi(z) = \sum C_{\mu_1\cdots\mu_m}N_m : \prod(\partial_z^{n_j}x^{\mu_j})e^{ik\cdot X(z)} :, \quad (2)$$

where $N_m = i^m\prod\{(n_j-1)!\}^{-1}$. In the OCFQ spectrum, physical states in Eq. (1) are subject to the following Virasoro conditions

$$(L_0-1)\,|\Psi\rangle = 0, \qquad L_1\,|\Psi\rangle = L_2\,|\Psi\rangle = 0, \quad (3a,b)$$

where

$$L_m = \frac{1}{2}\sum_{-\infty}^{\infty} : \alpha_{m-n}\cdot\alpha_n : \quad (4)$$

and $\alpha_0 \equiv k$. The solutions of Eqs. (3a,b) include positive-norm propagating states and two types of zero-norm states. The latter are[14)]

$$\text{Type I} : L_{-1}\,|x\rangle\,,\ \text{where } L_1\,|x\rangle = L_2\,|x\rangle = 0,\ L_0\,|x\rangle = 0; \quad (5)$$

$$\text{Type II} : \left(L_{-2}+\frac{3}{2}L_{-1}^2\right)|\widetilde{x}\rangle\,,\ \text{where } L_1\,|\widetilde{x}\rangle = L_2\,|\widetilde{x}\rangle = 0,\ (L_0+1)\,|\widetilde{x}\rangle = 0. \quad (6)$$

Equations (5) and (6) can be derived from Kac determinant in conformal field theory. While type I states have zero-norm at any spacetime dimension, type II states have zero-norm *only* at D=26. The existence of type II zero-norm states signals the importance of zero-norm states in the structure of the theory of string. It is straightforward to solve positive-norm state solutions of Eq. (3a, b) for some low-lying states, but soon becomes practically unmanageable. The authors of Ref. 12) gave a simple prescription to solve the positive-norm state solutions of Eqs. (3a, b). The strategy is to apply the Virasoro conditions only to purely transverse states, so that the zero-norm states will be got rid of at the very beginning. This prescription simplified a lot of computation although some complexities remained for low spin states at higher levels. Our aim here, on the contrary, is to generate zero-norm states in Eqs. (5) and (6), so that all physical state solutions of Eq. (3) will be completed.

Let us first assume we are given positive-norm state solutions of some mass level n. The number of positive-norm degree of freedom at mass level n ($M^2 = 2(n-1)$) is given by $N_{24}(n)$, where[15)]

$$N_D(n) = \frac{1}{2\pi i} \oint \frac{dx}{x^{n+1}} \Big(\prod_{k=1}^{\infty} \frac{1}{1-x^k} \Big)^D . \tag{7}$$

On the other hand, the number of physical state degree of freedom is given by $N_{25}(n)$ in view of the constraints in Eq. (3a,b). The discrepancy is of course due to physical zero-norm states given by solutions of Eqs. (5) and (6). That is, among 25 chains of α_m^μ oscillators one chain forms zero-norm states. Thus we can easily tabulate Young diagrams of zero-norm states at each mass level given Young diagrams of positive-norm states at the same mass level calculated by the simplified prescription in.[12)] For example, positive-norm state $\square\square\square\square$ at mass level $n=4$ gives zero-norm states $\square\square\square + \square\square + \square + \bullet$, posive-norm state $\begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix}$ gives zero-norm states $\begin{smallmatrix}\square \\ \square\end{smallmatrix} + \square\square + \square$ and positive-norm state $\square\square$ gives zero-norm states $\square + \bullet$. This completes the zero-norm states at mass level $n=4$. Young diagrams of zero-norm states up to mass level $M^2 = 10$, together with positive-norm states calculated in Ref. 12), are listed in Appendix A. A consistent check of counting of zero-norm states by using background ghost fields in WSFT was given in Ref. 6).

To explicitly calculate zero-norm states is another issue. Suppose we are given some low-lying positive-norm state solutions. It is interesting to see the similarity between Eqs. (3a, b) and Eqs. (5) and (6) for $|x\rangle$ and $|\widetilde{x}\rangle$. The only difference is the "mass shift" of L_0 equations. As is well-known, the L_1 and L_2 equations give the transverse and traceless conditions on the spin polarization. It turns out that, in many cases, the L_1 and L_2 equations will not refer to the L_0 equation or on-mass-shell condition. In these cases, a positive-norm state solution for $|\Psi\rangle$ at mass level n will give a zero-norm state solution $L_{-1}|x\rangle$ at mass level $n+1$ simply by taking $|x\rangle = |\Psi\rangle$ and shifting k^2 by one unit. Similarly, one can easily get a type II zero-norm state $(L_{-2} + \frac{3}{2}L_{-1}^2)|\widetilde{x}\rangle$ at mass level $n+2$ simply by taking $|\widetilde{x}\rangle = |\Psi\rangle$ and shifting k^2 by two units. For those cases where L_1 and L_2 equations do refer to L_0 equation, our prescription needs to be modified. We will give some examples to illustrate this method. Note that once we generate a zero-norm state, it soon becomes a candidate of physical state $|\Psi\rangle$ to generate two new zero-norm states at even higher levels.

1. The first zero-norm state begin at $k^2 = 0$. This state is suggested from the positive-norm tachyon state $|0,k\rangle$ with $k^2 = 2$. Taking $|x\rangle = |0,k\rangle$ and shifting k^2 by one unit to $k^2 = 0$, we get a type I zero-norm state.

$$L_{-1}\ |x\rangle = k\cdot\alpha_{-1}\,|0,k\rangle\,; |x\rangle = |0,k\rangle\,, -k^2 = M^2 = 0. \tag{8}$$

2. At the first massive level $k^2 = -2$, tachyon suggests a type II zero-norm state

$$\left(L_{-2}+\frac{3}{2}L_{-1}^2\right)|\widetilde{x}\rangle = \left[\frac{1}{2}\alpha_{-1}\cdot\alpha_{-1}+\frac{5}{2}k\cdot\alpha_{-2}+\frac{3}{2}(k\cdot\alpha_{-1})^2\right]|0,k\rangle\,; |\widetilde{x}\rangle = |0,k\rangle\,, -k^2 = 2. \tag{9}$$

Positive-norm massless vector state suggests a type I zero-norm state

$$L_{-1}\,|x\rangle = [\theta\cdot\alpha_{-2}+(k\cdot\alpha_{-1})(\theta\cdot\alpha_{-1})]\,|0,k\rangle\,; |x\rangle = \theta\cdot\alpha_{-1}\,|0,k\rangle\,, -k^2 = 2, \theta\cdot k = 0. \tag{10}$$

However, massless singlet zero-norm state (8) does not give a type I zero-norm state at the first massive level $k^2 = -2$ since L_1 equation on state (8) refers to L_0 equation, $k^2 = 0$. This means that L_1 will not annihilate state (8) if one shifts the mass to $k^2 = -2$.

3. At the second massive level $k^2 = -4$, positive-norm massless vector state suggests a type II zero-norm state

$$\begin{aligned}\left(L_{-2}+\frac{3}{2}L_{-1}^2\right)|\widetilde{x}\rangle &= \Big\{4\theta\cdot\alpha_{-3}+\frac{1}{2}(\alpha_{-1}\cdot\alpha_{-1})(\theta\cdot\alpha_{-1})+\frac{5}{2}(k\cdot\alpha_{-2})(\theta\cdot\alpha_{-1})\\ &\quad+\frac{3}{2}(k\cdot\alpha_{-1})^2(\theta\cdot\alpha_{-1})+3(k\cdot\alpha_{-1})(\theta\cdot\alpha_{-2})\Big\}\,|0,k\rangle\,;\\ |\widetilde{x}\rangle &= \theta\cdot\alpha_{-1}\,|0,k\rangle\,, -k^2 = 4, k\cdot\theta = 0.\end{aligned} \tag{11}$$

However, massless singlet zero-norm state (8) does not give a type II zero-norm state at mass level $k^2 = -4$ for the same reason stated after Eq. (10). Positive-norm spin-two state at $k^2 = -2$ suggests a type I zero-norm state

$$\begin{aligned}&L_{-1}\,|x\rangle = [2\theta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-2}^{\nu}+k_\lambda\theta_{\mu\nu}\alpha_{-1}^{\lambda\mu\nu}]\,|0,k\rangle\,; |x\rangle = \theta_{\mu\nu}\alpha_{-1}^{\mu\nu}\,|0,k\rangle\,, -k^2 = 4,\\ &\quad k\cdot\theta = \eta^{\mu\nu}\theta_{\mu\nu} = 0, \theta_{\mu\nu} = \theta_{\nu\mu},\end{aligned} \tag{12}$$

where $\alpha_{-1}^{\lambda\mu\nu} \equiv \alpha_{-1}^{\lambda}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}$. Similar notations will be used in the rest of this paper. Vector zero-norm state with $k^2 = -2$ in Eq. (10) does not give a type I zero-norm state for the same reason stated after Eq. (10). In this case, however, one can modify $|x\rangle$ to be

$$\text{Ansatz: } |x\rangle = [a\theta\cdot\alpha_{-2}+b(k\cdot\alpha_{-1})(\theta\cdot\alpha_{-1})]\,|0,k\rangle\,; -k^2 = 4, \theta\cdot k = 0, \tag{13}$$

where a, b are undetermined constants. L_0 equation is then trivially satisfied and L_1, L_2 equations give $a : b = 2 : 1$. This gives a type I zero-norm state

$$\begin{aligned}L_{-1}\,|x\rangle = \Big[&\frac{1}{2}(k\cdot\alpha_{-1})^2(\theta\cdot\alpha_{-1})+2\theta\cdot\alpha_{-3}+\frac{3}{2}(k\cdot\alpha_{-1})(\theta\cdot\alpha_{-2})\\ &+\frac{1}{2}(k\cdot\alpha_{-2})(\theta\cdot\alpha_{-1})\Big]\,|0,k\rangle\,; -k^2 = 4, \theta\cdot k = 0.\end{aligned} \tag{14}$$

Similarly, we modify the singlet zero-norm state with $k^2 = -2$ in Eq. (9) to be

$$\text{Ansatz: } |x\rangle = \left[\frac{5}{2}ak\cdot\alpha_{-2} + \frac{1}{2}b\alpha_{-1}\cdot\alpha_{-1} + \frac{3}{2}c(k\cdot\alpha_{-1})^2\right]|0,k\rangle\,; -k^2 = 4, \tag{15}$$

where a, b and c are undetermined constants. L_1 and L_2 equations give

$$5a + b + 3k^2c = 0, 5k^2a + 13b + \frac{3}{2}k^2c = 0. \tag{16}$$

For $k^2 = -4$, we have $a : b : c = 5 : 9 : \frac{17}{6}$. This gives a type I zero-norm state

$$\begin{aligned} L_{-1}|x\rangle = \Big[&\frac{17}{4}(k\cdot\alpha_{-1})^3 + \frac{9}{2}(k\cdot\alpha_{-1})(\alpha_{-1}\cdot\alpha_{-1}) + 9(\alpha_{-1}\cdot\alpha_{-2}) \\ &+21(k\cdot\alpha_{-1})(k\cdot\alpha_{-2}) + 25(k\cdot\alpha_{-3})\Big]|0,k\rangle\,; \\ -k^2 = 4.& \end{aligned} \tag{17}$$

This completes the four zero-norm states at the second massive level. Note that state (17) was calculated in Ref. 7) without modification. The coefficients there thus need to be corrected although the main results remain valid. It is interesting to note that the Young tableau of zero-norm states at level $M^2 = 4$ are the sum of those of all physical states at two lower levels, $M^2 = 2$ and $M^2 = 0$, *except* the singlet zero-norm state due to the dependence of L_1 and L_2 equations on L_0 condition in state (8). For those cases that L_1 and L_2 equations not referring to L_0 condition, our construction gives us a very simple way to calculate zero-norm states at any mass level n given those of positive-norm states at lower levels constructed by the simplified method in Ref. 12). When the modified method was needed to calculate a higher mass level zero-norm state from a lower mass level physical state like Eq. (8), an inconsistency may result and one gets no zero-norm state. This explains the discrepancy of singlet zero-norm states at levels $M^2 = 2, 4, 8$ and a vector zero-norm state at level $M^2 = 10$.

4. Similar method can be used to calculate zero-norm states at level $M^2 = 6$. We will just list those which are relevant for the discussion in section III. They are (from now on, unless otherwise stated, each spin polarization is assumed to be transverse, traceless and is symmetric with respect to each group of indices as in Ref. 12))

$$L_{-1}|x\rangle = \theta_{\mu\nu\lambda}(k_\beta\alpha_{-1}^{\mu\nu\lambda\beta} + 3\alpha_{-1}^{\mu\nu}\alpha_{-2}^{\lambda})|0,k\rangle\,; |x\rangle = \theta_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda}|0,k\rangle\,, \tag{18}$$

$$\begin{aligned} L_{-1}|x\rangle = [k_\lambda\theta_{\mu\nu}\alpha_{-1}^{\mu\lambda}\alpha_{-2}^{\nu} + 2\theta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-3}^{\nu}|0,k\rangle\,; |x\rangle] = \theta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-2}^{\nu}|0,k\rangle\,, \\ \text{where } \theta_{\mu\nu} = -\theta_{\nu\mu}, \end{aligned} \tag{19}$$

$$\begin{aligned} L_{-1}|x\rangle = \Big[&2\theta_{\mu\nu}\alpha_{-2}^{\mu\nu} + 4\theta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-3}^{\nu} + 2(k_\lambda\theta_{\mu\nu} + k_{(\lambda}\theta_{\mu\nu)})\alpha_{-1}^{\lambda\mu}\alpha_{-2}^{\nu} \\ &+\frac{2}{3}k_\lambda k_\beta\theta_{\mu\nu}\alpha_{-1}^{\mu\nu\lambda\beta}\Big]|0,k\rangle\,; \\ |x\rangle = \Big[&2\theta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-2}^{\nu} + \frac{2}{3}k_\lambda\theta_{\mu\nu}\alpha_{-1}^{\mu\nu\lambda}\Big]|0,k\rangle \end{aligned} \tag{20}$$

and

$$\left(L_{-2}+\frac{3}{2}L_{-1}^2\right)|\widetilde{x}\rangle=\Big[3\theta_{\mu\nu}\alpha_{-2}^{\mu\nu}+8\theta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-3}^{\nu}+\Big(k_\lambda\theta_{\mu\nu}+\frac{15}{2}k_{(\lambda}\theta_{\mu\nu)}\Big)\alpha_{-1}^{\lambda\mu}\alpha_{-2}^{\nu}$$
$$+\Big(\frac{1}{2}\eta_{\lambda\beta}\theta_{\mu\nu}+\frac{3}{2}k_\lambda k_\beta\theta_{\mu\nu}\Big)\alpha_{-1}^{\mu\nu\lambda\beta}\Big]|0,k\rangle\,;$$
$$|\widetilde{x}\rangle=\theta_{\mu\nu}\alpha_{-1}^{\mu\nu}|0,k\rangle\,. \tag{21}$$

Note that $|x\rangle$ in Eq. (20) has been modified as we did for Eq. (13). To further illustrate our method, we calculate the type I singlet zero-norm state from Eq. (17) as following

$$\text{Ansatz}\;:\;|x\rangle=[a(k\cdot\alpha_{-1})^3+b(k\cdot\alpha_{-1})(\alpha_{-1}\cdot\alpha_{-1})+c(k\cdot\alpha_{-1})(k\cdot\alpha_{-2})$$
$$+d(\alpha_{-1}\cdot\alpha_{-2})+f(k\cdot\alpha_{-3})]\,|0,k\rangle\,;$$
$$-k^2=6. \tag{22}$$

The L_1 and L_2 equations can be easily used to determine $a:b:c:d:f=37:72:261:216:450$. This gives the type I singlet zero-norm state

$$L_{-1}|x\rangle=[a(k\cdot\alpha_{-1})^4+b(k\cdot\alpha_{-1})^2(\alpha_{-1}\cdot\alpha_{-1})+(2b+d)(k\cdot\alpha_{-1})(\alpha_{-1}\cdot\alpha_{-2})$$
$$+(c+3a)(k\cdot\alpha_{-1})^2(k\cdot\alpha_{-2})+c(k\cdot\alpha_{-2})^2$$
$$+d(\alpha_{-2}\cdot\alpha_{-2})+b(k\cdot\alpha_{-2})(\alpha_{-1}\cdot\alpha_{-1})$$
$$+(2c+f)(k\cdot\alpha_{-3})(k\cdot\alpha_{-1})+2d(\alpha_{-1}\cdot\alpha_{-3})+3f(k\cdot\alpha_{-4})]\,|0,k\rangle\,,$$
$$-k^2=6. \tag{23}$$

5. We list relevant zero-norm states at level $M^2=8$ from the known positive-norm states and zero-norm states at level $M^2=4,6$. They are

$$L_{-1}|x\rangle=(k_\beta\theta_{\mu\nu\lambda\gamma}\alpha_{-1}^{\mu\nu\lambda\gamma\beta}+4\theta_{\mu\nu\lambda\gamma}\alpha_{-1}^{\mu\nu\lambda}\alpha_{-2}^{\gamma})\,|0,k\rangle\,;$$
$$|x\rangle=\theta_{\mu\nu\lambda\gamma}\alpha_{-1}^{\mu\nu\lambda\gamma}|0,k\rangle\,, \tag{24}$$

$$L_{-1}|x\rangle=\theta_{\mu\nu\lambda}\Big[\frac{3}{4}k_\beta k_\gamma\alpha_{-1}^{\mu\nu\lambda\gamma\beta}+3k_\beta\alpha_{-1}^{\mu\nu\beta}\alpha_{-2}^{\lambda}+3k_\beta\alpha_{-1}^{(\mu\nu\lambda}\alpha_{-2}^{\beta)}$$
$$+6\alpha_{-1}^{(\mu}\alpha_{-2}^{\nu\lambda)}+6\alpha_{-1}^{(\mu\nu}\alpha_{-3}^{\lambda)}\Big]\,|0,k\rangle\,;$$
$$|x\rangle=\theta_{\mu\nu\lambda}\Big(\frac{3}{4}k_\beta\alpha_{-1}^{\mu\nu\lambda\beta}+3\alpha_{-1}^{\mu\nu}\alpha_{-2}^{\lambda}\Big)\,|0,k\rangle\,, \tag{25}$$

$$\left(L_{-2}+\frac{3}{2}L_{-1}^2\right)|\widetilde{x}\rangle=\theta_{\mu\nu\lambda}\Big[\Big(\frac{3}{2}k_\beta k_\gamma+\frac{1}{2}\eta_{\gamma\beta}\Big)\alpha_{-1}^{\mu\nu\lambda\beta\gamma}+k_\gamma\Big(\frac{1}{2}\alpha_{-1}^{\mu\nu\lambda}\alpha_{-2}^{\gamma}+8\alpha_{-1}^{(\mu\nu\lambda}\alpha_{-2}^{\gamma)}\Big)$$
$$+3\alpha_{-1}^{(\mu}\alpha_{-2}^{\nu\lambda)}+6\alpha_{-1}^{(\mu\nu}\alpha_{-3}^{\lambda)}\Big]\,|0,k\rangle\,;$$
$$|\widetilde{x}\rangle=\theta_{\mu\nu\lambda}\alpha_{-1}^{\mu\nu\lambda}|0,k\rangle\,, \tag{26}$$

$$L_{-1}\left|x\right\rangle=\theta_{\mu\nu,\lambda}(k_\gamma\alpha_{-1}^{\gamma\mu\nu}\alpha_{-2}^{\lambda}+2\alpha_{-1}^{\mu}\alpha_{-2}^{\nu\lambda}+2\alpha_{-1}^{\mu\nu}\alpha_{-3}^{\lambda})\left|0,k\right\rangle;$$
$$\left|x\right\rangle=\theta_{\mu\nu,\lambda}\alpha_{-1}^{\mu\nu}\alpha_{-2}^{\lambda}\left|0,k\right\rangle,\ \text{where } \theta_{\mu\nu,\lambda} \text{ is mixed symmetric,} \tag{27}$$

$$L_{-1}\left|x\right\rangle=\theta_{\mu\nu}\Big(\frac{3}{4}k_\beta k_\lambda\alpha_{-1}^{\beta\lambda\mu}\alpha_{-2}^{\nu}+4k_\lambda\alpha_{-1}^{\lambda\mu}\alpha_{-3}^{\nu}+\frac{3}{4}k_\lambda\alpha_{-1}^{\mu}\alpha_{-2}^{\nu\lambda}+2\alpha_{-2}^{\mu}\alpha_{-3}^{\nu}+6\alpha_{-1}^{\mu}\alpha_{-4}^{\nu}\Big)\left|0,k\right\rangle;$$
$$\left|x\right\rangle=\Big(\frac{3}{4}k_\lambda\alpha_{-1}^{\lambda\mu}\alpha_{-2}^{\nu}+2\alpha_{-1}^{\mu}\alpha_{-3}^{\nu}\Big)\left|0,k\right\rangle,\ \text{where } \theta_{\mu\nu}=-\theta_{\nu\mu}, \tag{28}$$

and

$$\Big(L_{-2}+\frac{3}{2}L_{-1}^2\Big)\left|\widetilde{x}\right\rangle=\theta_{\mu\nu}\Big[\Big(\frac{3}{2}k_\gamma k_\lambda+\frac{1}{2}\eta_{\gamma\lambda}\Big)\alpha_{-1}^{\gamma\lambda\mu}\alpha_{-2}^{\nu}+6k_\lambda\alpha_{-1}^{\lambda\mu}\alpha_{-3}^{\nu}+\frac{5}{2}k_\lambda\alpha_{-1}^{\mu}\alpha_{-2}^{\nu\lambda}+2\alpha_{-2}^{\mu}\alpha_{-3}^{\nu}+\alpha_{-1}^{\mu}\alpha_{-4}^{\nu}\Big]\left|0,k\right\rangle,\left|\widetilde{x}\right\rangle=\theta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-2}^{\nu}\left|0,k\right\rangle,$$
$$\text{where } \theta_{\mu\nu}=-\theta_{\nu\mu}. \tag{29}$$

Note that the modified method was used in Eqs. (25) and (28).

6. Finally, we calculate general formulas of some zero-norm tensor states at arbitrary mass levels by making use of general formulas of some positive-norm states listed in Ref. 12).

a.

$$L_{-1}\theta_{\mu_1\cdots\mu_m}\alpha_{-1}^{\mu_1\cdots\mu_m}\left|0,k\right\rangle=\theta_{\mu_1\cdots\mu_m}(k_\lambda\alpha_{-1}^{\lambda\mu_1\cdots\mu_m}+m\alpha_{-2}^{\mu_1}\alpha_{-1}^{\mu_2\cdots\mu_m})\left|0,k\right\rangle, \tag{30}$$

where $-k^2=M^2=2m, m=0,1,2,3\cdots$. For example, $m=0,1$ give Eqs. (8) and (10).

b.

$$\Big(L_{-2}+\frac{3}{2}L_{-1}^2\Big)\theta_{\mu_1\cdots\mu_m}\alpha_{-1}^{\mu_1\cdots\mu_m}\left|0,k\right\rangle$$
$$=\Big\{\theta_{\mu_1\cdots\mu_m}\Big[\Big(\frac{3}{2}k_\nu k_\lambda+\frac{1}{2}\eta_{\nu\lambda}\Big)\alpha_{-1}^{\nu\lambda\mu_1\cdots\mu_m}+\frac{3}{2}m(m-1)\alpha_{-2}^{\mu_1\mu_2}\alpha_{-1}^{\mu_3\cdots\mu_m}+(1+3m)\alpha_{-1}^{\mu_1\cdots\mu_{m-1}}\alpha_{-3}^{\mu_m}\Big]+\Big[\frac{3}{2}(m+1)k_{(\lambda}\theta_{\mu_1\cdots\mu_m)}+\frac{3}{2}mk_{\mu_m}\theta_{\mu_1\cdots\mu_{m-1}\lambda}\Big]\alpha_{-1}^{\mu_1\cdots\mu_m}\alpha_{-2}^{\lambda}\Big\}\left|0,k\right\rangle, \tag{31}$$

where $-k^2=M^2=2m+2, m=0,1,2\cdots$. For example, $m=0,1$ give Eqs. (9) and (11).

c.

$$L_{-1}\theta_{\mu_1\cdots\mu_{m-2},\mu_{m-1}}\alpha_{-1}^{\mu_1\cdots\mu_{m-2}}\alpha_{-2}^{\mu_{m-1}}\left|0,k\right\rangle$$
$$=\theta_{\mu_1\cdots\mu_{m-2},\mu_{m-1}}[k_\lambda\alpha_{-1}^{\lambda\mu_1\cdots\mu_{m-2}}\alpha_{-2}^{\mu_{m-1}}+(m-2)\alpha_{-1}^{\mu_1\cdots\mu_{m-3}}\alpha_{-2}^{\mu_{m-2}\mu_m}+2\alpha_{-1}^{\mu_1\cdots\mu_{m-2}}\alpha_{-2}^{\mu_{m-1}}]\left|0,k\right\rangle, \tag{32}$$

where $-k^2 = M^2 = 2m, m = 3, 4, 5\cdots$. For example, $m = 3, 4$ give Eqs. (19) and (27).

d.

$$\begin{aligned}&\Big(L_{-2} + \frac{3}{2}L_{-1}^2\Big)\theta_{\mu_1\cdots\mu_{m-2},\mu_{m-1}}\alpha_{-1}^{\mu_1\cdots\mu_{m-2}}\alpha_{-2}^{\mu_{m-1}}\left|0,k\right\rangle \\ &= \theta_{\mu_1\cdots\mu_{m-2},\mu_{m-1}}\Big[\Big(\frac{3}{2}k_\lambda k_\nu + \frac{1}{2}\eta_{\lambda\nu}\Big)\alpha_{-1}^{\mu_1\cdots\mu_{m-2}\lambda\nu}\alpha_{-2}^{\mu_{m-1}} + 6k_\lambda\alpha_{-1}^{\mu_1\cdots\mu_{m-2}\lambda}\alpha_{-3}^{\mu_{m-1}} \\ &\quad+\Big(\frac{3}{2}m-2\Big)k_\lambda\alpha_{-1}^{\mu_1\cdots\mu_{m-2}}\alpha_{-2}^{\mu_{m-1}\lambda} + 2(m-2)\alpha_{-1}^{\mu_1\cdots\mu_{m-3}}\alpha_{-2}^{\mu_{m-2}}\alpha_{-3}^{\mu_{m-1}} \\ &\quad+11\alpha_{-1}^{\mu_1\cdots\mu_{m-2}}\alpha_{-4}^{\mu_{m-1}} + k_\lambda\alpha_{-1}^{\mu_1\cdots\mu_{m-3}\lambda}\alpha_{-2}^{\mu_{m-2}\mu_{m-1}} \\ &\quad+(m-3)\alpha_{-1}^{\mu_1\cdots\mu_{m-4}}\alpha_{-2}^{\mu_{m-3}\mu_{m-2}\mu_{m-1}}\Big]\left|0,k\right\rangle, \end{aligned} \tag{33}$$

where $-k^2 = M^2 = 2m+2, m = 3, 4, 5\ldots$. For example, $m = 3$ gives Eq. (29).

e.

$$\begin{aligned}&L_{-1}\theta_{\mu_1\cdots\mu_{m-4},\mu_{m-3}\mu_{m-2}}\Big(\alpha_{-1}^{\mu_1\cdots\mu_{m-4}}\alpha_{-2}^{\mu_{m-3}\mu_{m-2}} - \frac{4}{3}\alpha_{-1}^{\mu_1\cdots\mu_{m-3}}\alpha_{-3}^{\mu_{m-2}}\Big) \\ &= \theta_{\mu_1\cdots\mu_{m-4},\mu_{m-3}\mu_{m-2}}\Big[k_\lambda\alpha_{-1}^{\lambda\mu_1\cdots\mu_{m-4}}\alpha_{-2}^{\mu_{m-3}\mu_{m-2}} + (m-4)\alpha_{-1}^{\mu_1\cdots\mu_{m-3}}\alpha_{-2}^{\mu_{m-4}\mu_{m-3}\mu_{m-2}} \\ &\quad+\frac{16}{3}\alpha_{-1}^{\mu_1\cdots\mu_{m-4}}\alpha_{-3}^{\mu_{m-3}}\alpha_{-2}^{\mu_{m-2}} + \frac{4}{3}k_\lambda\alpha_{-1}^{\lambda\mu_1\cdots\mu_{m-3}}\alpha_{-3}^{\mu_{m-2}} + 4\alpha_{-1}^{\mu_1\cdots\mu_{m-3}}\alpha_{-4}^{\mu_{m-4}}\Big], \end{aligned} \tag{34}$$

where $-k^2 = M^2 = 2m, m = 5, 6\cdots$.

f. The zero-norm states of Eq. (30) can be used to generate new type I zero-norm states by the modified method as following

$$\begin{aligned}&L_{-1}\theta_{\mu_1\cdots\mu_m}\Big(\frac{m}{m+1}k_\lambda\alpha_{-1}^{\lambda\mu_1\cdots\mu_m} + \alpha_{-2}^{\mu_1}\alpha_{-1}^{\mu_2\cdots\mu_m}\Big)\left|0,k\right\rangle \\ &= \Big[\frac{m}{m+1}k_\nu k_\lambda\theta_{\mu_1\cdots\mu_m}\alpha_{-1}^{\nu\lambda\mu_1\cdots\mu_m} + m(k_{(\lambda}\theta_{\mu_1\cdots\mu_m)} + k_\lambda\theta_{\mu_1\cdots\mu_m})\alpha_{-2}^{\mu_1}\alpha_{-1}^{\lambda\mu_2\cdots\mu_m} \\ &\quad+m(m-1)\theta_{\mu_1\cdots\mu_m}\alpha_{-2}^{\mu_1\mu_2}\alpha_{-1}^{\mu_3\cdots\mu_m} + 2m\theta_{\mu_1\cdots\mu_m}\alpha_{-3}^{\mu_1}\alpha_{-1}^{\mu_2\cdots\mu_m}\Big]\left|0,k\right\rangle, \end{aligned} \tag{35}$$

where $-k^2 = M^2 = 2m+2, m = 1, 2, 3\cdots$. For example, $m = 1, 2$ and 3 give Eqs. (14), (20) and (25). Note that the coefficient of the first term in Eq. (35) has been modified to $\frac{m}{m+1}$. Similarly, new type II zero-norm states can also be constructed.

These are examples of some higher spin zero-norm states at arbitrary mass levels. As in the case of positive-norm states, the complexity of the calculation increases when calculating lower spin zero-norm states for higher levels. Fortunately, for our purpose in this paper, it is usually good enough to calculate higher spin zero-norm states as it will become clear in the next section. For those formulas with transverse trace[12)]

$$\eta_{\mu\nu}^T = \eta_{\mu\nu} - k_\mu k_\nu/k^2, \tag{36}$$

the modified method should be used, and we have no general formulas for them.

Each zero-norm state calculated in this section corresponds to an on-shell Ward identity, which can be easily written down. As an interesting example[8)] to illustrate the importance of zero-norm state, the *inter-particle Ward identity* for two propagating states at the second massive level ($M^2 = 4$) was calculated to be ($k \cdot \theta = 0$)

$$\left(\frac{1}{2}k_\mu k_\nu \theta_\lambda + 2\eta_{\mu\nu}\theta_\lambda\right)\mathcal{T}^{(\mu\nu\lambda)}_{2,\chi} + 9k_\mu\theta_\nu\mathcal{T}^{[\mu\nu]}_{2,\chi} - 6\theta_\mu\mathcal{T}^{\mu}_{2,\chi} = 0, \tag{37}$$

where we have chosen, say, $v_1(k_1)$ to be the vertex operator constructed from D_2 vector zero-norm state obtained by antisymmetrizing those terms which contain $\alpha^\mu_{-1}\alpha^\nu_{-2}$ in the original type I, Eq. (14), and type II, Eq. (11), vector zero-norm states and $k_\mu \equiv k_{1\mu}$. Note that v_2, v_3 and v_4 can be any string states (including zero-norm states), and we have omitted their tensor index for the cases of excited string states in Eq. (37). $\mathcal{T}'_{2,\chi}s$ in Eq. (37) are the second massive level, χ-th order string-loop amplitudes. At this point, $\{\mathcal{T}^{(\mu\nu\lambda)}_{2,\chi}, \mathcal{T}^{(\mu\nu)}_{2,\chi}, \mathcal{T}^{\mu}_{2,\chi}\}$ is identified to be the *amplitude triplet* of the spin-three state and $T^{[\mu\nu]}$ is identified to be the amplitude of the antisymmetric spin-two state.[8)] Eq. (37) thus relates the scattering amplitudes of two different string states at the second massive level. It is important to note that Eq. (37) is, in contrast to the high-energy $\alpha' \to \infty$ result of Gross, valid to all string-loop and *all* energy α', and its coefficients do depend on the center of mass scattering angle ϕ_{CM}, which is defined to be the angle between $\overrightarrow{k}_1$ and $\overrightarrow{k}_3$, through the dependence of momentum k . This angular dependence disappears in the high-energy limit of Eq. (37),[9)] which is consistent with Gross's result. The inter-particle gauge symmetry corresponding to Eq. (37) can be calculated to be[7)]

$$\delta C_{(\mu\nu\lambda)} = \left(\frac{1}{2}\partial_{(\mu}\partial_\nu\theta_{\lambda)} - 2\eta_{(\mu\nu}\theta_{\lambda)}\right), \quad \delta C_{[\mu\nu]} = 9\partial_{[\mu}\theta_{\nu]}, \tag{38}$$

where $\partial_\nu\theta^\nu = 0, (\partial^2 - 4)\theta^\nu = 0$ are the on-shell conditions of the D_2 vector zero-norm state. $C_{(\mu\nu\lambda)}$ and $C_{[\mu\nu]}$ are the background fields of the symmetric spin-three and antisymmetric spin-two states respectively at the second mass level. Equation (38) is the result of the first order weak field approximation but valid to *all* energy α' in the generalized σ-model approach. It is important to note that the decoupling of D_2 vector zero-norm state implies simultaneous change of both $C_{(\mu\nu\lambda)}$ and $C_{[\mu\nu]}$, thus they form a gauge multiplet. This important stringy phenomenon can also be justified in WSFT.[6),8)] A second order weak field calculation implies an even more interesting spontaneously broken inter-mass level symmetry in string theory.[16)]

§3. Reduction of degenerate state's amplitude

The decoupling of degenerate positive-norm states was first discovered in Ref. 3) by using generalized sigma-model approach. It was recently justified by using WSFT for the open string case up to the spin-five level.[6)] This stringy phenomenon begins to show up at spin-four level of open bosonic string. The explicit form of four positive-norm states at spin-four level can be found in Ref. 13). According to the decoupling conjecture, the spin-two and the scalar positive-norm states should be decoupled. That is, their amplitudes are determined from those of two other higher spin states. Let us begin the discussion by first making an important observation. According to Eq. (2), the vertex operator corresponding to $\alpha_{-1}^{\mu\nu\lambda\gamma}$ is $A^{\mu\nu\lambda\gamma} = :\partial x^\mu \partial x^\nu \partial x^\lambda \partial x^\gamma e^{ik\cdot x}:$. *Due to the factorization structure of this tensor vertex, which results from the strong constraint of 2D worldsheet conformal symmetry, the amplitude corresponding to $A^{\mu\nu\lambda\gamma}$ is fixed by its traceless, transverse spin part $\epsilon_{\mu\nu\lambda\gamma}$.* In particular, the longitudinal parts of $A^{\mu\nu\lambda\gamma}$ are determined by $\epsilon_{\mu\nu\lambda\gamma}$ through the Lorentz extension; and the trace parts of $A^{\mu\nu\lambda\gamma}$ are fixed by the *conformal extension.* This means that given the on-shell amplitude of $\epsilon_{\mu\nu\lambda\gamma}$, the amplitude $\mathcal{T}_{\mu\nu\lambda\gamma}$ of $A_{\mu\nu\lambda\gamma}$ is fixed. Here $\mathcal{T}^{\mu\nu\lambda\gamma}$ is defined to be the four-point function containing the *rank*-four tensor $:\partial x^\mu \partial x^\nu \partial x^\lambda \partial x^\gamma e^{ik\cdot x}:$ and three tachyons. Due to the factorization structure of stringy vertex operator, the string-tree scattering amplitude are factorized in the momentum k^μ carried by the vertex vertex. Let us use a simpler rank-two tensor to illustrate the *trace fixing* or conformal extension. Given a *factorized* symmetric rank-two tensor constructed from a D-vector k^μ (k^μ will correspond to momentum for the scattering amplitudes in the later discussion)

$$\begin{aligned} A^{\mu\nu} &= k^\mu k^\nu + c\eta^{\mu\nu} \\ &= \Big(k^\mu k^\nu - \frac{k^2}{D}\eta^{\mu\nu}\Big) + \Big(\frac{k^2}{D} + c\Big)\eta^{\mu\nu}, \end{aligned} \tag{39}$$

where we have decomposed $A^{\mu\nu}$ into a traceless spin part and a trace part containing a scalar c independent of the spin part, the trace part of $A^{\mu\nu}$ is not fixed by the spin part of $A^{\mu\nu}$. Now for the *homogeneous* factorized tensor, $c = 0$ in Eq. (39). The traceless spin part of Eq. (39) gives us $\frac{D(D+1)}{2}$ components which is of order D^2, while the *factorized* symmetric rank-two tensor $A^{\mu\nu}$ contains only D independent components which are components of k^μ. It is thus easy to see that the trace part of $A^{\mu\nu}$ is fixed by the spin part of the tensor. Thus, knowing the spin part of $A^{\mu\nu}$ means knowing the whole tensor. This result can be easily generalized to the decomposition of a homogeneous factorized tensor $A^{\mu\nu} = k_1^\mu k_2^\nu$, which contains only $2D$ independent components in contrast to the number of components of the spin part, which is of the order D^2. Similar results can be obtained for *homogeneous factorized* higher rank tensors. Note that this factorized property can only be seen in the first order weak field approximation[3)] (or vertex operator consideration), and does not show up in the zeroth order spectrum.

With the observation discussed above in mind, we can now discuss the decoupling phenomenon at level four. It was pointed out[3)] that the positive-norm spin-two state can be gauged to a gauge which contains only $\alpha_{-1}^{\mu\nu\lambda\gamma}$ and $\alpha_{-1}^{\mu\nu}\alpha_{-2}^{\lambda}$ terms by making use of the gauge transformations induced by the type I and the type II spin-two zero-norm states, Eqs. (20) and (21), to be

$$\Big[\Big(\frac{1}{3}k_\lambda\epsilon_{\mu\nu} + \frac{1}{2}k_{(\lambda}\epsilon_{\mu\nu)}\Big)\alpha_{-1}^{\lambda\mu}\alpha_{-2}^{\nu} + \Big(\frac{13}{174}k_\alpha k_\beta\epsilon_{\mu\nu} + \frac{3}{58}\eta_{\alpha\beta}\epsilon_{\mu\nu}\Big)\alpha_{-1}^{\mu\nu\alpha\beta}\Big]\left|0,k\right\rangle, \tag{40}$$

where $\epsilon_{\mu\nu}$ is a symmetric traceless and transverse spin-two tensor. Since the rank-four amplitude $\mathcal{T}_{3,\chi}^{\mu\nu\alpha\beta}$ is fixed by the spin-four amplitude and the mixed-symmetric rank-three amplitude $\mathcal{T}_{3,\chi}^{\lambda\mu\nu}$ is fixed by the mixed-symmetric spin-three amplitude, the amplitude of the spin-two state in Eq. (40) is determined by those of the spin-four and the mixed-symmetric spin-three states. (Note that $\mathcal{T}_{3,\chi}^{(\lambda\mu\nu)}$ is fixed by the spin-four amplitude $\mathcal{T}_{3,\chi}^{\mu\nu\alpha\beta}$ due to the existence of a totally symmetric spin-three zero-norm state Eq. (18) at this level.) In fact, $\mathcal{T}_{3,\chi}^{\mu\nu\alpha\beta}$ with $\chi = 1$ can be explicitly calculated to be[8)]

$$\begin{aligned}\mathcal{T}_{3,1}^{\mu\nu\lambda\gamma} = \frac{\Gamma(-\frac{s}{2}-1)\Gamma(-\frac{t}{2}-1)}{\Gamma(\frac{u}{2}+2)}\Big[&\Big(\frac{s^2}{4}-s\Big)\Big(\frac{s^2}{4}-1\Big)k_3^{\mu}k_3^{\nu}k_3^{\lambda}k_3^{\gamma} \\ &-t\Big(\frac{t^2}{4}-1\Big)(s+2)k_1^{(\mu}k_1^{\nu}k_1^{\lambda}k_3^{\gamma)} + \frac{3st}{2}\Big(\frac{s}{2}+1\Big)\Big(\frac{t}{2}+1\Big)k_1^{(\mu}k_1^{\nu}k_3^{\lambda}k_3^{\gamma)} \\ &-s\Big(\frac{s^2}{4}-1\Big)(t+2)k_1^{(\mu}k_3^{\nu}k_3^{\lambda}k_3^{\gamma)} + \Big(\frac{t^2}{4}-t\Big)\Big(\frac{t^2}{4}-1\Big)k_1^{\mu}k_1^{\nu}k_1^{\lambda}k_1^{\gamma}\Big], \end{aligned} \quad (41)$$

where $s = -(k_1 + k_2)^2, t = -(k_2 + k_3)^2$, and $u = -(k_1 + k_3)^2$ are the Mandelstam variables. We have chosen the second state to be the tensor and have done the $SL(2, R)$ gauge fixing and restricted to the $s - t$ channel by setting $x_1 = 0, 0 \leq x_2 \leq 1, x_3 = 1, x_4 = \infty$. One easily sees from Eq. (41) that there are no terms containing $\eta^{\mu\nu}$ on the right hand side of $\mathcal{T}_{3,1}^{\mu\nu\lambda\gamma}$. This is due to the normal ordering of the tensor vertex operator $:\partial x^{\mu}\partial x^{\nu}\partial x^{\lambda}\partial x^{\gamma}e^{ik\cdot x}:$, and there is no contribution of terms resulting from contraction within the tensor vertex when doing the amplitude calculation. Thus the trace part of the rank-four amplitude is fixed by the spin-four amplitude by the conformal extension mentioned in the beginning of this section. That is, the rank-four amplitude $\mathcal{T}_{3,1}^{\mu\nu\lambda\gamma}$ is fixed by the spin-four amplitude. This result can be easily generalized to N-point amplitudes containing more than one tensor state.

Take a representative of the positive-norm scalar state at this mass level to be[13)]

$$\begin{aligned}\Big[&\Big(\eta_{\mu\nu} + \frac{13}{3}k_{\mu}k_{\nu}\Big)\alpha_{-2}^{\mu\nu} + \Big(\frac{20}{9}k_{\mu}k_{\nu}k_{\rho} + \frac{2}{3}k_{\mu}\eta_{\nu\rho} + \frac{13}{3}k_{\rho}\eta_{\mu\nu}\Big)\alpha_{-1}^{\mu\nu}\alpha_{-2}^{\rho} \\ &+\Big(\frac{23}{81}k_{\mu}k_{\nu}k_{\rho}k_{\sigma} + \frac{32}{27}k_{\mu}k_{\nu}\eta_{\rho\sigma} + \frac{19}{18}\eta_{\mu\nu}\eta_{\rho\sigma}\Big)\alpha_{-1}^{\mu\nu\rho\sigma}\Big]|0,k\rangle\,. \end{aligned} \quad (42)$$

It turns out that one can not gauge away the first term in Eq. (42) by using the gauge transformations induced by the two singlet zero-norm states as in the case of positive-norm spin-two state. However, since the amplitude corresponding to $\alpha_{-2}^{\mu\nu}$ has been fixed by those of two higher spin states, we conclude that the positive-norm scalar state amplitude is again fixed by those of two higher spin states. This concludes the justification of decoupling conjecture for spin-four level. We stress here that the mechanisms that is responsible for this decoupling is the existence of two-types of zero-norm states and the factorization of stringy vertex, which are both due to 2D infinite dimensional worldsheet conformal symmetry.

The positive-norm states at level five were calculated in Ref. 12) to be

$$\epsilon_{\mu\nu\lambda\beta\gamma}\alpha_{-1}^{\mu\nu\lambda\beta\gamma}\left|0,k\right\rangle \quad \square\square\square\square\square, \tag{43}$$

$$\epsilon_{\mu\nu\lambda,\beta}\alpha_{-1}^{\mu\nu\lambda}\alpha_{-2}^{\beta}\left|0,k\right\rangle \quad \begin{array}{l}\square\square\square\\ \square\end{array}, \tag{44}$$

$$\epsilon_{\mu,\nu\lambda}\left(\alpha_{-1}^{\mu}\alpha_{-2}^{\nu\lambda}-\frac{4}{3}\alpha_{-1}^{\mu\nu}\alpha_{-3}^{\lambda}\right)\left|0,k\right\rangle \quad \begin{array}{l}\square\square\\ \square\end{array}, \tag{45}$$

$$\left[\frac{4}{5!(D+5)}\epsilon_{\mu\nu\lambda}\eta_{\beta\gamma}^{T}\alpha_{-1}^{\mu\nu\lambda\beta\gamma}+\epsilon_{\mu\nu\lambda}\left(\alpha_{-1}^{\mu}\alpha_{-2}^{\nu\lambda}-\frac{4}{3}\alpha_{-1}^{\mu\nu}\alpha_{-3}^{\lambda}\right)\right]\left|0,k\right\rangle \quad \square\square\square, \tag{46}$$

$$\left[\frac{5}{6(D+1)}\eta_{(\mu\nu}^{T}\epsilon_{\lambda)\beta}\alpha_{-1}^{\mu\nu\lambda}\alpha_{-2}^{\beta}+\epsilon_{\mu\nu}\left(\alpha_{-2}^{\mu}\alpha_{-3}^{\nu}-\frac{1}{2}\alpha_{-1}^{\mu}\alpha_{-4}^{\nu}\right)\right]\left|0,k\right\rangle \quad \begin{array}{l}\square\\ \square\end{array}, \tag{47}$$

and

$$\begin{aligned}&\left[\frac{D-2}{80(D+3)}\eta_{(\mu\nu}^{T}\eta_{\lambda\beta}^{T}\epsilon_{\gamma)}\alpha_{-1}^{\mu\nu\lambda\beta\gamma}+\left(\eta_{\mu\nu}^{T}\epsilon_{\lambda}-\frac{1}{2}(D-1)\epsilon_{(\mu}\eta_{\nu)\lambda}^{T}\right)\alpha_{-1}^{\mu\nu}\alpha_{-3}^{\lambda}\right.\\ &\left.+\frac{3}{4}(D\epsilon_{\mu}\eta_{\nu\lambda}^{T}-\eta_{\mu(\nu}^{T}\epsilon_{\lambda)})\alpha_{-1}^{\mu}\alpha_{-2}^{\nu\lambda})\right]\left|0,k\right\rangle \ \square.\end{aligned} \tag{48}$$

According to our decoupling conjecture, states (46), (47) and (48) should be decoupled. Note that states (27) and (45) are different in the $\alpha_i' s$ operator content although they share the same Young diagram. One corresponds to $\alpha_{-1}^{\mu}\alpha_{-2}^{\nu\lambda}$ and the other $\alpha_{-1}^{\mu\nu}\alpha_{-3}^{\lambda}$ or vice versa. With the explicit form of zero-norm states calculated in §3, we can now justify the decoupling conjecture at level five. The terms $\alpha_{-1}^{(\mu}\alpha_{-2}^{\nu\lambda)}$ and $\alpha_{-1}^{(\mu\nu}\alpha_{-3}^{\lambda)}$ in Eq. (46) can be gauged away by zero-norm states in Eqs. (25) and (26), and the amplitude corresponding to $\alpha_{-1}^{(\mu\nu\lambda}\alpha_{-2}^{\beta)}$ is fixed by that of $\alpha_{-1}^{\mu\nu\lambda\beta\gamma}$ through zero-norm state in Eq. (24) and our observation discussed in the beginning of this section. Thus the amplitude of state (46) is fixed by those of states (43) and (44). Now turn to state (47). The terms $\alpha_{-2}^{[\mu}\alpha_{-3}^{\nu]}$ and $\alpha_{-1}^{[\mu}\alpha_{-4}^{\nu]}$ can be gauged away by zero-norm states in Eqs. (28) and (29), the amplitudes corresponding to $\alpha_{-1}^{(\mu}\alpha_{-2}^{\nu\lambda)}$ and $\alpha_{-1}^{(\mu\nu}\alpha_{-3}^{\lambda)}$ are fixed by those of states in Eqs. (43) and (44) through zero-norm states in Eqs. (25) and (26). Finally the amplitude of mixed-symmetric $\alpha_{-1}^{\mu}\alpha_{-2}^{\nu\lambda}$ (or $\alpha_{-1}^{\mu\nu}\alpha_{-3}^{\lambda}$) is fixed by those of states (43), (44) and (45). Thus the amplitude of state (47) is fixed by those of states (43), (44) and (45). Similar analysis shows that the amplitude of state (48) is again fixed by those of states (43), (44) and (45). This completes the justification of our decoupling conjecture at level five.

The decoupling calculation presented in this paper by the S-matrix approach can be easily generalized to the closed string theory by making use of the simple relation between closed and open string amplitudes in Ref. 17). A similar generalization to the closed string theory can also be done for the massive worldsheet sigma-model approach. Our calculation in this section justifies two previous independent calculations based on the massive worldsheet sigma-model approach[3] and WSFT approach.[6]

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Appendix

The Young tabulations of all physical states solutions of Eq. (3) up to level six, including two types of zero-norm state solutions of Eqs. (5) and (6), are listed in the following table:

massive level	positive-norm states	zero-norm states
$M^2=-2$	$\bullet$	
$M^2=0$	$\ydiagram{1}$	$\bullet$ (singlet)
$M^2=2$	$\ydiagram{2}$	$\ydiagram{1}$, $\bullet$
$M^2=4$	$\ydiagram{3}$, $\ydiagram{1,1}$	$\ydiagram{2}$, $2\times\ydiagram{1}$, $\bullet$
$M^2=6$	$\ydiagram{4}$, $\ydiagram{2,1}$, $\ydiagram{2}$, $\bullet$	$\ydiagram{3}$, $\ydiagram{1,1}$, $2\times\ydiagram{2}$, $3\times\ydiagram{1}$, $2\times\bullet$
$M^2=8$	$\ydiagram{5}$, $\ydiagram{3,1}$, $\ydiagram{2,1}$, $\ydiagram{3}$, $\ydiagram{1,1}$, $\ydiagram{1}$	$\ydiagram{4}$, $\ydiagram{2,1}$, $2\times\ydiagram{3}$, $2\times\ydiagram{1,1}$, $4\times\ydiagram{2}$, $5\times\ydiagram{1}$, $3\times\bullet$
$M^2=10$	$\ydiagram{6}$, $\ydiagram{4,1}$, $\ydiagram{4}$, $\ydiagram{3,1}$, $\ydiagram{2,2}$, $\ydiagram{3}$, $\ydiagram{2,1}$, $\ydiagram{1,1,1}$, $2\times\ydiagram{2}$, $\ydiagram{1}$, $\bullet$	$\ydiagram{5}$, $2\times\ydiagram{4}$, $\ydiagram{3,1}$, $3\times\ydiagram{2,1}$, $4\times\ydiagram{3}$, $4\times\ydiagram{1,1}$, $7\times\ydiagram{2}$, $8\times\ydiagram{1}$, $6\times\bullet$

Note that the Young tabulations of zero-norm states at level n are subset of the sum of all physical states at levels n-1 and n-2.

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