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SYMPLECTIC INDUCTION AND SEMISIMPLE ORBITS

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Abstract

Symplectic induction was first introduced by Weinstein as the symplectic analogue of induced representations, and was further developed by Guillemin and Sternberg. This paper deals with the case where the symplectic manifold in question is a semisimple coadjoint orbit of a Lie group. In this case, the construction is generalized by adding a smooth mapping, in order to obtain various symplectic forms. In particular, when the orbit is elliptic, a study of the complex geometry shows that quantization commutes with induction.

1. *Introduction*

The idea of symplectic induction was introduced as the symplectic analogue of induced representations, where a symplectic manifold M induces another symplectic manifold $\mathrm{Ind}(M)$. It was first formalized by Weinstein [**12**], and was further developed by Guillemin and Sternberg [**3**]. More recent developments are summarized in [**2**]. In this paper, we consider the case where M is a semisimple coadjoint orbit of a Lie group G. We generalize the construction of $\mathrm{Ind}(M)$ by adding a smooth mapping ψ, and show that the various choices of ψ lead to different symplectic forms on $\mathrm{Ind}(M)$. When the coadjoint orbit M is elliptic, we study the complex geometries of M and $\mathrm{Ind}(M)$. In particular, if G has a compact Cartan subgroup, we quantize [**8**] M and $\mathrm{Ind}(M)$ to get the discrete series representations of G, and we show that quantization commutes with induction.

Let $\pi : E \longrightarrow M$ be a principal bundle with Lie group H acting along the fibre. We make use of the convention that the Lie algebra of a Lie group is denoted by the lower-case Gothic letter, so for instance the Lie algebra of H is $\mathfrak{h}$. Let θ be a connection form for the bundle. We use $\Omega^\bullet$ to denote differential forms, so $\theta \in \Omega^1(E, \mathfrak{h})$. Let $I : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ be the identity mapping. Extend π, θ, I naturally to $\pi : E \times \mathfrak{h}^* \longrightarrow M$, and $\theta \in \Omega^1(E \times \mathfrak{h}^*, \mathfrak{h})$, as well as $I : E \times \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$. Let ω be a symplectic form on M. Define the induced form [**4**, (40.1); **7**]

$$\mathrm{Ind}(\omega) = \pi^*\omega + d\langle I, \theta\rangle \in \Omega^2(E \times \mathfrak{h}^*).$$

It is certainly closed. It is known to be symplectic (that is, nondegenerate) on an open subset of $E \times \mathfrak{h}^*$, but not on the entire $E \times \mathfrak{h}^*$. The process $\omega \rightsquigarrow \mathrm{Ind}(\omega)$ is known as *symplectic induction*.

We can replace the identity mapping I with any smooth mapping $\psi : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$, and define the more general

$$\mathrm{Ind}_\psi(\omega) = \pi^*\omega + d\langle \psi, \theta\rangle. \tag{1.1}$$

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The advantage of $\mathrm{Ind}_\psi(\omega)$ over $\mathrm{Ind}(\omega)$ is that, for suitable choices of ψ, the induced form may be symplectic on the entire $E \times \mathfrak{h}^*$ and not just on an open set.

In this paper we carry out this idea on the semisimple coadjoint orbit $M = G/L$ and $E = G/L^{\mathrm{ss}}$. Here, L is the centralizer of a subgroup contained in the Cartan subgroup of G, $L^{\mathrm{ss}} = (L, L)$ is the commutator subgroup, and the centre of L acts on the fibre of $\pi : E \longrightarrow M$.

Let B be a Cartan subgroup of G, so that L is the centralizer of some subgroup H of B. Taking larger H if necessary, we may assume that H is the centre of L. Let $\Delta \subset \mathfrak{b}^*$ be the root system. By the Killing form, we can pair the elements of $\mathfrak{b}^*$. Let $\mathfrak{h}^*_{\mathrm{reg}}$ consist of all $\lambda \in \mathfrak{h}^*$ in which $(\alpha, \lambda) \neq 0$ whenever $(\alpha, \mathfrak{h}^*) \neq 0$, so $\mathfrak{h}^*_{\mathrm{reg}}$ is a union of open cones in $\mathfrak{h}^*$.

There certainly exist G-invariant symplectic forms on M, for instance the Kirillov–Kostant symplectic form. Since G is semisimple, these symplectic forms are Hamiltonian [**4**, Theorem 26.1], with moment map $\Phi : M \longrightarrow \mathfrak{g}^*$. Let $e \in M$ denote the identity coset. The moment map helps to classify the G-invariant symplectic forms on M.

THEOREM 1.1. *There is a one–one correspondence between $\mathfrak{h}^*_{\mathrm{reg}}$ and the G-invariant symplectic forms on M, given by $\Phi(e) \in \mathfrak{h}^*_{\mathrm{reg}}$.*

This result is an extension of [**1**, Theorem 5], which deals with compact B and elliptic coadjoint orbit M.

Let ω be a G-invariant symplectic form on M, with moment map Φ. Since H commutes with L^{ss}, it has a right action on $E = G/L^{\mathrm{ss}}$ and hence on $E \times \mathfrak{h}^*$. There exists a unique $G \times H$-invariant connection form θ (Proposition 3.3) on $E \longrightarrow M$. We use it to construct $\mathrm{Ind}_\psi(\omega)$ by (1.1), so $\mathrm{Ind}_\psi(\omega)$ is always $G \times H$-invariant. The next theorem gives the conditions for $\mathrm{Ind}_\psi(\omega)$ to be symplectic. Let $\mathrm{Im}(\psi) \subset \mathfrak{h}^*$ denote the image set of ψ.

THEOREM 1.2. *The 2-form $\mathrm{Ind}_\psi(\omega)$ is symplectic if and only if ψ is a local diffeomorphism and $\mathrm{Im}(\psi) + \Phi(e) \subset \mathfrak{h}^*_{\mathrm{reg}}$.*

Suppose from now on that $M = G/L$ is an elliptic coadjoint orbit. This assumption is in contrast to the case in [**9**], which deals with the hyperbolic coadjoint orbits. Since M is elliptic, there exists a G-invariant complex structure on M. We shall see that $E \times \mathfrak{h}^*$ also has a $G \times H$-invariant complex structure. We can study the conditions for $\mathrm{Ind}_\psi(\omega)$ to be pseudo-Kähler. By a *pseudo-Kähler form*, we mean a symplectic form which is preserved by the complex structure; that is, it satisfies all except the positivity of a Kähler form.

THEOREM 1.3. *The symplectic form $\mathrm{Ind}_\psi(\omega)$ is pseudo-Kähler if and only if ψ is a gradient function f'. In this case $\mathrm{Ind}_\psi(\omega) = 2\sqrt{-1}\partial\bar{\partial}F$ for $F(x) = f(x) + (\lambda, x)$. Conversely, every $G \times H$-invariant pseudo-Kähler form on $E \times \mathfrak{h}^*$ is given by $\mathrm{Ind}_\psi(\omega)$ for some ω and ψ.*

Here $f : \mathfrak{h}^* \longrightarrow \mathbb{R}$ and its gradient function is $f' : \mathfrak{h}^* \longrightarrow \mathfrak{h} \cong \mathfrak{h}^*$, where $\mathfrak{h} \cong \mathfrak{h}^*$ is given by the Killing form of $\mathfrak{g}$. Observe that if $\mathrm{Ind}_\psi(\omega)$ is pseudo-Kähler, then Theorem 1.2 says that $\psi = f'$ is a local diffeomorphism, or equivalently the

Hessian matrix f'' is nonsingular everywhere. If in particular f'' is positive definite everywhere, we say that f is *strictly convex*. We shall see that this is closely related to the condition for $\mathrm{Ind}_\psi(\omega)$ to be Kähler.

For $\mathrm{Ind}_\psi(\omega)$ to be Kähler, it is convenient that L be compact (see Proposition 5.1 and [**13**, §5.2]). Since $H \subset B \subset L$, this implies that H and B are compact. We assume from now on that G is a linear semisimple Lie group with compact Cartan subgroup B. Compactness of B implies that the roots Δ are divided into the compact roots Δ_{c} and noncompact roots Δ_{n}. Let Δ_{c}^+ and Δ_{n}^- denote the positive compact roots and negative noncompact roots, respectively. Let $\bar{\tau}$ be the positive roots that do not annihilate $\mathfrak{h}^*$. Then $\mathfrak{h}^*_{\mathrm{reg}}$ consists of $\lambda \in \mathfrak{h}^*$ in which $(\bar{\tau}, \lambda) \neq 0$. Define

$$\Sigma = \{\lambda \in \mathfrak{h}^*_{\mathrm{reg}} :\ ((\Delta_{\mathrm{c}}^+ \cup \Delta_{\mathrm{n}}^-) \cap \bar{\tau}, \lambda) > 0\}. \tag{1.2}$$

Observe that Σ is either an open cone in $\mathfrak{h}^*_{\mathrm{reg}}$ or an empty set. For instance, if $\Delta_{\mathrm{c}}^+ \cup \Delta_{\mathrm{n}}^-$ is another positive system, then Σ is not empty. Recall that Φ is the moment map of ω.

THEOREM 1.4. *Suppose that L is compact, and $\mathrm{Ind}_\psi(\omega)$ is pseudo-Kähler (so $\psi = f'$). Then $\mathrm{Ind}_\psi(\omega)$ is Kähler if and only if f is strictly convex and $\mathrm{Im}(\psi) + \Phi(e) \subset \Sigma$.*

The Kähler condition in Theorem 1.4 requires Σ to be nonempty, so if $\Sigma = \varnothing$, then $E \times \mathfrak{h}^*$ simply has no G-invariant Kähler structure. We suppose from now on that the conditions in Theorem 1.4 are satisfied, so that $E \times \mathfrak{h}^*$ has $G \times H$-invariant Kähler form. Also, by the compactness of the Cartan subgroup B, G has a nonempty discrete series [**6**]. We also assume that G is linear, so that we can utilize Schmid's construction [**10**] of the discrete series representation from the elliptic orbit M. As in [**1**], we use Harish-Chandra's notation $\Theta_{\nu+\rho}$ to denote the discrete series, where ν are the integral weights in $\mathfrak{h}^*$ and ρ is half the sum of positive roots.

Symplectic induction was originally motivated by being the classical analogue of induced representation. In representation theory, $\mathrm{Ind} = \mathrm{Ind}_H^G$ converts an H-representation to a G-representation. In (6.1), we also extend Ind to Ind_ψ on the representations. An important bridge between symplectic geometry and representation theory is supplied by geometric quantization [**8**]. We denote this by $\mathcal{H}$; in other words, $\mathcal{H}$ transforms a symplectic manifold to a representation. We shall explain Ind_ψ and $\mathcal{H}$ in more detail in Section 6. Recall that an integral symplectic form ω (or equivalently, a holomorphic hermitian line bundle) on M leads to a discrete series representation of G [**10**, **11**]. Further, $E \times \mathfrak{h}^*$ is a fibration over M. Therefore, we can expect a symplectic form $\mathrm{Ind}_\psi(\omega)$ on $E \times \mathfrak{h}^*$ to lead to several discrete series representations of G at once. We denote such a representation by $\mathcal{H} \cdot \mathrm{Ind}_\psi(\omega)$ in the next theorem. The next theorem shows that Ind_ψ on manifolds and on representations are analogous to each other. In other words, 'geometric quantization commutes with induction'.

THEOREM 1.5. *Suppose that ω and $\mathrm{Ind}_\psi(\omega)$ are G-invariant Kähler forms on M and $E \times \mathfrak{h}^*$ respectively. Then, as unitary G-representations, $\mathcal{H} \cdot \mathrm{Ind}_\psi(\omega) = \mathrm{Ind}_\psi \cdot \mathcal{H}(\omega)$. The discrete series $\Theta_{\nu+\rho}$ occurs in these representations if and only if $\nu \in \mathrm{Im}(\psi) + \lambda$. In that case it occurs with multiplicity one.*

As explained in Section 4, M has the complex structure of a domain in the complex flag manifold of the complex group G^{c}, so the above theorem is the complex analogue of [**9**, (4.6)], which quantizes the real flag manifolds and obtains the real parabolically induced representations in $\mathcal{H}$.

Let $\mathcal{H}$ denote these isomorphic representations. To demonstrate an application of Theorems 1.4 and 1.5, we obtain an example where $\Theta_{\nu+\rho}$ occurs in $\mathcal{H}$ for all $\nu \in \Sigma$.

This paper is organized as follows. In Section 2, we set up some notations on Lie algebras and prove Theorem 1.1. In Section 3, we construct $\mathrm{Ind}_\psi(\omega)$, study the conditions for it to be symplectic, and prove Theorem 1.2. In Section 4, we consider the case when M is an elliptic orbit, describe its complex structure, and prove Theorem 1.3 for $\mathrm{Ind}_\psi(\omega)$ to be pseudo-Kähler. In Section 5, we study the possibility for $\mathrm{Ind}_\psi(\omega)$ to be Kähler, and we prove Theorem 1.4. In Section 6, we define Ind_ψ on the representations and show that it commutes with $\mathcal{H}$ to prove Theorem 1.5. We also obtain the representation $\mathcal{H}$ as mentioned above.

2. *Lie algebras*

In this section we prove Theorem 1.1. We first review some basic facts and set up the notations on Lie algebras. A superscript **c** on Lie groups and Lie algebras denotes complexification, for example $\mathfrak{g}^{\mathbf{c}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$. Let $\mathfrak{g}^{\mathbf{c}} = \mathfrak{b}^{\mathbf{c}} + \sum_\Delta (\mathfrak{g}^{\mathbf{c}})_\alpha$ be the root space decomposition, with root system Δ. We may restrict the roots to $\mathfrak{b}$ and write $\Delta \subset \mathfrak{b}^*$. Let Δ^+ be a positive system. For each $\alpha \in \Delta^+$, let

$$\mathfrak{g}_\alpha = \mathfrak{g} \cap ((\mathfrak{g}^{\mathbf{c}})_\alpha + (\mathfrak{g}^{\mathbf{c}})_{-\alpha}).$$

Then $\mathfrak{g} = \mathfrak{b} + \sum_{\Delta^+} \mathfrak{g}_\alpha$. Each $\mathfrak{g}_\alpha$ is a real subspace of dimension 2.

Let $\Delta^s \subset \Delta^+$ be the simple roots. The subalgebra $\mathfrak{h} \subset \mathfrak{b}$ can be described by a subset τ of Δ^s, where $\mathfrak{h}$ lies in all the kernels of $\Delta^s \backslash \tau$. Equivalently, we define

$$\tau = \{\alpha \in \Delta^s : (\alpha, \mathfrak{h}) \neq 0\}, \quad \bar{\tau} = \{\alpha \in \Delta^+ : (\alpha, \mathfrak{h}) \neq 0\}. \tag{2.1}$$

The regular elements of $\mathfrak{h}^*$ are given by

$$\mathfrak{h}^*_{\mathrm{reg}} = \{\lambda \in \mathfrak{h}^* : (\alpha, \lambda) \neq 0 \text{ for all } \alpha \in \bar{\tau}\}. \tag{2.2}$$

Let $\mathfrak{l}$ be the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$, so $\mathfrak{l} = \mathfrak{b} + \sum_{\Delta^+ \backslash \bar{\tau}} \mathfrak{g}_\alpha$.

A positive root $\alpha \in \mathfrak{b}^*$ is identified with the coroot $V_\alpha \in \mathfrak{b}$ by the Killing form. Write

$$\{V_\alpha\}_{\Delta^s} \subset \mathfrak{b}, \quad \{v_\alpha\}_{\Delta^s} \subset \mathfrak{b}^*, \tag{2.3}$$

where $\{v_\alpha\}_{\Delta^s} \subset \mathfrak{b}^*$ is the dual basis of $\{V_\alpha\}_{\Delta^s} \subset \mathfrak{b}$. A basis of $\mathfrak{h}^*$ is given by $\{v_\alpha\}_\tau$. The Killing form of $\mathfrak{g}$ is nondegenerate, so it leads to inclusions of dual spaces of Lie algebras. For instance $\mathfrak{h}^* \subset \mathfrak{g}^*$, and so on. The distinct subspaces $\mathfrak{b}, \{\mathfrak{g}_\alpha\}_{\Delta^+}$ are mutually orthogonal with respect to the Killing form, so for example $\mathfrak{h}^*$ annihilates each $\mathfrak{g}_\alpha$.

For any closed subgroup $Z \subset G$, the G-invariant q-forms on G/Z can be identified with the subspace of $\bigwedge^q \mathfrak{g}^*$ defined by

$$\bigwedge^q (\mathfrak{g}, \mathfrak{z})^* = \left\{\alpha \in \bigwedge^q \mathfrak{g}^* : \mathrm{ad}^*_\xi \alpha = \iota(\xi)\alpha = 0 \text{ for all } \xi \in \mathfrak{z}\right\}.$$

Here $\mathrm{ad}^*_\xi : \bigwedge^q \mathfrak{g}^* \longrightarrow \bigwedge^q \mathfrak{g}^*$ is the natural extension of the coadjoint representation, while $\iota(\xi) : \bigwedge^q \mathfrak{g}^* \longrightarrow \bigwedge^{q-1} \mathfrak{g}^*$ is the interior product. Let $d : \bigwedge^q \mathfrak{g}^* \longrightarrow \bigwedge^{q+1} \mathfrak{g}^*$ be the exterior derivative.

PROPOSITION 2.1. *We have $d\mathfrak{h}^* \subset \bigwedge^2(\mathfrak{g},\mathfrak{l})^*$. In particular, if $\lambda \in \mathfrak{h}^*_{\mathrm{reg}}$, then $d\lambda$ is a symplectic form on G/L with moment map satisfying $\Phi(e) = \lambda$.*

Proof: Let $\lambda \in \mathfrak{h}^*$. We need to show that, given $\xi \in \mathfrak{l}$,

$$\text{(i) } \mathrm{ad}^*_\xi d\lambda = 0, \quad \text{(ii) } \iota(\xi) d\lambda = 0. \tag{2.4}$$

Since $\mathfrak{h}$ is the centre of $\mathfrak{l}$, $\mathrm{ad}^*_\xi \lambda = 0$. Let $\eta, \nu \in \mathfrak{g}$. By the Jacobi identity,

$$\begin{aligned}(\mathrm{ad}^*_\xi d\lambda)(\eta,\nu) &= (d\lambda)([\xi,\eta],\nu) + (d\lambda)(\eta,[\xi,\nu]) \\ &= \lambda([[\xi,\eta],\nu] + [\eta,[\xi,\nu]]) \\ &= (\lambda, [\xi,[\eta,\nu]]) \\ &= (\mathrm{ad}^*_\xi \lambda, [\eta,\nu]) = 0.\end{aligned}$$

This proves (2.4)(i). For all $\xi \in \mathfrak{l}$ and $\eta \in \mathfrak{g}$, we get $(\iota(\xi)d\lambda, \eta) = (d\lambda)(\xi,\eta) = (\mathrm{ad}^*_\xi \lambda, \eta) = 0$. This proves (2.4)(ii), and so $d\mathfrak{h}^* \subset \bigwedge^2(\mathfrak{g},\mathfrak{l})^*$.

It remains to show that each $\lambda \in \mathfrak{h}^*_{\mathrm{reg}}$ defines a symplectic form $d\lambda$ on $M = G/L$ with moment map $\Phi(e) = \lambda$, where $e \in G/L$ is the identity coset.

Let $\lambda \in \mathfrak{h}^*_{\mathrm{reg}}$, and we check that $d\lambda$ is nondegenerate. Note that $\mathfrak{g}/\mathfrak{l} \cong \sum_{\bar\tau} \mathfrak{g}_\alpha$. Let

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{h} \tag{2.5}$$

be the orthogonal projection induced by the Killing form. Given $\alpha \in \bar\tau$ and $\xi \in \mathfrak{g}_\alpha$, there exists $\eta \in \mathfrak{g}_\alpha$ such that $\rho[\xi,\eta] = V_\alpha$. Since λ is regular, $0 \neq (\lambda, V_\alpha) = (\lambda, [\xi,\eta]) = (d\lambda)(\xi,\eta)$, so $d\lambda$ is nondegenerate, and hence is symplectic. Let $\Phi : M \longrightarrow \mathfrak{g}^*$ be the moment map of $\omega = d\lambda$. Since $\mathfrak{g}$ is semisimple, up to linear combination, an element of $\mathfrak{g}$ can be written as $[\xi,\eta] \in \mathfrak{g}$, where $\xi,\eta \in \mathfrak{g}$. The G-action on M produces infinitesimal vector fields $\xi^\sharp, \eta^\sharp$ on M, and evaluation at the identity coset $e \in M$ gives $\xi^\sharp_e, \eta^\sharp_e \in T_eM \cong \mathfrak{g}/\mathfrak{l}$. They are the same as the images of ξ,η under the natural projection $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{l}$. Therefore, since we have identified $d\lambda \in \bigwedge^2(\mathfrak{g},\mathfrak{l})^*$ with the G-invariant 2-form ω on M, we get $\omega(\xi^\sharp,\eta^\sharp)_e = (d\lambda)(\xi,\eta)$. However, $\omega(\xi^\sharp,\eta^\sharp)_e = (\Phi(e),[\xi,\eta])$ and $(d\lambda)(\xi,\eta) = (\lambda,[\xi,\eta])$. We conclude that $\Phi(e) = \lambda$. This proves the proposition. □

Proof of Theorem 1.1. Part of the proof follows from Proposition 2.1. To complete the proof, let ω be a G-invariant symplectic form on M. We want to show that its moment map satisfies $\Phi(e) \in \mathfrak{h}^*_{\mathrm{reg}}$. Since Φ is G-equivariant, for all $x \in L$, $\mathrm{Ad}^*_x \Phi(e) = \Phi(xe) = \Phi(e)$. Therefore, $\Phi(e) \in \mathfrak{h}^*$.

Let ρ be the projection (2.5). Since $\Phi(e) \in \mathfrak{h}^*$,

$$\omega(\xi,\eta) = (\Phi(e),[\xi,\eta]) = (\Phi(e),\rho[\xi,\eta]) \tag{2.6}$$

for all $\xi,\eta \in \mathfrak{g}$.

We want to show that $\Phi(e)$ is regular. Suppose otherwise, namely that $(\Phi(e),\alpha) = 0$ for some root α which does not annihilate $\mathfrak{h}$. Then (2.6) says that for all $\xi \in \mathfrak{g}_\alpha$ and $\eta \in \mathfrak{g}$, $\omega(\xi,\eta) = 0$ because $\rho[\mathfrak{g}_\alpha,\mathfrak{g}] \subset \mathbb{R}(V_\alpha)$. This implies that $\iota(\xi)\omega = 0$, which contradicts the fact that ω is nondegenerate. We conclude that $\Phi(e)$ is regular.

Finally, suppose that the moment maps of ω_1,ω_2 satisfy $\Phi_1(e) = \Phi_2(e)$. Setting ξ,η in (2.6) to be a basis of $\mathfrak{g}_\alpha$, we get $\omega_1|_{\mathfrak{g}_\alpha} = \omega_2|_{\mathfrak{g}_\alpha}$. If $\alpha \neq \beta$, then $\rho[\mathfrak{g}_\alpha,\mathfrak{g}_\beta] = 0$, and (2.6) says that $\omega_i(\mathfrak{g}_\alpha,\mathfrak{g}_\beta) = 0$. We conclude that $\omega_1 = \omega_2$. This completes the proof of the theorem. □

3. *Symplectic induction*

In this section, we prove Theorem 1.2. The idea is to express ω, θ and ψ in terms of the v_α and V_α of (2.3), so that we can compute $\mathrm{Ind}_\psi(\omega)$. We start with the following property of v_α. Recall from (2.1) that τ and $\bar\tau$ are respectively the simple and positive roots that do not annihilate $\mathfrak{h}$. Also, $v_\alpha \in \mathfrak{h}^*$ for all $\alpha \in \tau$.

PROPOSITION 3.1. *If $\alpha \in \tau$, then $v_\alpha \in \bigwedge^1(\mathfrak{g}, \mathfrak{l}^{ss})^*$.*

Proof: Let $\alpha \in \tau$, and consider $v_\alpha \in \mathfrak{h}^*$. We need to show that for all $\xi \in \mathfrak{l}^{ss}$,

$$\text{(i) } \mathrm{ad}_\xi^* v_\alpha = 0, \quad \text{(ii) } (v_\alpha, \xi) = 0. \tag{3.1}$$

Write

$$\xi = \xi_0 + \sum_{\Delta^+\setminus\bar\tau} \xi_\beta \in \mathfrak{h}^\perp + \sum_{\Delta^+\setminus\bar\tau} \mathfrak{g}_\beta = \mathfrak{l}^{ss}, \quad \eta = \eta_0 + \sum_{\Delta^+} \eta_\beta \in \mathfrak{h} + \sum_{\Delta^+} \mathfrak{g}_\beta = \mathfrak{g}.$$

The projection ρ of (2.5) annihilates all the $\mathfrak{g}_\alpha$, so $\rho[\mathfrak{h}, \mathfrak{g}_\alpha] = 0$, and $\rho[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ for distinct α, β. Then

$$\begin{aligned}(v_\alpha, [\xi, \eta]) &= (v_\alpha, \rho[\xi, \eta]) \qquad\qquad \text{since } (v_\alpha, \mathfrak{g}_\beta) = 0\\ &= \big(v_\alpha, \rho \textstyle\sum_{\Delta^+\setminus\bar\tau} [\xi_\beta, \eta_\beta]\big)\\ &= \big(v_\alpha, \textstyle\sum_{\Delta^+\setminus\bar\tau} c_\beta V_\beta\big)\end{aligned} \tag{3.2}$$

for some coefficients c_β. Since α differs from all the $\beta \in \Delta^+\setminus\bar\tau$ in (3.2), we get $(v_\alpha, V_\beta) = 0$ and (3.2) vanishes. This proves that $\mathrm{ad}_\xi^* v_\alpha = 0$, and (3.1)(i) follows.

We next prove (3.1)(ii). Since $\mathfrak{l}^{ss}$ is semisimple, up to linear combination, we can write $\xi = [\eta, \nu] \in \mathfrak{l}^{ss}$ for some $\eta, \nu \in \mathfrak{l}^{ss}$. By (3.1)(i), $\mathrm{ad}_\eta^* v_\alpha = 0$, and so $(v_\alpha, \xi) = (v_\alpha, [\eta, \nu]) = (\mathrm{ad}_\eta^* v_\alpha, \nu) = 0$. This proves (3.1)(ii). We have shown that if $\alpha \in \tau$, then $v_\alpha \in \bigwedge^1(\mathfrak{g}, \mathfrak{l}^{ss})^*$. The proposition follows. □

Fix a G-invariant symplectic form ω on M. By Theorem 1.1, it is determined by its moment map via $\Phi(e) = \lambda \in \mathfrak{h}^*_{\mathrm{reg}}$. We next give a formula for ω.

PROPOSITION 3.2. *The symplectic form is given by $\omega = d \sum_\tau (\lambda, V_\alpha) v_\alpha$.*

Proof. Here, $\omega \in \bigwedge^2(\mathfrak{g}, \mathfrak{l})^*$. By Proposition 2.1, $d\sum_\tau(\lambda, V_\alpha)v_\alpha \in \bigwedge^2(\mathfrak{g}, \mathfrak{l})^*$, so it makes sense to compare it with ω. By Theorem 1.1, $\omega = d\lambda$. For all $\xi, \eta \in \mathfrak{g}$,

$$\begin{aligned}\omega(\xi, \eta) &= (\lambda, [\xi, \eta])\\ &= \big(\textstyle\sum_\tau (\lambda, V_\alpha) v_\alpha, [\xi, \eta]\big)\\ &= \big(d \textstyle\sum_\tau (\lambda, V_\alpha) v_\alpha\big)(\xi, \eta).\end{aligned}$$

This completes the proof. □

The fibration $\pi : E \longrightarrow M$ leads to an injective map $\pi^* : \Omega^2(M) \hookrightarrow \Omega^2(E)$. If we restrict it to the G-invariant forms, then π^* is simply the inclusion $\bigwedge^2(\mathfrak{g}, \mathfrak{l})^* \subset \bigwedge^2(\mathfrak{g}, \mathfrak{l}^{ss})^*$. By Proposition 3.1, $v_\alpha \in \bigwedge^1(\mathfrak{g}, \mathfrak{l}^{ss})^*$, so Proposition 3.2 also says that $\pi^*\omega$ is exact. By extending π to $E \times \mathfrak{h}^*$, we see that $\mathrm{Ind}_\psi(\omega)$ is exact too.

Let $\alpha \in \tau$. In the next proposition, we use $v_\alpha \in \bigwedge^1(\mathfrak{g}, \mathfrak{l}^{ss})^* \subset \Omega^1(E)$ to describe the connection form θ of the H-bundle $E \longrightarrow M$.

Proposition 3.3. *There exists a unique $G \times H$-invariant connection for the principal bundle $E \longrightarrow M$, given by the connection form $\theta = \sum_\tau v_\alpha \otimes V_\alpha \in \mathfrak{h}^* \otimes \mathfrak{h} \subset \Omega^1(E, \mathfrak{h})$.*

Proof. Let θ be a $G \times H$-invariant connection form for the bundle. By definition, θ is a 1-form on E with values in $\mathfrak{h}$. Note that $E = G/L^{\mathrm{ss}}$, and the only component in $(\mathfrak{g}/\mathfrak{l}^{\mathrm{ss}})^* = \mathfrak{h}^* + \sum_{\bar\tau} \mathfrak{g}_\alpha^*$ which is invariant under the coadjoint representation $\mathrm{ad}_{\mathfrak{h}}^*$ is $\mathfrak{h}^*$. Thus $\mathfrak{h}^*$ is the only component in $\Omega^1(E)$ which is $G \times H$-invariant. Therefore,

$$\theta \in \mathfrak{h}^* \otimes \mathfrak{h}. \tag{3.3}$$

Since H commutes with L^{ss}, there is a right H-action on $E = G/L^{\mathrm{ss}}$, so each $\xi \in \mathfrak{h}$ leads to an infinitesimal vector field ξ^r on E. Given $p \in E$, let $E_p \subset E$ denote the fibre in the bundle $E \longrightarrow M$ which contains p. The tangent space of $p \in E_p$ is a vertical subspace $T_p E_p$ in the bundle. Each p determines an isomorphism $J_p : \mathfrak{h} \longrightarrow T_p E_p$ by $J_p(\xi) = \xi_p^r$. The definition of the connection form requires $\theta_p J_p(\xi) = \xi$. This becomes $\theta_p(\xi_p^r) = \xi$. Therefore, the connection form (3.3) has to have the expression $\theta = \sum_\tau v_\alpha \otimes V_\alpha$. This proves the proposition. □

To prove Theorem 1.2, we set up some notations for the differential forms on $E \times \mathfrak{h}^*$. Let $\alpha \in \tau$. By Proposition 3.1, write

$$v_\alpha \in \mathfrak{h}^* \subset \bigwedge^1 (\mathfrak{g}, \mathfrak{l}^{\mathrm{ss}})^* \subset \Omega^1(E) \subset \Omega^1(E \times \mathfrak{h}^*). \tag{3.4}$$

By Proposition 2.1,

$$dv_\alpha \in \bigwedge^2 (\mathfrak{g}, \mathfrak{l})^* \subset \bigwedge^2 (\mathfrak{g}, \mathfrak{l}^{\mathrm{ss}})^* \subset \Omega^2(E) \subset \Omega^2(E \times \mathfrak{h}^*). \tag{3.5}$$

Let $\psi : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ be a smooth mapping, and let $\psi_\alpha = (\psi, V_\alpha) \in C^\infty(\mathfrak{h}^*)$ for all $\alpha \in \tau$. Extending ψ_α to $C^\infty(E \times \mathfrak{h}^*)$, we have

$$\begin{gathered} \psi_\alpha = (\psi, V_\alpha) \in C^\infty(\mathfrak{h}^*) \subset C^\infty(E \times \mathfrak{h}^*), \\ d\psi_\alpha \in \Omega^1(\mathfrak{h}^*) \subset \Omega^1(E \times \mathfrak{h}^*). \end{gathered} \tag{3.6}$$

Note that $\psi = \sum_\tau \psi_\alpha v_\alpha$. Extend θ to $\Omega^1(E \times \mathfrak{h}^*, \mathfrak{h})$. Then by Proposition 3.3,

$$\langle \psi, \theta \rangle = \sum_\tau \psi_\alpha v_\alpha. \tag{3.7}$$

Proof of Theorem 1.2. It is clear that $\mathrm{Ind}_\psi(\omega)$ is closed, and in fact Proposition 3.2 shows that it is exact, so the main issue is whether it is nondegenerate. Here

$$\begin{aligned} \mathrm{Ind}_\psi(\omega) &= \pi^*\omega + d\langle \psi, \theta \rangle \\ &= d \textstyle\sum_\tau (\lambda, V_\alpha) v_\alpha + d \sum_\tau \psi_\alpha v_\alpha && \text{by Prop. 3.2, (3.7)} \\ &= \textstyle\sum_\tau (\psi_\alpha + (\lambda, V_\alpha)) dv_\alpha + \sum_\tau d\psi_\alpha \wedge v_\alpha \\ &\in C^\infty(\mathfrak{h}^*) \otimes \textstyle\bigwedge^2 (\mathfrak{g}, \mathfrak{l}^{\mathrm{ss}})^* + C^\infty(\mathfrak{h}^*) \otimes \bigwedge^2 (\mathfrak{h} \oplus \mathfrak{h}^*)^* && \text{by (3.4), (3.5), (3.6)} \\ &\subset \Omega^2(E \times \mathfrak{h}^*). \end{aligned} \tag{3.8}$$

Since $\mathrm{Ind}_\psi(\omega)$ is G-invariant, it suffices to check the nondegenerate condition on $x \in e \times \mathfrak{h}^* \subset E \times \mathfrak{h}^*$, where e is the identity coset. The tangent space at x is

$$T_x(E \times \mathfrak{h}^*) \cong \mathfrak{g}/\mathfrak{l}^{\mathrm{ss}} \cong \sum_{\bar\tau} \mathfrak{g}_\alpha + (\mathfrak{h} \oplus \mathfrak{h}^*). \tag{3.9}$$

By (3.8) and (3.9), $\mathrm{Ind}_\psi(\omega)$ is nondegenerate at $x \in e \times \mathfrak{h}^*$ if and only if the expressions

$$\sum_\tau (\psi_\alpha(x) + (\lambda, V_\alpha)) dv_\alpha \qquad \text{and} \qquad \sum_\tau d\psi_\alpha(x) \wedge v_\alpha$$

are nondegenerate when restricted to $\sum_{\bar\tau} \mathfrak{g}_\alpha$ and $\mathfrak{h} \oplus \mathfrak{h}^*$ respectively. By Proposition 2.1, $\sum_\tau (\psi_\alpha(x) + (\lambda, V_\alpha)) dv_\alpha$ is nondegenerate if and only if $\sum_\tau (\psi_\alpha(x) + (\lambda, V_\alpha)) v_\alpha \in \mathfrak{h}^*_{\mathrm{reg}}$, or equivalently $\psi(x) + \lambda \in \mathfrak{h}^*_{\mathrm{reg}}$. The second expression, $\sum_\tau d\psi_\alpha(x) \wedge v_\alpha$, is nondegenerate if and only if $\{d\psi_\alpha(x)\}_\tau$ are linearly independent, which is equivalent to $\psi = \sum_\tau \psi_\alpha v_\alpha$ being a local diffeomorphism at x. This proves Theorem 1.2. □

4. *Elliptic orbits*

Suppose from now on that M is an elliptic coadjoint orbit (compare this with [**3**, Theorem 5.7], which treats hyperbolic coadjoint orbits). It is then well known that M has a G-invariant complex structure. In this section, we show that this induces a $G \times H$-invariant complex structure on $E \times \mathfrak{h}^*$ as a holomorphic fibration over M. We also consider the condition for existence of pseudo-Kähler structure on $E \times \mathfrak{h}^*$, and prove Theorem 1.3.

To say that $M = G/L$ is an elliptic orbit means that L is the centralizer of some torus. We may assume that H is a torus and L is the centralizer of H. A positive system $\Delta^+ \subset \mathfrak{b}^*$ leads to a complex parabolic subalgebra $\mathfrak{p} = \mathfrak{l}^{\mathbf{c}} + \mathfrak{u}^{\mathbf{c}}$, where $\mathfrak{u}^{\mathbf{c}}$ are the root spaces of $\bar\tau$ of (2.1). The natural mappings $G \hookrightarrow G^{\mathbf{c}} \longrightarrow G^{\mathbf{c}}/P$ then lead to the imbedding $G/L \hookrightarrow G^{\mathbf{c}}/P$ as an open domain. In this way M is complex. It is known as the flag domain, and in particular $M = G^{\mathbf{c}}/P$ is the flag manifold if G is compact.

Let (P, P) be the commutator subgroup, and consider the natural mapping $\pi : G^{\mathbf{c}}/(P,P) \longrightarrow G^{\mathbf{c}}/P$. Then

$$\pi^{-1}(M) = G/L^{\mathrm{ss}} \times \exp(\sqrt{-1}\mathfrak{h}) \tag{4.1}$$

is a $G \times H$-invariant complex submanifold of $G^{\mathbf{c}}/(P,P)$. By the covering $\mathfrak{h}^* \cong \sqrt{-1}\mathfrak{h} \longrightarrow \exp(\sqrt{-1}\mathfrak{h})$, it follows from (4.1) that $E \times \mathfrak{h}^*$ is a covering of $\pi^{-1}(M)$. This equips $E \times \mathfrak{h}^*$ with a $G \times H$-invariant complex structure.

We shall consider when $\mathrm{Ind}_\psi(\omega)$ is pseudo-Kähler under this complex structure on $E \times \mathfrak{h}^*$. Some arguments in the next two propositions will rely on [**1**], which deals with the special case where the Cartan subgroup B is compact. In that case the covering $\sqrt{-1}\mathfrak{h} \longrightarrow \exp(\sqrt{-1}\mathfrak{h})$ is just a diffeomorphism. However, many of its results are still valid in the present situation.

PROPOSITION 4.1. *Every $G \times H$-invariant real closed* (1,1)*-form on $E \times \mathfrak{h}^*$ has the expression $\sqrt{-1}\partial\bar\partial F$.*

Proof. The arguments resemble [**1**, Propositions 3.2, 3.3 and 3.4]. We consider the deRham cohomology $H^2_G(E \times \mathfrak{h}^*)$ and the Dolbeault cohomology $H^{0,1}_{GH}(E \times \mathfrak{h}^*)$, with subscripts G and GH indicating G- and $G \times H$-invariance respectively. Here

$$H^2_G(E \times \mathfrak{h}^*) = 0 \tag{4.2}$$

follows from the arguments of [**1**, Proposition 3.2].

The arguments for

$$H^{0,1}_{GH}(E \times \mathfrak{h}^*) = 0 \tag{4.3}$$

follow almost similarly as in [**1**, Proposition 3.3], but since H is not necessarily compact here, we shall make sure that the H-invariant subcomplex of the Dolbeault complex of the Stein manifold $H \times \mathfrak{h}^*$ is trivial at degree (0,1). Let δ be a closed $G \times H$-invariant (0,1)-form on $E \times \mathfrak{h}^*$. As pointed out in [**1**, Proposition 3.3], the component $\bigwedge^{0,1} \sum_{\bar{\tau}} \mathfrak{g}_\alpha^*$ is not right H-invariant, so we get

$$\delta \in C^\infty(\mathfrak{h}^*) \otimes \bigwedge^{0,1}(\mathfrak{h} \oplus \mathfrak{h}^*)^* = \Omega_{GH}^{0,1}(E \times \mathfrak{h}^*).$$

In this way it is an H-invariant (0,1)-form on $H \times \mathfrak{h}^*$. Let x_α be the linear coordinates on $\mathfrak{h}^*$ with respect to the basis $\{v_\alpha\}_\tau$. Together with the invariant coordinates y_α on H, we get holomorphic coordinates $z_\alpha = x_\alpha + \sqrt{-1}y_\alpha$ on $H \times \mathfrak{h}^*$. Write $\delta = \sum_\alpha f_\alpha d\bar{z}_\alpha$. Here $f_\alpha = f_\alpha(x)$ because δ is H-invariant. Since

$$0 = \bar{\partial}\delta = \sum_{\alpha\beta} \frac{\partial f_\alpha}{\partial \bar{z}_\beta} d\bar{z}_\beta \wedge d\bar{z}_\alpha = \frac{1}{2} \sum_{\alpha\beta} \frac{\partial f_\alpha}{\partial x_\beta} d\bar{z}_\beta \wedge d\bar{z}_\alpha,$$

we get $\partial f_\alpha / \partial x_\beta = \partial f_\beta / \partial x_\alpha$. Each $f_\alpha(x)$ is just a function on $\mathfrak{h}^*$, so there exists $h \in C^\infty(\mathfrak{h}^*)$ such that $\partial h / \partial x_\alpha = f_\alpha$. One checks that $\bar{\partial}(2h) = \alpha$. This proves (4.3).

Let σ be a $G \times H$-invariant real closed (1,1)-form on $E \times \mathfrak{h}^*$. We apply (4.2) and (4.3), and we follow the arguments of [**1**, Proposition 3.4] to get $\sigma = \sqrt{-1}\partial\bar{\partial}F$. This proves the proposition. □

Just like the above proposition, the next proposition can be proved by many ideas in [**1**]. Whenever appropriate, we shall apply them without giving separate arguments.

PROPOSITION 4.2. *The 2-form* $\mathrm{Ind}_\psi(\omega)$ *is of type* (1,1) *if and only if* ψ *is a gradient function* f'. *In this case* $\mathrm{Ind}_\psi(\omega) = 2\sqrt{-1}\partial\bar{\partial}F$ *for* $F(x) = f(x) + (\lambda, x)$.

Proof. Suppose that ψ is a gradient function. Write $\psi = f'$. Let x_α be the linear coordinates on $\mathfrak{h}^*$ with respect to the basis $\{v_\alpha\}_\tau \subset \mathfrak{h}^*$. Define $F \in C^\infty(\mathfrak{h}^*)$ by $F(x) = f(x) + \lambda \cdot x = f(x) + \sum_\tau (\lambda, V_\alpha)x_\alpha$. Then $F'(x) = \psi(x) + \lambda$; or equivalently

$$\frac{\partial F}{\partial x_\alpha} = \psi_\alpha(x) + (\lambda, V_\alpha), \tag{4.4}$$

where $\psi_\alpha = (\psi, V_\alpha)$. Extend F to $C^\infty(E \times \mathfrak{h}^*)$ by G-invariance. We claim that $\mathrm{Ind}_\psi(\omega) = 2\sqrt{-1}\partial\bar{\partial}F$. By G-invariance, it suffices to check that

$$(\mathrm{Ind}_\psi(\omega))_x = (2\sqrt{-1}\partial\bar{\partial}F)_x \tag{4.5}$$

for $x \in e \times \mathfrak{h}^* \subset E \times \mathfrak{h}^*$.

By (3.9), $\{\mathfrak{g}_\alpha\}_{\bar{\tau}}$ and $\mathfrak{h} \oplus \mathfrak{h}^*$ are subspaces of the tangent space $T_x(E \times \mathfrak{h}^*)$. In fact, they are complex subspaces. By (3.8), these complex subspaces are mutually orthocomplementary with respect to $\mathrm{Ind}_\psi(\omega)$. Similarly, by [**1**, Proposition 3.5], they are mutually orthocomplementary with respect to $2\sqrt{-1}\partial\bar{\partial}F$. Therefore, to prove (4.5), it suffices to show that they agree on the complex subspaces $\mathfrak{g}_\alpha$ and $\mathfrak{h} \oplus \mathfrak{h}^*$; in other words, that

$$\begin{aligned} &\text{(i)} \quad (\mathrm{Ind}_\psi(\omega))_x \,|\, \mathfrak{g}_\alpha = (2\sqrt{-1}\partial\bar{\partial}F)_x \,|\, \mathfrak{g}_\alpha; \\ &\text{(ii)} \quad (\mathrm{Ind}_\psi(\omega))_x \,|\, (\mathfrak{h} + \mathfrak{h}^*) = (2\sqrt{-1}\partial\bar{\partial}F)_x \,|\, (\mathfrak{h} \oplus \mathfrak{h}^*). \end{aligned} \tag{4.6}$$

We first prove (4.6)(i). By (3.8),

$$(\mathrm{Ind}_\psi(\omega))_x \,|\, \mathfrak{g}_\alpha = (\psi_\alpha(x) + (\lambda, V_\alpha))dv_\alpha.$$

By (4.4), this simplifies to $(\partial F/\partial x_\alpha)dv_\alpha$. Then [**1**, (3.21)] says that

$$\frac{\partial F}{\partial x_\alpha}dv_\alpha = (2\sqrt{-1}\partial\bar{\partial}F)_x \,|\, \mathfrak{g}_\alpha.$$

This proves (4.6)(i).

Let y_α be the left invariant coordinates on H such that, together with the coordinates x_α on $\mathfrak{h}^*$, we have the holomorphic coordinates $\sqrt{-1}y_\alpha + x_\alpha$ on the complex submanifold $H \times \mathfrak{h}^* \subset E \times \mathfrak{h}^*$. Note that $dy_\alpha = v_\alpha$. We now prove (4.6)(ii). By (3.8),

$$(\mathrm{Ind}_\psi(\omega))_x \,|\, (\mathfrak{h} \oplus \mathfrak{h}^*) = \sum_\tau d\psi_\alpha \wedge v_\alpha.$$

By (4.4), this simplifies to $\sum_\tau(\partial^2 F/\partial x_\alpha \partial x_\beta)dx_\beta \wedge dy_\alpha$. Then [**1**, (3.20)] says that

$$\sum_\tau \frac{\partial^2 F}{\partial x_\alpha \partial x_\beta}dx_\beta \wedge dy_\alpha = (2\sqrt{-1}\partial\bar{\partial}F)_x \,|\, (\mathfrak{h} + \mathfrak{h}^*).$$

This proves (4.6)(ii).

We have proved (4.6). This leads to (4.5), and so $\mathrm{Ind}_\psi(\omega) = 2\sqrt{-1}\partial\bar{\partial}F$.

Conversely, suppose that $\mathrm{Ind}_\psi(\omega)$ is of type (1,1). By Proposition 4.1, there exists $F \in C^\infty(\mathfrak{h}^*)$ such that $\mathrm{Ind}_\psi(\omega) = 2\sqrt{-1}\partial\bar{\partial}F$. In particular, they satisfy (4.6)(i). By a computation similar to the arguments for (4.6)(i), we get $\psi(x) + (\lambda, x) = F'(x)$, or equivalently ψ is the gradient function of $F(x) - (\lambda, x)$. This proves the proposition. □

Proof of Theorem 1.3. Since a pseudo-Kähler form is a symplectic form of type (1,1), the first part of the theorem follows from Proposition 4.2. It remains to show that every $G \times H$-invariant pseudo-Kähler form on $E \times \mathfrak{h}^*$ can be constructed via symplectic induction. By Proposition 4.1, such a form has the expression $2\sqrt{-1}\partial\bar{\partial}F$, where $F \in C^\infty(\mathfrak{h}^*)$. By [**1**, Theorem 1], $\mathrm{Im}(F') \subset \mathfrak{h}^*_{\mathrm{reg}}$. Pick any $\lambda \in \mathfrak{h}^*_{\mathrm{reg}}$. By Theorem 1.1, it determines a G-invariant symplectic form ω on M with moment map $\Phi(e) = \lambda$. Define $\psi : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ by $\psi(x) = F'(x) - \lambda$. Then $\mathrm{Ind}_\psi(\omega) = 2\sqrt{-1}\partial\bar{\partial}F$. Theorem 1.3 follows. □

5. *Kähler induction*

Finally, we give the condition for $\mathrm{Ind}_\psi(\omega)$ to be Kähler, and we prove Theorem 1.4. We start with the following proposition on the existence of Kähler structures.

PROPOSITION 5.1. *If* $\mathrm{Ind}_\psi(\omega)$ *is Kähler, then* M *has* G*-invariant Kähler structures.*

Proof. Suppose that $\mathrm{Ind}_\psi(\omega)$ is a Kähler form on $E \times \mathfrak{h}^*$. Recall from (3.8) that

$$\mathrm{Ind}_\psi(\omega) = \sum_\tau(\psi_\alpha + (\lambda, V_\alpha))dv_\alpha + \sum_\tau d\psi_\alpha \wedge v_\alpha.$$

For each $x \in e \times \mathfrak{h}^* \subset E \times \mathfrak{h}^*$, $\sum_\tau (\psi_\alpha(x) + (\lambda, V_\alpha)) dv_\alpha$ is positive on $\sum_{\bar{\tau}} \mathfrak{g}_\alpha$. By Proposition 2.1, $dv_\alpha \in \bigwedge^2(\mathfrak{g}, \mathfrak{l})^*$. Therefore, $\sum_\tau (\psi_\alpha(x) + (\lambda, V_\alpha)) dv_\alpha$ defines a G-invariant Kähler form on M. □

This proposition says that an obstruction to $\mathrm{Ind}_\psi(\omega)$ being Kähler is the absence of Kähler structures on M, for instance if L is not compact [**13**, §5.2]. For convenience, we assume from now on that L is compact. Since $H \subset B \subset L$, this implies that H and B are compact too. Then the roots Δ take on imaginary values on $\mathfrak{b}$. Recall that the complex structure comes from a choice of positive system Δ^+. We can choose Δ^+ to be stable under a Cartan involution, whose Cartan decomposition is written as $\mathfrak{g} = \mathfrak{k} + \mathfrak{q}$. For each $\alpha \in \Delta^+$, we say that $\pm\alpha$ are compact or noncompact depending on whether $\mathfrak{g}_\alpha \subset \mathfrak{k}$ or $\mathfrak{g}_\alpha \subset \mathfrak{q}$. We use the notation Δ_c and Δ_n accordingly. Let Δ_c^+ and Δ_n^- denote the positive compact roots and negative noncompact roots respectively.

Recall that $\bar{\tau}$ is defined in (2.1), $\mathfrak{h}^*_{\mathrm{reg}}$ is defined in (2.2), and the open cone $\Sigma \subset \mathfrak{h}^*_{\mathrm{reg}}$ is defined in (1.2).

Proof of Theorem 1.4. Suppose that $\mathrm{Ind}_\psi(\omega)$ is pseudo-Kähler. By Theorem 1.3, $\mathrm{Ind}_\psi(\omega) = 2\sqrt{-1}\partial\bar{\partial}F$, where $\psi = f'$ and $F(x) = f(x) + (\lambda, x)$. Since F and f have the same Hessian matrices, clearly F is strictly convex if and only if f is. Also, $F'(x) = f'(x) + \lambda$. By [**1**, Theorem 1], $2\sqrt{-1}\partial\bar{\partial}F$ is Kähler if and only if F is strictly convex and $\mathrm{Im}(F') \in \Sigma$. This is equivalent to f being strictly convex and $\mathrm{Im}(\psi) + \lambda \in \Sigma$. The theorem follows. □

6. *Geometric quantization*

In this section, we show an analogy between induction in symplectic geometry and in representation theory. We prove that 'quantization commutes with induction' in Theorem 1.5. This principle is shown for the real flag manifolds in [**9**, (4.6)], which construct the real parabolically induced representation, but since we deal with the complex flag domains (that is, elliptic orbits) here, the corresponding representations are the discrete series representations. We shall apply Schmid's construction [**10**] of the discrete series representations, so that Theorems 1.4 and 1.5 lead to a unitary G-representation where all the discrete series $\Theta_{\nu+\rho}$ with $\nu \in \Sigma$ occur in $\mathcal{H}$, and Σ is the open cone in (1.2).

We assume that G is a linear semisimple Lie group with compact Cartan subgroup B, so that it has nonempty discrete series [**6**]. We also assume that the conditions in Theorem 1.4 are satisfied, so that $E \times \mathfrak{h}^*$ has $G \times H$-invariant Kähler forms. By Proposition 5.1, M has G-invariant Kähler form ω. Given a G-invariant Kähler form σ on $E \times \mathfrak{h}^*$, we use the language of geometric quantization [**8**] to express a unitary G-representation $\mathcal{H}(\sigma)$ [**1**, (1.8)]. We say that σ leads to a holomorphic hermitian line bundle over $E \times \mathfrak{h}^*$, and $\mathcal{H}(\sigma)$ consists of its holomorphic sections which are square-integrable with respect to the $G \times \mathfrak{h}^*$-invariant measure on $E \times \mathfrak{h}^*$. Then $\mathcal{H}(\sigma)$ is a unitary G-representation. We shall of course be concerned with $\mathcal{H}(\mathrm{Ind}_\psi(\omega))$.

Recall that Theorem 1.1 gives a bijection

$$\mathcal{H} : \{G\text{-invariant symplectic forms on } M\} \longrightarrow \mathfrak{h}^*_{\mathrm{reg}},$$

where $\mathcal{H}(\omega) = \Phi(e)$ and Φ is the moment map of ω.

If $\nu \in \mathfrak{h}^*$ is integral, then it is the differential of a multiplicative homomorphism $\chi : H \longrightarrow S^1$. We shall always identify χ with its differential ν. The H-representation χ induces a G-representation $\mathrm{Ind}_H^G(\nu) = \mathrm{Ind}(\nu)$. One way to realize $\mathrm{Ind}(\nu)$ is to consider the following homogeneous line bundle. Extend χ holomorphically to $\chi : H^{\mathbf{c}} \longrightarrow \mathbb{C}^\times$. Since $H^{\mathbf{c}}$ is the centre of $L^{\mathbf{c}}$, χ extends to the complex parabolic subgroup $P = L^{\mathbf{c}}U^{\mathbf{c}}$ given in Section 4, where it acts trivially on $(L^{\mathbf{c}}, L^{\mathbf{c}})$ and $U^{\mathbf{c}}$. Let $(G^{\mathbf{c}} \times \mathbb{C})/\nu \longrightarrow G^{\mathbf{c}}/P$ be the homogeneous line bundle, where $[gp, z] = [g, \chi(p)z]$ for all $g \in G^{\mathbf{c}}$, $p \in P$ and $z \in \mathbb{C}$. Restrict it to $M \subset G^{\mathbf{c}}/P$; then $\mathrm{Ind}(\lambda)$ consists of its holomorphic sections over M. It either vanishes or is an irreducible G-representation in the discrete series [**5**]. From $\psi : \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ and $\lambda \in \mathfrak{h}^*$, let

$$\mathrm{Ind}_\psi(\lambda) = \sum_{\nu \in \mathbf{Z}(\mathrm{Im}(\psi)+\lambda)} \mathrm{Ind}(\nu). \tag{6.1}$$

Here, $\mathbb{Z}(\mathrm{Im}(\psi) + \lambda)$ denotes the integral weights in $\mathrm{Im}(\psi) + \lambda$.

We now prove that if ω is a G-invariant Kähler form on M, then $\mathcal{H} \cdot \mathrm{Ind}_\psi(\omega) = \mathrm{Ind}_\psi \cdot \mathcal{H}(\omega)$ as G-representations.

Proof of Theorem 1.5. Let ρ denote half the sum of positive roots. We parametrize the holomorphic discrete series representations of G by Harish-Chandra's notation $\Theta_{\lambda+\rho}$, where $\lambda \in \mathfrak{h}^*$ are the integral weights in Σ.

Let ω be a G-invariant Kähler form on M, with $\mathcal{H}(\omega) = \lambda$. Suppose that $\mathrm{Ind}_\psi(\omega)$ is Kähler, so that by Theorem 1.4, $\mathrm{Im}(\psi) + \lambda \subset \Sigma$. By Theorem 1.3, $\mathrm{Ind}_\psi(\omega) = 2\sqrt{-1}\partial\bar{\partial}F$, where $F'(x) = \psi(x) + \lambda$. By [**1**, Theorem 2], $\mathcal{H} \cdot \mathrm{Ind}_\psi(\omega)$ contains all the $\Theta_{\nu+\rho}$ with integral weights ν in $\mathrm{Im}(F') = \mathrm{Im}(\psi) + \lambda$, each of which has multiplicity one. By (6.1), this is exactly the case for $\mathrm{Ind}_\psi \cdot \mathcal{H}(\omega)$. This proves the theorem. □

Let $\{\lambda_i\}$ be the dominant fundamental weights of Σ. That is, all the weights in Σ are of the form $\sum_i c_i \lambda_i$, where c_i are positive integers. Let ω be a G-invariant symplectic form on M, with corresponding $\lambda = \Phi(e) \in \mathfrak{h}^*_{\mathrm{reg}}$ as given in Theorem 1.1. Let

$$f : \mathfrak{h}^* \longrightarrow \mathbb{R}, \quad f(x) = \left(\sum_i e^{\lambda_i(x)} \right) - \lambda(x).$$

Then $f'(x) = (\sum_i e^{\lambda_i(x)} \lambda_i) - \lambda$, and so $\mathrm{Im}(f') + \lambda = \Sigma$. Also, f is strictly convex. Let $\psi = f'$. It follows from Theorem 1.4 that $\mathrm{Ind}_\psi(\omega)$ is Kähler. By Theorem 1.5, all the discrete series representations $\Theta_{\nu+\rho}$ with $\nu \in \Sigma$ occur in $\mathcal{H} \cdot \mathrm{Ind}_\psi(\omega)$.

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