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Maximum entropy analysis to the N policy M/G/1 queueing system with server breakdowns and general startup times

Kuo-Hsiung Wang [a,*], Tsung-Yin Wang [b], Wen-Lea Pearn [b]

[a] Department of Applied Mathematics, National Chung-Hsing University, Taichung 402, Taiwan
[b] Department of Industrial Engineering and Management, National Chiao Tung University, Hsinchu 30050, Taiwan

Abstract

We study a single removable and unreliable server in the *N* policy M/G/1 queueing system with general startup times where arrivals form a Poisson process and service times are generally distributed. When *N* customers are accumulated in the system, the server is immediately turned on but is temporarily unavailable to the waiting customers. He needs a startup time before providing service until the system becomes empty. The server is subject to breakdowns according to a Poisson process and his repair time obeys an arbitrary distribution. We use maximum entropy principle to derive the approximate formulas for the steady-state probability distributions of the queue length. We perform a comparative analysis between the approximate results with established exact results for various distributions, such as exponential (M),

* Corresponding author.
E-mail address: khwang@amath.nchu.edu.tw (K.-H. Wang).

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fc-stage Erlang (E_k), and deterministic (D). We demonstrate that the maximum entropy approach is accurate enough for practical purposes and is a useful method for solving complex queueing systems.

Keywords: Control; Lagrange's method; Maximum entropy; M/G/1 queue; Startup; Unreliable server

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1. Introduction

This paper deals with a single removable and unreliable server in the N policy M/G/1 queueing system in which the breakdown times of the server follow the negative exponential distribution, the repair times of the server obey a general distribution, and the startup times are generally distributed. The term 'removable server' is just an abbreviation for the system of turning on and turning off the server, depending on the number of customers in the system. An unreliable server means that the server is typically subject to unpredictable breakdowns. When the queue length reaches the threshold N ($N \geqslant 1$), the server is immediately turned on but is temporarily unavailable to the waiting customers. He requires for preservice work (i.e. begin startup) before starting service. Once the startup is over, the server immediately starts serving the waiting customers.

It is assumed that customers arrive according to a Poisson process with parameter λ and service times are independent and identically distributed (i.i.d.) random variables having a general distribution function $S(t)$ ($t \geqslant 0$) with a mean service time μ_S and a finite variance σ_S^2. The server is subject to breakdowns at any time with Poisson breakdown rate α when he is turned on and working. When the server fails, he is immediately repaired at a repair facility, where the repair times are independent and identically distributed random variables obeying a general distribution function $R(t)$ ($t \geqslant 0$) with a mean repair time μ_R and a finite variance σ_R^2. Arriving-customers form a single waiting line based on the first-come, first-served (FCFS) discipline. The server can serve only one customer at a time and the service is independent of the arrival of the customers. A customer who arrives and finds the server busy or broken down must wait in the queue until a server is available. Although no service occurs during the repair period of the server, customers continue to arrive following a Poisson process. Furthermore, when the number of customers in the system reaches a specific level, denoted by N, the server is immediately turned on (i.e. begin startup) but is temporarily unavailable to the waiting customers. He requires a startup time with random length before starting service. The startup times are independent and identically distributed random variables obeying a general distribution function $U(t)$ ($t \geqslant 0$) with a mean startup time μ_U and a finite variance σ_U^2. Once the startup is over, the server begins serving the waiting customers until there are no customers in the system. Service is allowed to be interrupted if the server breaks down, and the server is immediately repaired. Once the server is repaired, he immediately returns to serve customers until the system becomes empty.

The previous work is divided into two parts according to whether the server is reliable or unreliable. In the first part we review previous papers which deal with reliable server. For a reliable server, the concept of the *N* policy was first proposed by Yadin and Naor [29]. The *N* policy M/G/1 queueing system was first studied by Heyman [6] and was investigated by several researchers such as Bell [3], Tijms [18], Wang and Ke [24], and others. Exact steady-state solutions of the *N* policy M/E_k/1 queueing system were first developed by Wang and Huang [23]. Recently, Wang and Yen [27] proposed the *N* policy M/H_k/1 queueing system. Exact steady-state solutions of the *N* policy M/M/1 queueing system with exponential startup times were first derived by Baker [2]. Borthakur et al. [4] extended Baker's model to general startup times. The *N* policy M/G/1 queueing system with startup times was examined by several researchers such as Medhi and Templeton [14], Takagi [17], Lee and Park [13], Krishna et al. [12], Hur and Paik [7] and others. Recently, Ke [8] analyzed the *N* policy G/M/1/K queueing system with exponential startup times. The second part considers previous papers dealing with the unreliable server. For an unreliable server, Wang [19,20], and Wang et al. [22] derived exact steady-state solutions of the *N* policy M/M/1, the *N* policy M/E_k/1, and the *N* policy M/H_2/1 queueing systems, respectively. Wang and Ke [25] analyzed three control policies in an M/G/1 queueing system and proved that in three control policies, the probability that the server is busy in the steady-state is equal to the traffic intensity. Recently, Ke [9] investigated the *N* policy M/G/1 queueing system with server vacations, startup and breakdowns. Exact steady-state solutions of the *N* policy M/M/1 queueing system with exponential startup times were first developed by Wang [21].

The maximum entropy principle is applied to analyze the ordinary queueing systems by several researchers such as Shore [15,16], Arizono et al. [1], Wu and Chan [28], El-Affendi and Kouvatsos [5], Kouvatsos [11], and so on. The maximum entropy principle has been widely applied to the study of more complicated ordinary queueing systems having general interarrival times, or general service times, or general interarrival times and general service times. Wang et al. [26] used the maximum entropy principle to examine the *N* policy M/G/1 queueing system with a reliable server. Many of the exact steady-state solutions to the control queueing problems with service times or repair times or startup times distribution of the general type have not been found. It is extremely difficult, if not impossible, to obtain the explicit formulas such as the steady-state probability mass function of the number of customers and the expected waiting time for the N policy M/G/1 queueing system with repair times and startup times are generally distributed. However, one can utilize the maximum entropy principle to approximate the N policy M/G/1 queueing system with general repair times and general startup times. This becomes particularly helpful when some system performance measures (for instance, the expected number of customers in the system, the probability that the server is busy, broken down, etc.) are known. In this paper, we utilize the maximum entropy principle associated with five basic known results from the literature to study the N policy M/G/1 queueing system with general repair times and general startup times.

The purpose of this paper is

(i) to provide the maximum entropy formalism for the N policy M/G/1 queueing system with general repair times and general startup times;
(ii) to develop the maximum entropy (approximate) solutions for the N policy M/G/1 queueing system with general repair times and general startup times by using Lagrange's method;
(iii) to obtain approximate results for the expected waiting time in queue;
(iv) to perform a comparative analysis between the exact results and the approximate results obtained through maximum entropy principle.

2. The expected number of customers in the system

Let $G_O(z)$ denote the probability generating function (p.g.f.) of the number of customers in the ordinary M/G/1 queueing system with reliable server. From Kleinrock [10, p. 194], we have

$$G_O(z) = \frac{(1-\rho)(1-z)B^*(\lambda - \lambda z)}{B^*(\lambda - \lambda z) - z}, \tag{1}$$

where $\rho = \lambda\mu_S$ and $B^*(\cdot)$ is the Laplace–Stieltjes transform (abbreviated LST) of service time.

Let H be a random variable representing the completion time of a customer, which includes both the service time of a customer and the repair time of a server. Applying the well-known formula for the p.g.f. of the number of customers in the ordinary M/G/1 queueing system with reliable server, the p.g.f. of the number of customers in ordinary M/G/1 queueing system with unreliable server is given by

$$G(z) = \frac{(1-\rho_H)(1-z)H^*(\lambda - \lambda z)}{H^*(\lambda - \lambda z) - z}, \tag{2}$$

where $\rho_H = \lambda E[H]$ ($E[H]$ is the mean completion time) and $H^*(\cdot)$ is the LST of completion time. Note that ρ_H is traffic intensity and it should be assumed to be less than unity. It should be noted that expression (2) is obtained only by replacing service times by completion times in the formula of the ordinary M/G/1 queueing system with reliable server.

We consider that the server is on 'extended vacation' during the turned-off period F_p and startup period S_p, the lengths of which equal $(F_p + S_p)$. Following the result of Medhi and Templeton [14], we obtain

$$G_N(z) = \frac{[1-\beta(z)]G(z)}{\beta'(1)(1-z)}, \tag{3}$$

where

$G_N(z)$	$\equiv$ the p.g.f. of number of customers in the N policy M/G/1 queueing system with server breakdowns and general startup times;
$\beta(z)$	$\equiv$ the p.g.f. of the number of customers that arrive during a vacation of length $F_p + S_p$;
	$\equiv$ [the p.g.f. of the number of customers that arrive during F_p] $\times$ [the p.g.f. of the number of customers that arrive during S_p]
	$\equiv z^N U^*(\lambda - \lambda z)$, where $U^*(\cdot)$ is the LST of startup time.

We have $\beta'(z) = Nz^{N-1}U^*(\lambda - \lambda z) + z^N U^{*\prime}(\lambda - \lambda z)(-\lambda)$. It follows that $\beta'(1) = N + \lambda\mu_U = N + \rho_U$, where $\rho_U = \lambda\mu_U$. From (2) and (3), we obtain

$$G_N(z) = \frac{[1 - z^N U^*(\lambda - \lambda z)](1-\rho_H)H^*(\lambda - \lambda z)}{(N+\rho_U)[H^*(\lambda - \lambda z) - z]}.$$

Let L_N denote the expected number of customers in the N policy M/G/1 queueing system with server breakdowns and general startup times. Thus we have

$$\begin{aligned} L_N &= G_N'(z)|_{z=1} \\ &= \frac{1}{N+\rho_U}\left[\frac{N(N-1)}{2} + N\rho_U + \frac{\lambda^2 E[U^2]}{2}\right] + \lambda E[H] + \frac{\lambda^2 E[H^2]}{2[1-\lambda E[H]]}, \end{aligned} \tag{4}$$

where $E[H] = \mu_S(1 + \alpha\mu_R)$ and $E[H^2] = (1 + \alpha\mu_R)^2 E[S^2] + \alpha\mu_S E[R^2]$.

3. The maximum entropy results

In this section, we will develop the maximum entropy solutions for the steady-state probabilities of the N policy M/G/1 queueing system with server breakdowns and general startup times. Let us define

$P_{0,I}(n)$	$\equiv$ probability that there are n customers in the system when the server is turned off, where $n = 0, 1, \ldots, N-1$.
$P_{0,S}(n)$	$\equiv$ probability that there are n customers in the system when the server is startup, where $n = N, N+1, N+2, \ldots$
$P_1(n)$	$\equiv$ probability that there are n customers in the system when the server is turned on and working, where $n = 1, 2, 3, \ldots$
$P_2(n)$	$\equiv$ probability that there are n customers in the system when the server is in operation but found to be broken down, where $n = 1, 2, 3, \ldots$

In order to derive the steady-state probabilities $P_{0,I}(n)$, $P_{0,S}(n)$ and $P_i(n)$ ($i = 1, 2$) by using the maximum entropy principle, we formulate the maximum entropy model in the following. Following El-Affendi and Kouvatsos [5], the entropy function Y can be illustrated mathematically as

$$Y = -\sum_{n=0}^{N-1} P_{0,I}(n)\ln P_{0,I}(n) - \sum_{n=N}^{\infty} P_{0,S}(n)\ln P_{0,S}(n) - \sum_{n=1}^{\infty} P_1(n)\ln P_1(n) - \sum_{n=1}^{\infty} P_2(n)\ln P_2(n)$$

or equivalently

$$Y = -NP_{0,I}(0)\ln P_{0,I}(0) - \sum_{n=N}^{\infty} P_{0,S}(n)\ln P_{0,S}(n) - \sum_{n=1}^{\infty} P_1(n)\ln P_1(n) - \sum_{n=1}^{\infty} P_2(n)\ln P_2(n). \tag{5}$$

There are five basic known results from the literature (see [4] and [25]) that facilitate the application of the maximum entropy formalism to study the N policy M/G/1 queueing system with server breakdowns and general startup times. The maximum entropy solutions are obtained by maximizing (5) subject to the following five constraints, written as,

(i) normalizing condition

$$NP_{0,I}(0) + \sum_{n=N}^{\infty} P_{0,S}(n) + \sum_{n=1}^{\infty} P_1(n) + \sum_{n=1}^{\infty} P_2(n) = 1, \tag{6}$$

(ii) the probability that the server is startup

$$\sum_{n=N}^{\infty} P_{0,S}(n) = \frac{\rho_U}{N+\rho_U}[1 - \rho(1 + \alpha\mu_R)] = \rho_U\Theta(1 - \rho_H), \tag{7}$$

where $\Theta = \frac{1}{N+\rho_U}$ and $\rho_H = \rho(1 + \alpha\mu_R)$.

(iii) the probability that the server is busy

$$\sum_{n=1}^{\infty} P_1(n) = \rho \tag{8}$$

(iv) the probability that the server is broken down

$$\sum_{n=1}^{\infty} P_2(n) = \rho\alpha\mu_R, \tag{9}$$

(v) the expected number of customers in the system

$$\sum_{n=0}^{N-1} nP_{0,I}(n) + \sum_{n=N}^{\infty} nP_{0,S}(n) + \sum_{n=1}^{\infty} nP_1(n) + \sum_{n=1}^{\infty} nP_2(n) = L_N, \tag{10}$$

where L_N is given by (4).

It yields from (6) to (9)

$$P_{0,I}(0) = P_{0,I}(n) = \Theta(1-\rho_H), \quad n = 1,2,\ldots,N-1. \tag{11}$$

In (6)–(10), (6) is multiplied by θ_1 (7) is multiplied by θ_2, (8) is multiplied by θ_3, (9) is multiplied by θ_4 and (10) is multiplied by θ_5. Thus the Lagrangian function y is given by

$$\begin{aligned} y = {} & -NP_{0,I}(0)\ln P_{0,I}(0) - \sum_{n=N}^{\infty} P_{0,S}(n)\ln P_{0,S}(n) \\ & - \sum_{n=1}^{\infty} P_1(n)\ln P_1(n) - \sum_{n=1}^{\infty} P_2(n)\ln P_2(n) \\ & - \theta_1\left[NP_{0,I}(0) + \sum_{n=N}^{\infty} P_{0,S}(n) + \sum_{n=1}^{\infty} P_1(n) + \sum_{n=1}^{\infty} P_2(n) - 1\right] \\ & - \theta_2\left[\sum_{n=N}^{\infty} P_{0,S}(n) - \rho_U\Theta(1-\rho_H)\right] - \theta_3\left[\sum_{n=1}^{\infty} P_1(n) - \rho\right] \\ & - \theta_4\left[\sum_{n=1}^{\infty} P_2(n) - \rho\alpha\mu_R\right] \\ & - \theta_5\left[\frac{N(N-1)}{2}P_{0,I}(0) + \sum_{n=N}^{\infty} nP_{0,S}(n) + \sum_{n=1}^{\infty} nP_1(n) + \sum_{n=1}^{\infty} nP_2(n) - L_N\right], \end{aligned} \tag{12}$$

where θ_1–θ_5 are the Lagrangian multipliers corresponding to constraints (6)–(10), respectively.

3.1. The maximum entropy solutions

To get the maximum entropy solutions $P_{0,S}(n)$, $P_1(n)$, $P_2(n)$, maximizing in (5) subject to constraints (6)–(10) is equivalent to maximizing (12).

The maximum entropy solutions are obtained by taking the partial derivatives of y with respect to $P_{0,I}(0)$, $P_{0,S}(n)$, $P_i(n)$ ($i = 1, 2$), and setting the results equal to zero, namely,

$$\frac{\partial y}{\partial P_{0,I}(0)} = -N\ln P_{0,I}(0) - N - N\theta_1 - \frac{N(N-1)}{2}\theta_5 = 0, \tag{13}$$

$$\frac{\partial y}{\partial P_{0,S}(n)} = -\ln P_{0,S}(n) - 1 - \theta_1 - \theta_2 - n\theta_5 = 0, \tag{14}$$

$$\frac{\partial y}{\partial P_1(n)} = -\ln P_1(n) - 1 - \theta_1 - \theta_3 - n\theta_5 = 0, \tag{15}$$

$$\frac{\partial y}{\partial P_2(n)} = -\ln P_2(n) - 1 - \theta_1 - \theta_4 - n\theta_5 = 0. \tag{16}$$

It implies from (13)–(16) that we obtain

$$P_{0,I}(0) = P_{0,I}(n) = \mathrm{e}^{-(1+\theta_1)}\mathrm{e}^{(-(N-1)\theta_5)/2}, \quad n = 1, 2, \ldots, N-1, \tag{17}$$

$$P_{0,S}(n) = \mathrm{e}^{-(1+\theta_1+\theta_2)}\mathrm{e}^{-n\theta_5}, \quad n = N, N+1, \ldots \tag{18}$$

$$P_1(n) = \mathrm{e}^{-(1+\theta_1+\theta_3)}\mathrm{e}^{-n\theta_5}, \quad n = 1, 2, \ldots \tag{19}$$

$$P_2(n) = \mathrm{e}^{-(1+\theta_1+\theta_4)}\mathrm{e}^{-n\theta_5}, \quad , n = 1, 2, \ldots \tag{20}$$

Let $\phi_1 = \mathrm{e}^{-(1+\theta_1)}$, $\phi_2 = \mathrm{e}^{-\theta_2}$, $\phi_3 = \mathrm{e}^{-\theta_3}$, $\phi_4 = \mathrm{e}^{-\theta_4}$ and $\phi_5 = \mathrm{e}^{-\theta_5}$.
We transform (17)–(20) in terms ϕ_1, ϕ_2, ϕ_3, ϕ_4 and ϕ_5 given by

$$P_{0,I}(n) = \phi_1\phi_5^{(N-1)/2}, \quad n = 0, 1, \ldots, N-1, \tag{21}$$

$$P_{0,S}(n) = \phi_1\phi_2\phi_5^n, \quad n = N, N+1, \ldots \tag{22}$$

$$P_1(n) = \phi_1\phi_3\phi_5^n, \quad n = 1, 2, \ldots \tag{23}$$

$$P_2(n) = \phi_1\phi_4\phi_5^n, \quad n = 1, 2, \ldots \tag{24}$$

Substituting (22)–(24) into (7)–(9), respectively, yields

$$\phi_1\phi_2 = \frac{\rho_U\Theta(1-\rho_H)(1-\phi_5)}{\phi_5^N}, \tag{25}$$

$$\phi_1\phi_3 = \frac{\rho(1-\phi_5)}{\phi_5}, \tag{26}$$

$$\phi_1\phi_4 = \frac{\rho\alpha\mu_R(1-\phi_5)}{\phi_5}. \tag{27}$$

Substituting (11) and (22)–(24) into (10) and taking the algebraic manipulations, we obtain

$$\phi_5 = 1 - \frac{\rho_H + \rho_U\Theta(1-\rho_H)}{L_N - \Theta(N-1)(1-\rho_H)\left(\frac{N}{2}+\rho_U\right)}. \tag{28}$$

Finally, we get

$$P_{0,I}(n) = \Theta(1-\rho_H), \quad n = 0,1,2,\ldots,N-1, \tag{29}$$

$$P_{0,S}(n) = \rho_U\Theta(1-\rho_H)(1-\phi_5)\phi_5^{n-N}, \quad n = N, N+1,\ldots \tag{30}$$

$$P_1(n) = \rho(1-\phi_5)\phi_5^{n-1}, \quad n = 1,2,\ldots \tag{31}$$

$$P_2(n) = \rho\alpha\mu_R(1-\phi_5)\phi_5^{n-1}, \quad n = 1,2,\ldots \tag{32}$$

4. The exact and approximate expected waiting time in the queue

In this section, we develop the exact and the approximate formulae for the expected waiting time in the N policy M/G/1 queueing system with server breakdowns and general startup times as follows.

4.1. The exact expected waiting time in the queue

Let W_q denote the exact expected waiting time in the queue. Using (4) and Little's formula, we obtain

$$W_q = \frac{L_N}{\lambda} - E[H] = \Theta\left[\frac{N(N-1)}{2\lambda} + N\mu_U + \frac{\lambda E[U^2]}{2}\right] + \frac{\lambda E[H^2]}{2[1-\lambda E[H]]}. \tag{33}$$

4.2. The approximate expected waiting time in the queue

We define the idle state, the startup state, the busy state, and the repair state as follows:

(i) Idle state denoted by I: the server is turned off and the number of customers waiting in the system is less than or equal to $N-1$.
(ii) Startup state denoted by U: the server begins startup and the number of customers waiting in the system is greater than or equal to N.

(iii) Busy state denoted by B: the server is busy and provides service to a customer.
(iv) Repair state denoted by R: the server is broken down and being repaired.

Following Borthakur et al. [4], we find the expected waiting time of customer C at the states I, U, B and R as follows. Suppose that a customer C finds n customers waiting in the queue for service in front of him, while the system is at any one of the states I, U, B and R are described, respectively, as follows:

(i) In idle state I: The server will begin startup after $(N - n - 1)$ customers arrive in the system. Thus customer C will be served until $(N - n - 1)$ customers arrive and n customers in front of him waiting for service. The expected waiting time of customer C at the idle state is $(N - n - 1)/\lambda + \mu_U + n\mu_S$.
(ii) In startup state U: We derive the expected waiting time of customer C at the startup state in the following. Let us define

$U_r(t) \equiv$ remaining startup time for the server begin startup.

Following Borthakur et al. [4], the cumulative distribution function (c.d.f.) of $U_r(t)$ is given by

$$F_{Ur}(t) = P\{U_r(t) \leqslant t\} = \frac{1}{\mu_U}\int_0^t [1 - D(x)]\,\mathrm{d}x,$$

where D(x) is the c.d.f. of startup time. Let $E(U_r)$ be the mean remaining startup time. It implies that $E[U_r] = E[U^2]/2\mu_U$. Thus we obtain the expected waiting time of customer C at the startup state is $n\mu_S + E[U^2]/2\mu_U$.
(iii) In busy state B: Since the server is turned on and working, customer C only waits n customers in front of him to be served. The expected waiting time of customer C at the busy state is $n\mu_S$.
(iv) In repair state R: Using the same argument as (ii), we have the expected waiting time of customer C at the repair state is $n\mu_S + E[R^2]/2\mu_R$.

Finally, using the listed above results, we obtain the approximate expected waiting time in the queue given by

$$\begin{aligned} W_\mathrm{q}^* = {} & \sum_{n=0}^{N-1}\left(\frac{N-n-1}{\lambda} + \mu_U + n\mu_S\right)P_{0,I}(0) \\ & + \sum_{n=N}^{\infty}\left(n\mu_S + \frac{E[U^2]}{2\mu_U}\right)P_{0,S}(n) + \sum_{n=1}^{\infty}(n\mu_S)P_1(n) \\ & + \sum_{n=1}^{\infty}\left(n\mu_S + \frac{E[R^2]}{2\mu_R}\right)P_2(n), \end{aligned} \tag{34}$$

where $P_{0,I}(0)$, $P_{0,S}(n)$, $P_1(n)$, and $P_2(n)$ are given in (29)–(32), respectively.

5. Comparative analysis

The primary objective of this section is to examine the accuracy of the maximum entropy results. We present specific numerical comparisons between the exact results and the maximum entropy (approximate) results for the *N* policy M/G/1 queueing system with general service times, general repair times and general startup times. Conveniently, we represent this queueing system as the *N* policy M/G(G, G)/l queueing system where the second, third, fourth symbols denote the general distribution of service time, repair time, and startup time, respectively.

This section includes the following three subsections:

(i) Comparative analysis for the *N* policy M/M(M, M)/1 and M/D(D, D)/1 queueing systems.
(ii) Comparative analysis for the *N* policy $M/E_3(E_4, E_3)/1$ and $M/M(E_3, E_2)/1$ queueing systems.
(iii) Comparative analysis for the *N* policy $M/E_3(E_4, D)/1$ and $M/E_3(E_4, M)/1$ queueing systems.

5.1. Comparative analysis for the N policy M/M(M, M)/1 and M/D(D, D)/1 queueing systems

Here we perform a comparative analysis between the exact W_q and the approximate (maximum entropy) W_q^* for the *N* policy M/M(M, M)/1 and M/D(D, D)/1 queueing systems. For the *N* policy M/M(M, M)/1 queueing system, we obtain $\mu_S = 1$, $E[S^2] = 2/\mu^2$, $\mu_R = 1/\beta$, $E[R^2] = 2/\beta^2$, $\mu_U = 1/\gamma$, and $E[U^2] = 2/\gamma^2$. For the *N* policy M/D(D, D)/1 queueing system, we have $\mu_S = 1/\mu$, $E[S^2] = 1/\mu^2$, $\mu_R = 1/\beta$, $E[R^2] = 1/\beta^2$, $\mu_U = 1/\gamma$, and $E[U^2] = 1/\gamma^2$.

We set $N = 5$ and $N = 10$, and choose the various values of λ, μ, α, β, and γ. The numerical results are obtained by considering the following parameters:

Case 1: We fix $\mu = 1.0$, $\alpha = 0.05$, $\beta = 3.0$, $\gamma = 3.0$, and vary the values of λ from 0.2 to 0.8.
Case 2: We fix $\lambda = 0.3$, $\alpha = 0.05$, $\beta = 3.0$, $\gamma = 3.0$, and vary the values of μ from 0.5 to 2.0.
Case 3: We fix $\lambda = 0.3$, $\alpha = 1.0$, $\beta = 3.0$, $\gamma = 3.0$, and vary the values of α from 0.05 to 0.2.
Case 4: We fix $\lambda = 0.3$, $\mu = 1.0$, $\alpha = 0.05$, $\gamma = 3.0$, and vary the values of β from 2.0 to 6.0.
Case 5: We fix $\lambda = 0.3$, $\mu = 1.0$, $\alpha = 0.05$, $\beta = 3.0$, and vary the values of γ from 2.0 to 5.0.

Table 1
Comparison of exact W_q and approximate W_q^* for the N policy M/M(M, M)/1 and M/D(D, D)/1 queueing systems

	M/M(M, M)/1						M/D(D, D)/1					
	$N = 5$			$N = 10$			$N = 5$			$N = 10$		
	W_q	W_q^*	% Error	W_q	W_q^*	% Error	W_q	W_q^*	% Error	W_q	W_q^*	% Error
λ	Case 1. ($\mu = 1.0$, $\alpha = 0.05$, $\beta = 3.0$, $\gamma = 3.0$)											
0.2	10.4626	10.4244	0.3657	22.9452	22.8653	0.3481	10.3300	10.3955	0.6345	22.8137	22.8376	0.1049
0.4	5.9040	5.8579	0.7815	12.1359	12.0482	0.7225	5.5494	5.7124	2.9363	11.7834	11.9048	1.0302
0.6	5.1372	5.0756	1.1979	9.2850	9.1820	1.1095	4.3314	4.5880	5.9242	8.4824	8.6975	2.5358
0.8	7.1603	7.0513	1.5226	10.2658	10.1154	1.4654	4.9251	5.2593	6.7861	8.0347	8.3274	3.6437
μ	Case 2. ($\lambda = 0.3$, $\alpha = 0.05$, $\beta = 3.0$, $\gamma = 3.0$)											
0.5	10.0582	9.9373	1.2022	18.3737	18.1696	1.1107	8.4605	8.9757	6.0898	16.7776	17.2097	2.5752
1.0	7.3178	7.2762	0.5695	15.6334	15.5501	0.5325	7.0903	7.2048	1.6150	15.4074	15.4804	0.4733
1.5	7.0437	7.0179	0.3654	15.3592	15.3057	0.3480	6.9532	6.9967	0.6252	15.2704	15.2861	0.1031
2.0	6.9617	6.9431	0.2683	15.2773	15.2378	0.2583	6.9122	6.9324	0.2924	15.2294	15.2288	0.0038
α	Case 3. ($\lambda = 0.3$, $\mu = 1.0$, $\beta = 3.0$, $\gamma = 3.0$)											
0.05	7.3178	7.2762	0.5695	15.6334	15.5501	0.5325	7.0903	7.2048	1.6150	15.4074	15.4804	0.4733
0.10	7.3384	7.2546	1.1408	15.6539	15.4870	1.0660	7.1006	7.1794	1.1102	15.4177	15.4134	0.0282
0.15	7.3594	7.2333	1.7140	15.6750	15.4241	1.6005	7.1111	7.1540	0.6041	15.4282	15.3464	0.5302
0.20	7.3810	7.2121	2.2890	15.6966	15.3613	2.1359	7.1219	7.1288	0.0968	15.4391	15.2796	1.0328
β	Case 4. ($\lambda = 0.3$, $\mu = 1.0$, $\alpha = 0.05$, $\gamma = 3.0$)											
2.0	7.3298	7.2672	0.8549	15.6454	15.5204	0.7991	7.0963	7.1930	1.3625	15.4134	15.4478	0.2226
3.0	7.3178	7.2762	0.5695	15.6334	15.5501	0.5325	7.0903	7.2048	1.6150	15.4074	15.4804	0.4733
4.0	7.3123	7.2811	0.4269	15.6279	15.5655	0.3993	7.0875	7.2109	1.7411	15.4047	15.4969	0.5986
6.0	7.3072	7.2864	0.2845	15.6227	15.5811	0.2661	7.0850	7.2172	1.8670	15.4021	15.5136	0.7238
γ	Case 5. ($\lambda = 0.3$, $\mu = 1.0$, $\alpha = 0.05$, $\beta = 3.0$)											
2.0	7.4211	7.3789	0.5685	15.7269	15.6432	0.5323	7.1895	7.3035	1.5858	15.4989	15.5714	0.4675
3.0	7.3178	7.2762	0.5695	15.6334	15.5501	0.5325	7.0903	7.2048	1.6150	15.4074	15.4804	0.4733
4.0	7.2667	7.2253	0.5700	15.5869	15.5039	0.5326	7.0406	7.1553	1.6299	15.3617	15.4348	0.4762
5.0	7.2362	7.1949	0.5702	15.5591	15.4762	0.5327	7.0107	7.1256	1.6390	15.3342	15.4075	0.4779

Table 2
Comparison of exact W_q and approximate W_q^* for the N policy $M/E_3(E_4, E_3)/1$ and $M/M(E_3, E_2)/1$ queueing systems

	$M/E_3(E_4, E_3)/1$						$M/M(E_3, E_2)/1$					
	$N = 5$			$N = 10$			$N = 5$			$N = 10$		
	w_q	W_q^*	% Error	W_q	W_q^*	% Error	W_q	W_q^*	% Error	W_q	W_q^*	% Error
λ	Case 1. ($\mu = 1.0$, $\alpha = 0.05$, $\beta = 3.0$, $\gamma = 3.0$)											
0.2	10.3742	10.4051	0.2982	22.8575	22.8468	0.0467	10.4611	10.4228	0.3657	22.9442	22.8643	0.3481
0.4	5.6675	5.7607	1.6454	11.9007	11.9524	0.4344	5.9006	5.8545	0.7815	12.1335	12.0459	0.7225
0.6	4.5996	4.7502	3.2733	8.7496	8.8586	1.2465	5.1311	5.0696	1.1981	9.2805	9.1775	1.1095
0.8	5.6692	5.8557	3.2895	8.7774	8.9224	1.6524	7.1482	7.0393	1.5230	10.2557	10.1054	1.4655
μ	Case 2. ($\lambda = 0.3$, $\alpha = 0.05$, $\beta = 3.0$, $\gamma = 3.0$)											
0.5	8.9927	9.2959	3.3715	17.3093	17.5293	1.2711	10.0537	9.9328	1.2022	18.3700	18.1660	1.1107
1.0	7.1660	7.2285	0.8714	15.4827	15.5035	0.1348	7.3154	7.2737	0.5695	15.6318	15.5485	0.5325
1.5	6.9833	7.0037	0.2921	15.2999	15.2926	0.0478	7.0416	7.0158	0.3654	15.3579	15.3045	0.3480
2.0	6.9287	6.9359	0.1046	15.2453	15.2318	0.0888	6.9598	6.9411	0.2683	15.2761	15.2367	0.2583
α	Case 3. ($\lambda = 0.3$, $\mu = 1.0$, $\beta = 3.0$, $\gamma = 3.0$)											
0.05	7.1660	7.2285	0.8714	15.4827	15.5035	0.1348	7.3154	7.2737	0.5695	15.6318	15.5485	0.5325
0.10	7.1796	7.2043	0.3433	15.4962	15.4377	0.3776	7.3351	7.2514	1.1409	15.6515	15.4846	1.0660
0.15	7.1936	7.1802	0.1863	15.5102	15.3720	0.8907	7.3554	7.2293	1.7141	15.6717	15.4209	1.6005
0.20	7.2079	7.1562	0.7175	15.5245	15.3064	1.4045	7.3761	7.2073	2.2892	15.6925	15.3573	2.1359
β	Case 4. ($\lambda = 0.3$, $\mu = 1.0$, $\alpha = 0.05$, $\gamma = 3.0$)											
2.0	7.1739	7.2175	0.6073	15.4905	15.4717	0.1214	7.3264	7.2638	0.8549	15.6427	15.5177	0.7991
3.0	7.1660	7.2285	0.8714	15.4827	15.5035	0.1348	7.3154	7.2737	0.5695	15.6318	15.5485	0.5325
4.0	7.1624	7.2343	1.0033	15.4790	15.5197	0.2628	7.3103	7.2791	0.4269	15.6266	15.5642	0.3993
6.0	7.1590	7.2403	1.1350	15.4756	15.5361	0.3907	7.3054	7.2846	0.2845	15.6217	15.5801	0.2661
γ	Case 5. ($\lambda = 0.3$, $\mu = 1.0$, $\alpha = 0.05$, $\beta = 3.0$)											
2.0	7.2666	7.3285	0.8525	15.5748	15.5952	0.1310	7.4166	7.3744	0.5685	15.7242	15.6405	0.5323
3.0	7.1660	7.2285	0.8714	15.4827	15.5035	0.1348	7.3154	7.2737	0.5695	15.6318	15.5485	0.5325
4.0	7.1159	7.1786	0.8811	15.4367	15.4578	0.1367	7.2650	7.2236	0.5700	15.5856	15.5026	0.5326
5.0	7.0858	7.1486	0.8870	15.4091	15.4303	0.1378	7.2348	7.1935	0.5703	15.5580	15.4751	0.5327

Table 3
Comparison of exact W_q and approximate W_q^* for the N policy $M/E_3(E_4, D)/1$ and $M/E_3(E_4, M)/1$ queueing systems

	$M/E_3(E_4, D)/1$						$M/E_3(E_4, M)/1$					
	$N = 5$			$N = 10$			$N = 5$			$N = 10$		
	w_q	W_q^*	% Error	w_q	W_q^*	% Error	W_q	W_q^*	% Error	W_q	W_q^*	% Error
λ	Case 1. ($\mu = 1.0$, $\alpha = 0.05$, $\beta = 3.0$, $\gamma = 3.0$)											
0.2	10.3734	10.4044	0.2983	22.8571	22.8464	0.0467	10.3756	10.4065	0.2981	22.8582	22.8475	0.0467
0.4	5.6660	5.7593	1.6460	11.9000	11.9517	0.4345	5.6703	5.7636	1.6443	11.9022	11.9539	0.4343
0.6	4.5975	4.7481	3.2753	8.7485	8.8575	1.2467	4.6039	4.7544	3.2693	8.7517	8.8608	1.2459
0.8	5.6664	5.8529	3.2917	8.7760	8.9210	1.6529	5.6748	5.8612	3.2849	8.7803	8.9253	1.6515
μ	Case 2. ($\lambda = 0.3$, $\alpha = 0.05$, $\beta = 3.0$, $\gamma = 3.0$)											
0.5	8.9916	9.2948	3.3720	17.3087	17.5288	1.2712	8.9949	9.2980	3.3704	17.3104	17.5304	1.2710
1.0	7.1650	7.2274	0.8716	15.4821	15.5030	0.1348	7.1682	7.2307	0.8710	15.4838	15.5046	0.1347
1.5	6.9822	7.0026	0.2922	15.2994	15.2920	0.0478	6.9855	7.0059	0.2919	15.3010	15.2937	0.0479
2.0	6.9276	6.9349	0.1047	15.2448	15.2312	0.0888	6.9309	6.9381	0.1045	15.2464	15.2329	0.0889
α	Case 3. ($\lambda = 0.3$, $\mu = 1.0$, $\beta = 3.0$, $\gamma = 3.0$)											
0.05	7.1650	7.2274	0.8716	15.4821	15.5030	0.1348	7.1682	7.2307	0.8710	15.4838	15.5046	0.1347
0.10	7.1785	7.2032	0.3435	15.4957	15.4372	0.3776	7.1818	7.2064	0.3429	15.4973	15.4388	0.3777
0.15	7.1925	7.1791	0.1861	15.5096	15.3715	0.8907	7.1957	7.1823	0.1867	15.5113	15.3731	0.8908
0.20	7.2068	7.1551	0.7173	15.5239	15.3059	1.4045	7.2100	7.1583	0.7178	15.5256	15.3075	1.4046
β	Case 4. ($\lambda = 0.3$, $\mu = 1.0$, $\alpha = 0.05$, $\gamma = 3.0$)											
2.0	7.1728	7.2164	0.6076	15.4900	15.4712	0.1214	7.1761	7.2196	0.6069	15.4916	15.4728	0.1214
3.0	7.1650	7.2274	0.8716	15.4821	15.5030	0.1348	7.1682	7.2307	0.8710	15.4838	15.5046	0.1347
4.0	7.1613	7.2332	1.0035	15.4785	15.5192	0.2628	7.1646	7.2365	1.0029	15.4801	15.5208	0.2627
6.0	7.1579	7.2392	1.1352	15.4751	15.5355	0.3907	7.1612	7.2424	1.1346	15.4767	15.5372	0.3907
γ	Case 5. ($\lambda = 0.3$, $\mu = 1.0$, $\alpha = 0.05$, $\beta = 3.0$)											
2.0	7.2642	7.3261	0.8529	15.5736	15.5940	0.1311	7.2714	7.3334	0.8516	15.5773	15.5977	0.1309
3.0	7.1650	7.2274	0.8716	15.4821	15.5030	0.1348	7.1682	7.2307	0.8710	15.4838	15.5046	0.1347
4.0	7.1152	7.1779	0.8812	15.4363	15.4574	0.1367	7.1171	7.1798	0.8809	15.4373	15.4584	0.1366
5.0	7.0854	7.1482	0.8871	15.4089	15.4301	0.1378	7.0866	7.1494	0.8868	15.4095	15.4307	0.1378

Numerical results for the N policy M/M(M, M)/1 and M/D(D, D)/1 queueing systems are shown in Table 1 for the above five cases. The relative error percentages are very small (0–6.8%).

5.2. Comparative analysis for the N policy $M/E_3(E_4, E_3)/1$ and $M/M(E_3, E_2)/1$ queueing systems

Here we perform a comparative analysis between the exact W_q and the approximate (maximum entropy) W_q^* for the N policy $M/E_3(E_4, E_3)/1$ and $M/M(E_3, E_2)/1$ queueing systems. For the N policy $M/E_3(E_4, E_3)/1$ queueing system, we have $\mu_S = 1/\mu$, $E[S^2] = 4/3\mu^2$, $\mu_R = 1/\beta$, $E[R^2] = 5/4/\beta^2$, $\mu_U = 1/\gamma$, and $E[U^2] = 4/3\gamma^2$. For the N policy $M/M(E_3, E_2)/1$ queueing system, we get $\mu_S = 1/\mu$, $E[S^2] = 2/\mu^2$, $\mu_R = 1/\beta$, $E[R^2] = 4/3\beta^2$, $\mu_U = 1/\gamma$, and $E[U^2] = 3/2\gamma^2$.

Numerical results for the N policy $M/E_3(E_4, E_3)/1$ and $M/M(E_3, E_2)/1$ queueing systems are shown in Table 2 for the above five cases. The relative error percentages are also very small (0–3.5%).

5.3. Comparative analysis for the N policy $M/E_3(E_4, D)/1$ and $M/E_3(E_4, M)/1$ queueing systems

Here we perform a comparative analysis between the exact W_q and the approximate (maximum entropy) W_q^* for the N policy $M/E_3(E_4, D)/1$ and $M/E_3(E_4, M)/1$ queueing systems. For the N policy $M/E_3(E_4, D)/1$ queueing system, we get $\mu_S = 1/\mu$, $E[S^2] = 4/3/\mu^2$, $\mu_R = 1/\beta$, $E[R^2] = 5/4\beta^2$, $\mu_U = 1/\gamma$, and $E[U^2] = 1/\gamma^2$. For the N policy $M/E_3(E_4, M)/1$ queueing system, we obtain $\mu_S = 1/\mu$, $E[S^2] = 4/3/\mu^2$, $\mu_R = 1/\beta$, $E[R^2] = 5/4\beta^2$, $\mu_U = 1/\gamma$ and $E[U^2] = 2/\gamma^2$.

Numerical results for the N policy $M/E_3(E_4, D)/1$ and $M/E_3(E_4, M)/1$ queueing systems are shown in Table 3 for the above five cases. Again, the relative error percentages are very small (0–3.5%).

6. Conclusion

We have utilized maximum entropy principle to develop the maximum entropy (approximate) solutions for the N policy M/G/1 queueing system with general service times, general repair times, and general startup times. We perform a comparative analysis between the approximate results obtained using maximum entropy principle and established exact results. We have demonstrated that the relative error percentages are very small (below 6.8%). The numerical results indicate that the use of maximum entropy principle is accurate enough for practical purposes and provides a helpful method for analyzing complex queueing systems.

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