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# Packing $\lambda$-Fold Complete Multipartite Graphs with 4-Cycles

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**Abstract.** A maximum packing of any $\lambda$-fold complete multipartite graph (where there are $\lambda$ edges between any two vertices in different parts) with edge-disjoint 4-cycles is obtained and the size of each minimum leave is given. Moreover, when $\lambda = 2$, maximum 4-cycle packings are found for all possible leaves.

## 1. Introduction and Preliminaries

A *k-cycle packing* of a graph $G$ is a set $C$ of edge-disjoint $k$-cycles in $G$. Such a packing is *maximum* if $|C| \geq |C'|$ for all other $k$-cycle packings $C'$ of $G$. The *leave* $L$ of a packing is the set of edges of $G$ that occur in no $k$-cycle of the packing; we also refer to the subgraph induced by the edges in $L$ as the leave. The leave of a maximum packing is referred to as a minimum leave. A *k-cycle system* of $G$ is a $k$-cycle packing of $G$ with leave $L = \emptyset$.

Let $K(v_1, v_2, \ldots, v_n)$ denote the *complete multipartite graph* with vertex set $V$ partitioned into *n parts* $V_i$ of size $v_i$ for $1 \leq i \leq n$, and edge set consisting of all edges between all vertices in $V_i$ and $V_j$, for $1 \leq i < j \leq n$, but no edges between any two vertices in the same part. The complete bipartite graph $K(v_1, v_2)$ is also denoted by the more common $K_{v_1,v_2}$. If $G$ denotes a simple graph (with no loops or multiple edges), then $\lambda G$ denotes the multigraph obtained from $G$ by

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replicating each edge of $G$ precisely $\lambda$ times. The term "$\lambda$-fold" graph is also used.

The existence problem for $k$-cycle systems of complete graphs $K_n$ has been actively studied over the past 35 years, and this recently resulted in a complete solution of the problem by Alspach, Gavlas, and Šajna [1,12] that was partially based on some work of Hoffman, Lindner and Rodger [7]. Maximum packings of $K_n$ were also found in [1,12] in all cases where the leave is a 1-factor (this restricts the possible values of $n$), and have been found for all values of $n$ when $k \in \{3,4,5,6\}$ [6,8,9,11,13]. For a survey, see [10].

The existence problem for $k$-cycle systems of complete multipartite graphs, even when $k = 3$, is proving to be an extremely difficult problem to solve, partly because so many different kinds of graphs have to be considered. For example, one excellent paper deals exclusively with the case where $k = 3$ and all parts except one have the same size [5]. However, it turns out that this myriad of complete multipartite graphs can be handled when looking for 4-cycle systems [4]. Furthermore, perhaps surprisingly, the existence problem for maximum 4-cycle packings of complete multipartite graphs was also completely solved [3], producing the following result.

**Theorem 1.1.** *Let $G$ be a complete multipartite graph with $\eta$ vertices of odd degree and $v$ vertices in the largest part containing vertices of odd degree (if such a part exists). If $C$ is a 4-cycle packing of $G$ with leave $L$ then $C$ is a maximum 4-cycle packing if and only if*

*(i)* $max\{\eta/2, v\} \leq |L| \leq max\{\eta/2+3, v+3\}$, *or*
*(ii)* *$G$ has an odd number of parts, $n$, all of odd size, with $n \equiv 5$ or $7 \pmod 8$, in which case $|L| = 6$ or $5$ respectively.*

In this paper we extend this work to the case of any $\lambda$-fold complete multipartite graph. That is, we solve the problem of finding a maximum 4-cycle packing of any $\lambda$-fold complete multipartite graph, for all $\lambda > 1$. Moreover, in the case $\lambda = 2$, we exhibit not just a single minimum leave, but all possible minimum leaves.

The graph theoretic notation not defined here can be found in [15]. Sets in this paper are considered to be multisets, and the union of sets requires each element to occur the the number of times equal to the sum of the numbers of times it occurs in the sets themselves. If $G$ and $H$ are two vertex-disjoint graphs, then $G \vee H$ is formed from the union of $G$ and $H$ by joining each vertex in $G$ to each vertex in $H$ with exactly one edge. It will cause no confusion if we also refer to a $k$-cycle packing $C$ as an ordered pair $(V, C)$, where $V$ is the set of vertices on which the cycles in $C$ are defined.

Section 2 deals with the case $\lambda = 2$, while Section 3 deals with $\lambda = 3$. Then Section 4 completes all remaining values of $\lambda$. We may summarise our results as follows (see Theorems 1.1, 2.7, 3.4, and 4.10).

**Main Theorem** *Let $G$ be a complete multipartite graph. Let $\eta(\lambda)$ be the number of vertices of odd degree in $\lambda G$, and let $v(\lambda)$ be the number of vertices in the largest part of $\lambda G$ containing vertices of odd degree. There exists a maximum* 4*-cycle packing of $\lambda G$ with some leave $L_\lambda$ satisfying $|L_\lambda| = l$ if and only if*

*(i)* *if $G = K(1,n)$ or $K(1,1,1)$, then $L_\lambda = E(\lambda G)$,*
*(ii)* *if $G = K(1,1,n)$ and $n > 1$ then*
$$L_\lambda = \begin{cases} \lambda K_2 \vee K_1 \text{ if } n \text{ and } \lambda \text{ are odd, and} \\ \lambda K_2 \text{ otherwise,} \end{cases}$$
*(iii)* *if $\eta(\lambda) = 0$, $|E(\lambda G)| \equiv 1 \pmod 4$, and $G \neq K(1,1,n)$, then $|L_\lambda| = 5$,*
*(iv)* *if $\eta(\lambda) = 0$, $|E(\lambda G)| \equiv 2 \pmod 4$, and $\lambda = 1$, then $|L_\lambda| = 6$, and otherwise*
*(v)* *$l$ is the unique integer satisfying*

*(1)* $\max\{\eta(\lambda)/2, v(\lambda)\} \le l \le \max\{\eta(\lambda)/2 + 3, v(\lambda) + 3\}$, *and*
*(2)* 4 *divides* $|E(\lambda G)| - l$.

The following result addresses the necessity of the conditions *(i)–(v)* for the existence of a maximum 4-cycle packing of $\lambda G$ in the Main Theorem.

Given that the lower bound in condition *(v)(1)* and condition *(v)(2)* are proved below to be necessary conditions for the existence of a maximum packing, it is clear that any 4-cycle packing which also satisfies the upper bound in condition *(v)(1)* must be a maximum 4-cycle packing.

**Lemma 1.2.** *If there exists a maximum* 4*-cycle packing $C$ of $\lambda G$ then, (referring to conditions in the Main Theorem above), $C$ satisifies conditions (i) and (ii), $C$ has a leave $L_\lambda$ which must satisfy $|L_\lambda| \ge 5$ or* 6 *in conditions (iii) and (iv) respectively, and $C$ satisfies the lower bound in condition (v(1)) and condition (v(2)) of the Main Theorem.*

*Proof.* Let $C$ be a maximum 4-cycle packing of $\lambda G$, and let $L$ be its leave. Each 4-cycle in $C$ accounts for either 0 or 2 edges incident with each vertex in $\lambda G$, and therefore each vertex of odd degree in $\lambda G$ must have odd degree in $L$. Therefore $|L| \ge \eta(\lambda)/2$, and since clearly $L$ contains no edge joining two vertices in the same part of $\lambda G$, the condition $|L| \ge v(\lambda)$ also follows. Therefore $|L| \ge \max\{\eta(\lambda)/2, v(\lambda)\}$. Also, each 4-cycle in $C$ accounts for four edges in $\lambda G$, so clearly 4 divides $|E(\lambda G)| - |L|$. So the lower bound in condition $(v(1))$ is necessary, as is condition $(v(2))$.

If $G = K(1,n)$ or $G = K(1,1,1) = K_3$ then $\lambda G$ contains no 4-cycles, so $C = \emptyset$ is the only 4-cycle packing of $G$. So condition $(i)$ is necessary.

Similarly, if $G = K(1,1,n)$ then $\lambda G$ contains no 4-cycles which contain an edge joining the vertices in $V_1 \cup V_2$. So since 4 divides $|E(\lambda G)| - |L|$, it follows that condition $(ii)$ is necessary.

If all vertices in $L$ have even degree, then clearly $|L| \neq 1$. So if $|E(\lambda G)| \equiv 1 \pmod 4$ then $|L| \ge 5$ since $|L| \equiv |E(\lambda G)| \pmod 4$. So condition $(iii)$ is necessary.

Similarly, if all vertices in $L$ have even degree and $\lambda = 1$, then $|L| \neq 2$ (since $G$ contains no multiple edges). So if $|E(\lambda G)| \equiv 2 \pmod 4$ then $|L| \geq 6$ since $|L| \equiv |E(\lambda G)| \pmod 4$. So condition $(iv)$ is necessary. □

There is one further result that we need.

**Lemma 1.3.** *There exists a* 4*-cycle system of* $\lambda K_{x,y}$ *if and only if min*$\{x, y\} \geq 2$, $\lambda x$ *and* $\lambda y$ *are even, and* 4 *divides* $\lambda xy$.

*Proof.* This is easy to prove, and also follows from a more general result of Sotteau [14]. □

We shall use Lemma 1.3 often, so we adopt the following notation. Let $B_\lambda(X, Y)$ denote a 4-cycle system of $\lambda K_{x,y}$ with bipartition $\{X, Y\}$ of the vertex set, where $x = |X|$ and $y = |Y|$. Often the value of $\lambda$ will be clear from the context, in which case we shall simply use $B(X, Y)$.

Subsequently we also use the existence of a 4-cycle decomposition of $\lambda K_v$ for appropriate $\lambda$ and $v$; see [2].

## 2. The Case $\lambda = 2$

We begin with some useful lemmas.

**Lemma 2.1.** *There exists a* 4*-cycle packing of* $2K(1,1,1,2)$ *with leave* $\{\{u_1, u_2\}, \{u_1, u_2\}\}$ *in each of the following cases:*

*(a)* $u_1 \in V_1$ *and* $u_2 \in V_2$, *and*
*(b)* $u_1 \in V_1$ *and* $u_2 \in V_4$.

*Proof.* Let $V_i = \{i\}$ for $1 \leq i \leq 3$ and $V_4 = \{4, 5\}$. Then $(\{1, 2, 3, 4, 5\}, B)$ is the required packing, where in case $(a)$,

$$B = \{(1, 3, 2, 4), (1, 3, 2, 5), (1, 4, 3, 5), (2, 4, 3, 5)\},$$

and in case $(b)$,

$$B = \{(1, 2, 4, 3), (1, 2, 3, 5), (1, 3, 2, 5), (2, 4, 3, 5)\}.$$

□

**Lemma 2.2.** *There exists a* 4*-cycle packing of* $2K(1,1,2,2)$ *with leave* $\{\{u_1, u_2\}, \{u_1, u_2\}\}$ *in each of the following cases:*

*(a)* $u_1 \in V_1$ *and* $u_2 \in V_2$,
*(b)* $u_1 \in V_1$ *and* $u_2 \in V_3$, *and*
*(c)* $u_1 \in V_3$ *and* $u_2 \in V_4$.

*Proof.* Let $V_1 = \{1\}$, $V_2 = \{2\}$, $V_3 = \{3,4\}$ and $V_4 = \{5,6\}$. Then $(\{1,2,3,4,5,6\}, B)$ is the required packing, where in case $(a)$,

$$B = \{(1,3,2,4),(1,3,2,4),(1,5,2,6),(1,5,2,6),(3,5,4,6),(3,5,4,6)\},$$

in case $(b)$,

$$B = \{(1,2,3,5),(1,2,5,4),(1,4,2,6),(1,5,4,6),(2,3,6,4),(2,5,3,6)\},$$

and in case $(c)$,

$$B = \{(1,2,3,6),(1,2,6,3),(1,3,2,4),(1,4,2,5),(1,5,4,6),(2,5,4,6)\}.$$

□

**Lemma 2.3.** *There exists a* 4-*cycle packing of* 2*K*(3,3) *with leave* 2*K*(1,1).

*Proof.* A suitable packing is given by $((\{1,2,3\} \cup \{4,5,6\}), B)$ where

$$B = \{(1,5,2,6),(1,5,3,6),(2,4,3,5),(2,4,3,6)\}.$$

The leave here is $\{\{1,4\},\{1,4\}\}$. □

**Lemma 2.4.** *There exists a* 4-*cycle packing of* 2*K*(1,3,2) *with leave* $\{\{u_1,u_2\}, \{u_1,u_2\}\}$ *in each of the following cases:*

*(a)* $u_1 \in V_1$ *and* $u_2 \in V_2$, *and*
*(b)* $u_1 \in V_2$ *and* $u_2 \in V_3$.

*Proof.* Let $V_1 = \{1\}$, $V_2 = \{2,3,4\}$ and $V_3 = \{5,6\}$. Then $(\{1,2,3,4,5,6\}, B)$ is the required packing, where in case $(a)$,

$$B = \{(1,3,5,4),(1,3,6,4),(1,5,2,6),(1,5,3,6),(2,5,4,6)\},$$

and in case $(b)$,

$$B = \{(1,2,6,3),(1,2,6,4),(1,3,5,4),(1,5,3,6),(1,5,4,6)\}.$$

□

**Lemma 2.5.** *There exists a* 4-*cycle packing of* 2*K*(3,3,2) *with leave* $\{\{u_1,u_2\}, \{u_1,u_2\}\}$ *in each of the following cases:*

*(a)* $u_1 \in V_1$ *and* $u_2 \in V_2$, *and*
*(b)* $u_1 \in V_1$ *and* $u_2 \in V_3$.

*Proof.* Let $V_1 = \{1,2,3\}$, $V_2 = \{4,5,6\}$ and $V_3 = \{7,8\}$. Then $(\{1,2,\ldots,8\}, B)$ is the required packing, where in case $(a)$, $B = B_1 \cup B(V_1 \cup V_2, V_3)$, where $(V_1 \cup V_2, B_1)$ is a 4-cycle packing of $2K(3,3)$ (see Lemma 2.3), and in case $(b)$,

$B = B_2 \cup B(\{5,6\}, V_1 \cup V_3)$ where $((V_1, \{4\}, V_3), B_2)$ is a 4-cycle packing of $2K(3,1,2)$ (see Lemma 2.4($b$)). □

**Lemma 2.6.** *There exists a* 4*-cycle packing of* $2K(1,3,3)$ *with leave* $\{\{u_1, u_2\}, \{u_1, u_2\}\}$ *in each of the following cases:*

*(a)* $u_1 \in V_1$ and $u_2 \in V_3$, *and*
*(b)* $u_1 \in V_2$ *and* $u_2 \in V_3$.

*Proof.* Let $V_1 = \{1\}$, $V_2 = \{2,3,4\}$ and $V_3 = \{5,6,7\}$. Then $(\{1,2,\dots,7\}, B)$ is the required packing, where in case ($a$),

$$B = \{(1,3,5,4),(1,3,6,4),(1,5,2,6),(1,5,3,7),(1,6,4,7),(2,5,4,7),(2,6,3,7)\},$$

and in case ($b$),

$$B = \{(1,2,6,3),(1,2,7,4),(1,3,5,4),(1,5,3,7),(1,5,4,6),(1,6,3,7),(2,6,4,7)\}.$$

□

**Theorem 2.7.** *Let* $G = 2K(v_1, v_2, \dots, v_{s+t})$ *be the* 2*-fold complete multipartite graph with parts* $V_1, V_2, \dots, V_{s+t}$, *where* $v_i = |V_i|$ *is odd for* $1 \le i \le t$ *and is even for* $t+1 \le i \le s+t$. *Let* $V = \bigcup_{i=1}^{s+t} V_i$ *and* $v = |V|$. *There exists a* 4*-cycle packing of* $G$ *with leave* $L$ *that is a maximum packing if and only if*

*(a)* $L = E(G)$ *if* $G = K(1,n)$ *or* $G = K(1,1,1)$, *and otherwise*
*(b)* *if* $t \equiv 0$ *or* 1 (*mod* 4) *then* $L = \emptyset$, *and*
*(c)* *if* $t \equiv 2$ *or* 3 *(mod* 4*) then* $L = \{\{u_1,u_2\},\{u_1,u_2\}\}$, *where*
*(i) if* $G = K(1,1,n)$ *then each of* $u_1$ *and* $u_2$ *is in a part of size* 1;
*(ii) if* $G = K(1, 2n+1, 2)$ *then exactly one of* $u_1$ *or* $u_2$ *occurs in the part of size* $2n+1$*; and*
*(iii) for all other* $G$, $u_1$ *and* $u_2$ *occur in any two different parts.*

*Proof.* We begin by showing that no 4-cycle packings of $G$ with smaller leaves exist, and that if $|L| = 2$ then no other choices for $u_1$ and $u_2$ are possible.

If $t \equiv 2$ or 3 (mod 4) then $|E(G)| \equiv 2$ (mod 4), so necessarily $|L| \ge 2$. If $G = K(1,n)$ or $G = K(1,1,1)$, then $G$ contains no 4-cycles, so clearly $L = E(G)$. If $|L| = 2$ then since each vertex in $G$ has even degree, $L = \{\{u_1,u_2\},\{u_1,u_2\}\}$ for some vertices $u_1$ and $u_2$ which, being adjacent vertices, must occur in different parts.

If $G = K(1,1,n)$ and $n > 1$ with $V_1 = \{u_1\}$ and $V_2 = \{u_2\}$ then no 4-cycle contains an edge joining $u_1$ to $u_2$. So $L = \{\{u_1,u_2\},\{u_1,u_2\}\}$, since $|L| = 2$.

If $G = K(1,2n+1,2)$ with say $V_1 = \{u_1\}$ and $V_3 = \{u_2, z\}$ then there is no 4-cycle in $G \setminus \{\{u_1,u_2\},\{u_1,u_2\}\}$ that contains an edge $\{u_1, z\}$. But this cannot happen since $|L| = 2$. So one of $u_1$ and $u_2$ must occur in $V_2$.

So we now turn to the construction of a maximum 4-cycle packing of $G$ with leave $L$ that satisfies conditions $(a),(b)$ and $(c)$. Clearly we can assume that $s+t \geq 2$, and by $(b)$ we can assume that $v \geq 4$ and that if $s+t=2$ then $\min\{v_1, v_2\} > 1$. We consider various situations in turn.

*Case 1.* Suppose that $t \equiv 0$ or 1 (mod 4).

For $1 \leq i \leq t$, let $W_i \subseteq V_i$ with $|W_i| = 1$; for $t+1 \leq i \leq s+t$ it is convenient to define $W_i = \emptyset$. Let $W = \bigcup_{i=1}^{t} W_i$, and let $(W, B_1)$ be a 4-cycle system of $2K_t$. For $1 \leq i \leq s+t$, clearly we have that $|V_i \setminus W_i|$ is even (possibly zero), and if $X_i = (\bigcup_{j=i+1}^{s+t} V_j) \cup (\bigcup_{j=1}^{i-1} W_j)$, then $|X_i| \geq 2$ (by the assumptions on $s$, $t$ and $v$). Therefore, by Lemma 1.3, we can define the set of 4-cycles

$$B_0 = \bigcup_{i=1}^{s+t} B(V_i \setminus W_i, X_i),$$

(where we take $B(\emptyset, X_i) = \emptyset$). Then $(V, B_1 \cup B_0)$ is the required 4-cycle system of $G$.

*Case 2.* Suppose that $t \equiv 2$ or 3 (mod 4).

Let $u_i \in V_{\alpha_i}$ for $1 \leq i \leq 2$; since it only matters if $\alpha_i$ is in a part of even or odd size, we can assume that $(\alpha_1, \alpha_2) \in \{(1,2), (1, t+1), (t+1, t+2)\}$ for notational convenience; so in particular $\alpha_1 < \alpha_2$. In each of the following cases, let $B_0$ be as defined in Case 1, where the sets $W_i$ are defined below. Also, as in Case 1, we assume that $W_i = \emptyset$ unless otherwise defined.

(a) Suppose that either $t \geq 4$, or $t = 3$ and $s \geq 1$. For $1 \leq i \leq t$ let $W_i \subseteq V_i$ with $|W_i| = 1$, where for $1 \leq j \leq 2$ we choose $W_j = \{u_j\}$ if $\alpha_j \leq t$. Let $W = \bigcup_{i=1}^{t} W_i$. We consider three cases in turn.

Suppose $\alpha_2 \leq t$ and $t \geq 4$. Let $(W, B_1)$ be a 4-cycle packing of $2K_t$ with leave $\{\{u_1, u_2\}, \{u_1, u_2\}\}$. It follows that $(V, B_1 \cup B_0)$ is the required maximum 4-cycle packing of $G$.

Suppose $\alpha_1 \leq t$ and $\alpha_2 > t$, or $\alpha_2 \leq t$ and $t = 3$. Since $t - 2 \equiv 0$ or 1 (mod 4), let $(\bigcup_{i=3}^{t} W_i, B_1)$ be a 4-cycle system of $2K_{t-2}$. Since $\alpha_2 > t$ or $t = 3$, we know that $s \geq 1$, so choose $W_{t+1} \subseteq V_{t+1}$ with $|W_{t+1}| = 2$ and with $u_2 \in W_{t+1}$ if $\alpha_2 > t$. Let $((W_1, W_2, W_3, W_{t+1}), B_2)$ be a 4-cycle packing of $2K(1,1,1,2)$ with leave $\{\{u_1, u_2\}, \{u_1, u_2\}\}$ (see Lemma 2.1 $(a)$ or $(b)$ if $u_2 \in V_2$ or $u_2 \in V_{t+1}$ respectively). Then $(V, B_2 \cup B_1 \cup B_0 \cup B(W_1 \cup W_2 \cup W_{t+1}, \bigcup_{i=4}^{t} W_i))$ is a maximum 4-cycle packing of $G$ with leave $\{\{u_1, u_2\}, \{u_1, u_2\}\}$.

Finally, suppose that $\alpha_1 > t$. Let $(\bigcup_{i=3}^{t} W_i, B_1)$ be a 4-cycle system of $K_{t-2}$. For $1 \leq j \leq 2$ choose $W_{t+j} \subseteq V_{t+j}$ with $|W_{t+j}| = 2$ and with $u_j \in W_{t+j}$. Let $((W_1, W_2, W_{t+1}, W_{t+2}), B_2)$ be a 4-cycle packing of $2K(1,1,2,2)$ with leave $\{\{u_1, u_2\}, \{u_1, u_2\}\}$ (see Lemma 2.2$(c)$). Now $(V, B_2 \cup B_1 \cup B_0 \cup B(W_1 \cup W_2 \cup W_{t+1} \cup W_{t+2}, \bigcup_{i=3}^{t} W_i))$ is the required 4-cycle packing of $G$.

(b) Suppose that $t = 3$ and $s = 0$. If two of the parts have size 1, then by *(c)(i)*, $V_1 \cup V_2 = \{u_1, u_2\}$, so $(V, B(V_1 \cup V_2, V_3))$ is the required maximum 4-cycle packing of $G$. Otherwise at most one of $v_1$, $v_2$ or $v_3$ is 1. Therefore we can select three vertices from each of two parts and one vertex from the third part, ensuring that $u_1$ and $u_2$ are among the selected vertices; for $1 \leq i \leq 3$ let $W_i$ be the set of vertices selected from $V_i$. Let $((W_1, W_2, W_3), B_1)$ be a 4-cycle packing of $2K(|W_1|, |W_2|, |W_3|)$ with leave $\{\{u_1, u_2\}, \{u_1, u_2\}\}$ (see Lemma 2.6 (*a*) or (*b*)). Then $(V, B_1 \cup B_0)$ is the required 4-cycle packing of $G$.

(c) Finally, suppose that $t = 2$. If $s = 0$ then $\min\{v_1, v_2\} \geq 3$, so for $1 \leq i \leq 2$ let $W_i \subseteq V_i$ with $u_i \in W_i$ and $|W_i| = 3$. Let $((W_1, W_2), B_1)$ be a 4-cycle packing of $2K(3, 3)$ with leave $\{\{u_1, u_2\}, \{u_1, u_2\}\}$ (see Lemma 2.3). Then $(V, B_1 \cup B_0)$ is the required maximum 4-cycle packing.

If $s \geq 2$ then for $1 \leq i \leq 4$ let $W_i \subseteq V_i$ with $|W_1| = |W_2| = 1$, $|W_3| = |W_4| = 2$, and $\{u_1, u_2\} \subseteq \bigcup_{i=1}^{4} W_i$. By Lemma 2.2(*a*), (*b*) or (*c*) there exists a 4-cycle packing $((W_1, W_2, W_3, W_4), B_1)$ of $2K(1, 1, 2, 2)$ with leave $\{\{u_1, u_2\}, \{u_1, u_2\}\}$. Then $(V, B_1 \cup B_0)$ is the required maximum 4-cycle packing.

Finally, suppose that $s = 1$. If $G = 2K(1, 1, n)$ then condition *(c)(i)* requires $u_1$ and $u_2$ to be in the parts of size 1, so $(V, B(\{u_1, u_2\}, V \setminus \{u_1, u_2\}))$ is the required maximum 4-cycle packing.

Otherwise, we now assume that $v_2 \geq 3$. If $u_1 \in V_1$ and $u_2 \in V_2$, or if $u_1 \in V_2$ and $u_2 \in V_3$, then let $W_1 \subseteq V_1$ with $|W_1| = 1$, $W_2 \subseteq V_2$ with $|W_2| = 3$, and $W_3 \subseteq V_3$ with $|W_3| = 2$, where $\{u_1, u_2\} \subseteq \bigcup_{i=1}^{3} W_i$. By Lemma 2.4 (*a*) or (*b*), there exists a 4-cycle packing $((W_1, W_2, W_3), B_1)$ of $2K(1, 3, 2)$ with leave $\{\{u_1, u_2\}, \{u_1, u_2\}\}$, so $(V, B_1 \cup B_0)$ is the required maximum 4-cycle packing. It remains to consider the possibility of $u_1 \in V_1$ and $u_2 \in V_3$, in which case $v_1 \geq 3$ by condition *(c)(ii)*. For $1 \leq i \leq 3$ let $W_i \subseteq V_i$ with $|W_i| = 2$ or 3 if $i = 3$ or $i \leq 2$ respectively. By Lemma 2.5(*b*) there exists a 4-cycle packing $((W_1, W_2, W_3), B_1)$ of $2K(3, 3, 2)$ with leave $\{\{u_1, u_2\}, \{u_1, u_2\}\}$, so $(V, B_1 \cup B_0)$ is the required maximum 4-cycle packing. □

## 3. The Case $\lambda = 3$

In this section we shall use the following notation. A maximum 4-cycle packing of the complete multipartite graph $\lambda G$ with vertex set $V$ will be denoted by $(V, \mathscr{B}_\lambda)$, with leave $L_\lambda$, for $\lambda = 1, 2, 3$.

We shall also refer to the graph $D$ where $V(D) = \{a_1, a_2, u_1, u_2, u_3, u_4\}$ and $E(D) = \{\{a_1, a_2\}, \{a_1, u_2\}, \{a_1, u_4\}, \{a_2, u_1\}, \{a_2, u_3\}\}$ as $D(a_1, a_2, u_1, u_2, u_3, u_4)$. This graph is sometimes a subgraph of a leave $L_1$.

We begin with three lemmas.

**Lemma 3.1.** *If there exists a* 4*-cycle packing* $(V, \mathscr{B}_1)$ *of a complete multipartite graph G with leave* $L_1$ *such that* $D \subseteq L_1$, *and if there exists a* 4*-cycle packing* $(V, \mathscr{B}_2)$ *of* 2G *with leave* $\{\{u_1, u_2\}, \{u_1, u_2\}\}$, *then there exists a* 4*-cycle packing of* 3G *with leave* $L_3$ *where* $|L_3| = |L_1| - 2$.

*Proof.* Let $\mathscr{B}_3 = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \{(a_1, a_2, u_1, u_2)\}$, so that

$$L_3 = (L_1 \setminus E(D)) \cup \{\{a_1, u_4\}, \{a_2, u_3\}, \{u_1, u_2\}\} \quad \textit{and} \quad |L_3| = |L_1| - 2.$$

□

**Lemma 3.2.** *If there exists a* 4*-cycle packing* $(V, \mathscr{B}_1)$ *of a complete multipartite graph* $G$ *with leave* $L_1$ *such that*

*(i)* $Q = \{\{u_5, u_1\}, \{u_5, u_2\}, \{u_5, u_3\}, \{u_5, u_4\}\} \subseteq L_1$ *and*
*(ii)* $b = (u_1, u_2, u_3, u_4) \in \mathscr{B}_1$,

*and if there exists a* 4*-cycle packing* $(V, \mathscr{B}_2)$ *of* $2G$ *with leave* $\{\{(u_1, u_2\}, \{(u_1, u_2\}\}$, *then there exists a* 4*-cycle packing of* $3G$ *with leave* $L_3$ *where* $|L_3| = |L_1| - 2$.

*Proof.* Let $\mathscr{B}_3 = (\mathscr{B}_1 \setminus \{b\}) \cup \mathscr{B}_2 \cup \{(u_1, u_2, u_3, u_5), (u_1, u_2, u_5, u_4)\}$. Then $L_3 = L_1 \setminus Q \cup \{\{u_1, u_2\}, \{u_3, u_4\}\}$. □

**Lemma 3.3.** *Let* $H(1, 2, \ldots, 12)$ *be the graph with vertex set* $\{1, 2, \ldots, 12\}$ *and edge set* $E(H) = \{\{6, 10\}, \{6, 10\}, \{1, 2\}, \{1, 3\}\} \cup \{\{i, j\} \mid 3 \le i \le 6,\ 7 \le j \le 12\} \cup \{\{i, j\} \mid 7 \le i \le 10,\ 11 \le j \le 12\} \cup \{\{2, j\} \mid 4 \le j \le 12\}$. *There exists a* 4*-cycle packing* $B(H(1, 2, \ldots, 12))$ *of* $H(1, 2, \ldots, 12)$ *with leave the matching* $L = \{\{3, 11\}, \{4, 8\}, \{5, 9\}, \{6, 10\}, \{7, 12\}\}$ *saturating the vertices of odd degree.*

*Proof.* The 4-cycles in {(1,2,8,3), (2,4,9,6), (2,5,11,7), (2,9,11,10), (2,11,8,12), (3,7,4,12), (3,9,12,10), (4,10,6,11), (5,7,6,8), (5,10,6,12)} provide the required 4-cycle packing. □

**Theorem 3.4.** *Let* $G$ *be a complete multipartite graph with parts* $V_1, V_2, \ldots, V_{s+t}$, *where* $v_i = |V_i|$ *is odd for* $1 \le i \le t$ *and is even for* $t + 1 \le i \le s + t$. *Let* $\eta$ *be the number of vertices of odd degree and* $v$ *be the size of the largest part having vertices of odd degree. There exists a maximum* 4*-cycle packing of* 3G *with leave* $L_3$ *where:*

*(i)* *if* $\eta = 0$ *then* $|L_3| \in \{0, 2, 3, 5\}$, *and in particular if* $G = K(1, 1, n)$ *then*

$$L_3 = \begin{cases} 3K_2 & \textit{when } n \textit{ is even ,} \\ 3K_2 \vee K_1 & \textit{when } n \textit{ is odd, and} \end{cases}$$

*(ii)* *if* $\eta \ge 1$ *then* $|L_3| \le max\{\eta/2 + 3, v + 3\}$, *except if* $G = K_3$ *or* $G = K(1, n)$, *in which cases* $L_3 = E(3G)$.

**Remark.** Note that $|L_3|$ is completely determined by (i) and (ii), since if $L'$ is the leave of any 4-cycle packing of $3G$, then clearly $|L'| \ge \max\{\eta/2, v\}$ and $|L'| - |L_3|$ is divisible by 4.

*Proof.* **(1) Sporadic Cases**. If $G = K_3$ or if $G = K(1, n)$ then $3G$ contains no 4-cycles, so clearly $L_3 = E(3G)$ in the only 4-cycle packing of $3G$.

If $G = K(1, 2n+1, 2)$, then the union of a maximum 4-cycle packing of $G$ (which has leave of size $v = 2n+1$) and a maximum 4-cycle packing of $2G$ (which has leave of size 2) produces a maximum 4-cycle packing of $3G$ with leave $L_3$, where $|L_3| = v + 2$.

If $G = K(1, 1, n)$, then the union of a maximum 4-cycle packing of $G$ (which has leave of size 1 if $n$ is even and 3 if $n$ is odd) and a maximum 4-cycle packing of $2G$ (which has leave of size 2) produces a maximum 4-cycle packing of $3G$ with leave $L_3$, where $|L_3| = 3$ if $n$ is even and $|L_3| = 5$ if $n$ is odd (so $\eta = 0$).

Therefore, by Theorem 2.7, throughout the rest of the proof we can assume that for any $\{u_1, u_2\} \in E(G)$, if the number of edges in $2G$ is congruent to 2 (mod 4), then there exists a 4-cycle packing of $2G$ with leave $\{\{u_1, u_2\}, \{u_1, u_2\}\}$.

**(2) $t \equiv$ 0 or 1 (mod 4)**. In this case, when $\lambda = 2$ the leave is $\emptyset$, and so a maximum 4-cycle packing in which the leave when $\lambda = 3$ is exactly the same as when $\lambda = 1$, so the result follows from Theorem 1.1.

**(3) $t \equiv$ 2 or 3 (mod 4)**. Let $M = \max\{v, \eta/2\}$. If $|L_1| \leq M + 1$, we can add the repeated edge leave $L_2$, and the new leave $L_3$ satisfies (ii), so is a minimum leave as required. So henceforth we assume that $|L_1| \in \{M+2, M+3\}$.

We shall follow the order of the sections and adopt the notation used in [3].

The bipartite case

Both parts must have odd size since $t \equiv 2$ or 3 (mod 4). Let $v_1 \geq v_2$. If $v_2 \equiv 1$ (mod 4) then $|L_1| = v_1 = M$. If $v_2 \equiv 3$ (mod 4), then we can use Lemma 3.1 since $L_1$ contains $D$; $L_1$ is shown in Fig. 1 of [3]. So in this case, we have $|L_1| = v_1 + 2$ and $|L_3| = |L_1| - 2 = v_1$.

An odd number of parts, all of odd size

We must have $t \equiv 3$ (mod 4). We have two cases.

If $t \equiv 3$ (mod 8) then let $(V, \mathscr{B}_1)$ be a maximum 4-cycle packing of $G$ with leave $L_1 = K_3$, and let $(V, \mathscr{B}_2)$ be a maximum 4-cycle packing of $2G$ with leave $L_2 = C_2$. (Here $C_2$ denotes a pair of vertices joined by two edges.) Then $(V, \mathscr{B}_1 \cup \mathscr{B}_2)$ is a 4-cycle packing of $3G$ with leave $L_3 = K_3 \cup C_2$ of size 5 as required.

If $t \equiv 7$ (mod 8), then let $(V, \mathscr{B}_1)$ be a 4-cycle packing of $G$ with minimum leave the 5-cycle $L_1 = (u_1, u_4, u_3, u_2, u_5)$, and let $(V, \mathscr{B}_2)$ be a 4-cycle packing of $2G$ with leave $L_2 = \{\{u_1, u_2\}, \{u_1, u_2\}\}$. Then $(V, \mathscr{B}_1 \cup \mathscr{B}_2 \cup \{(u_1, u_2, u_3, u_4)\})$ is a 4-cycle packing of $3G$ with leave $L_3 = \{(u_1, u_2, u_5)\} = K_3$, so $|L_3| = 3$.

An even number of parts, all of odd size

We refer the reader to Fig. 5 of [3], where the possibilities for the leave $L_1$ of $(V, B_1)$ are listed.

*Case (i).*

In this case, $|L_1| \in \{v_1 + 2, v_1 + 3\}$, so 2 or 3 copies of $K_2$ in $L_1$ do not contain a vertex in $V_1$; let $\{u_1, u_3\}$ and $\{u_2, u_4\}$ induce 2 such copies. Also, since $|L_1| \geq \eta/2 + 2$, in $L_1$ there are 4 vertices in $V_1$, say $u_5, u_6, u_7$, and $u_8$ that have a common neighbour, say $a_2 \in V_2$. So we have:

$$Q = \{\{u_1, u_3\}, \{u_2, u_4\}, \{a_2, u_5\}, \{a_2, u_6\}, \{a_2, u_7\}, \{a_2, u_8\}\} \subseteq L_1.$$

Also, the partition on the last line in page 115 in [3] ensures that

$$T_1 = \{(u_1, u_5, u_3, u_6), (u_2, u_5, u_4, u_6), (u_1, u_7, u_3, u_8), (u_2, u_7, u_4, u_8)\} \subseteq \mathscr{B}_1.$$

Let $(V, \mathscr{B}_2)$ be a 4-cycle packing of $2G$ with $L_2 = \{\{u_1, u_2\}, \{u_1, u_2\}\}$. Also let $T_2 = \{(u_3, u_5, u_4, u_6), (u_1, u_2, u_8, u_3), (u_1, u_2, u_4, u_7), (a_2, u_5, u_2, u_7), (a_2, u_6, u_1, u_8)\}$. Then $(V, ((\mathscr{B}_1 \setminus T_1) \cup \mathscr{B}_2 \cup T_2))$ is a 4-cycle packing of $3G$ with leave $L_3 = (L_1 \setminus Q) \cup \{\{u_i, u_{i+4}\} \mid 1 \le i \le 4\}$. So $|L_3| = |L_1| - 2$.

*Cases (ii) and (iv.3).*
We can use Lemma 3.1 in this case, since $L_1$ contains $D$, and since $B_1$ contains a 4-cycle that uses 4 edges joining the vertices of degree 1 in $D$ (see the first sentence on page 115 of [3]).

*Cases (iii) and (iv.1–2).*
Here, $L_1$ contains two copies of $K_{1,3}$ (in Case (iii), this follows as described in Case (i) since again $|L_1| \ge \eta/2 + 2$, so in $L_1$ there are 3 vertices of degree 1 in $V_1$ that have a common neighbour). Furthermore, each such copy of $K_{1,3}$ contains a pair of vertices of degree 1, say $p_1 = \{u_1, u_3\}$ and $p_2 = \{u_2, u_4\}$ respectively, such that $\mathscr{B}_1$ contains the 4-cycle $(u_1, u_2, u_3, u_4)$. Now use Lemma 3.2.

Parts of both even and odd sizes

In this case we refer constantly to Section 5 of [3], adopting the notation defined therein. In particular, $S$ is a set of pairs of vertices, each pair containing two vertices from the same part; and if two such pairs contain vertices that occur in different parts, then $\mathscr{B}_1$ contains the 4-cycle induced by these pairs. Moreover, there exists a vertex $z$ such that $\{z, u\} \in L_1$ for each vertex $u$ in each pair in $S$.

As remarked in the second paragraph of Section 5 in [3], if $t$ is even then the leave is identical to the leave formed if all vertices in parts of even size are deleted. So the result follows from the previous case (an even number of parts, all of odd size). So we can assume that $t$ is odd, so therefore $t \equiv 3 \pmod 4$.

Following the cases in [3], we obtain the following.

$s = 1$ (page 120 of [3])

If $t \equiv 3 \pmod 8$ then $L_1$ contains the edges $\{z, u_1\}, \{z, u_2\}, \{z, u_3\}$, and $\{u_2, u_3\}$, where $\{z, u_2, u_3\} \subset O$ and $\{u_1\} \subset E$. So choose $L_2 = \{\{u_1, u_2\}, \{u_1, u_2\}\}$ and let $B_3 = B_1 \cup B_2 \cup \{(a, u_1, u_2, u_3)\}$. Then $|L_3| = |L_1| - 2$, so $(V, B_3)$ provides the required 4-cycle packing.

If $t \equiv 7 \pmod 8$ then we can simply use $B_2 \cup B_2$ since $|L_1| = v + 1$, so then $|L_3| = v + 3$ as required.

$s \ne 1$ and $t \equiv 3 \pmod 8$

If $|S| = 0$ then $L_1$ contains a copy of $K_3$ (see the last line on page 121 of [3]) with vertex $z$ joined to a vertex $u_1 \in E$, so the result follows here by using exactly the same argument in the case where $s = 1, t \equiv 3 \pmod 8$.

If $S$ contains two pairs from different parts, then

$$|L_1| = \begin{cases} \eta/2 + |S| - 1 & \text{if } |S| \in \{2,3\}, \text{ and} \\ v_1 + 1 & \text{if } |S| \geq 4. \end{cases}$$

So we can assume $|S| = 3$. Then by (ai) on page 121 of [3], all three pairs in $S$, say $\{u_1, u_5\}, \{u_2, u_4\}$, and $\{u_3, u_6\}$, occur in different parts.

Let $T_1 = \{(u_1, u_2, u_5, u_4), (u_1, u_3, u_5, u_6)\} \subseteq \mathscr{B}_1$. And let

$$T_2 = \{(u_1, u_2, z, u_6), (u_1, u_2, u_5, z), (z, u_3, u_5, u_4)\}.$$

Let $(V, \mathscr{B}_2)$ be a 4-cycle packing of $2G$ with leave $L_2 = \{\{u_1, u_2\}, \{u_1, u_2\}\}$. Then $(V, \mathscr{B}_3)$ is a 4-cycle packing of $3G$ where $\mathscr{B}_3 = (\mathscr{B}_1 \setminus T_1) \cup \mathscr{B}_2 \cup T_2$ with

$$L_3 = (L_1 \setminus \{\{z, u_i\} \mid 1 \leq i \leq 6\}) \cup \{\{\{u_1, u_i\} \mid 2 \leq i \leq 4\}, \{u_5, u_6\}\}.$$

Now suppose all pairs in $S$ belong to one part. Then page 123 of [3] details two cases. One results in the leave described in equation (1) of [3] in which $|L_1| = \eta/2 = M$ so has already been considered (just take $B_1 \cup B_2$), and the other has leave $L_1$ consisting of a copy of $K_3$ containing a vertex $z$ joined to a vertex $u_1 \in E$, so this again reverts to the case where $s = 1$ and $t \equiv 3 \pmod 8$, handled above.

$s \neq 1$ and $t \equiv 7 \pmod 8$

If $|S| = 2$ or $|S| \geq 4$ and $S$ contains two pairs from two different parts (see page 124 of [3]), instead of applying Lemma 5.6 of [3] to the graph formed from $K_{11}$ by deleting the two disjoint edges joining vertices in $p_1$ and $p_2$, with vertex set $\{z_i \mid 1 \leq i \leq 7\} \cup p_1 \cup p_2$, to form $B'_4$, we supplement the set $\tilde{B}_1$ of 4-cycles defined so far, in the following way.

Let $p_1 = \{u_1, u_3\}$ and $p_2 = \{u_2, u_4\}$. Let $(\{z_i \mid 1 \leq i \leq 7\} \cup \{u_1, u_2\}, T_1)$ be a 4-cycle system of $K_9$, let $((\{z_i \mid 2 \leq i \leq 7\}, \{u_3, u_4\}), T_2)$ be a 4-cycle system of $K_{6,2}$, and let $T_3 = \{(u_1, u_2, u_3, u_4)\}$.

Let $(V, \mathscr{B}_2)$ be a 4-cycle packing of $2G$ with leave $L_2 = \{\{u_1, u_2\}, \{u_1, u_2\}\}$. Then $(V, \tilde{B}_1 \cup T_1 \cup T_2 \cup T_3 \cup \mathscr{B}_2)$ is a 4-cycle packing of $3G$ with leave

$$L_3 = (L' \setminus \{\{z_1, z_2\}, \{z_1, z_3\}, \{z_2, z_3\}, \{u_3, u_4\}\}) \cup \{\{z_1, u_3\}, \{z_1, u_4\}\}.$$

So $|L_3| = |L'| - 2$. (See Fig. 1).

If $|S| \neq 3$ then let $(V, \mathscr{B}_1)$ be the 4-cycle packing of $G$ with leave $L_1$ where $\{\{y_1, z_2\}, \{y_2, z_1\}, \{z_1, z_2\}\} \subseteq L_1$. Then since we are assuming that $|L_1| \geq M + 2$, it follows from the top of page 125 in [3] that $|S| = 1$, say $S = \{p\}$, and one of the sets $S_1, \ldots, S_l$, say $S_1$, contains at most one pair that is in the same part as $p$. Then

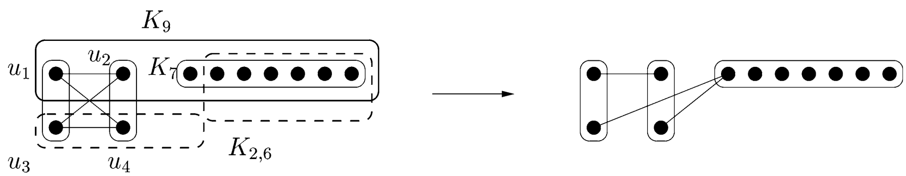


**Fig. 1.**

we may let $p = \{y_1, y_2\}$ and $S_1 = \{p_i = \{y_{2i+1}, y_{2i+2}\} \mid 1 \leq i \leq 4\}$, named so that possibly $p$ and $p_1$ occur in the same part, possibly $p_2$ and $p_3$ occur in the same part, but no other two pairs occur in the same part. If $p$ and $p_1$ occur in the same or different parts then let $b_1 = \emptyset$ or $\{(y_1, y_3, y_2, y_4)\}$ respectively, if $p_2$ or $p_3$ occur in the same or different parts then let $b_2 = \emptyset$ or $\{(y_5, y_7, y_6, y_8)\}$ respectively, and in any case let $b_3 = \{(y_1, y_{2i+1}, y_2, y_{2i+2}) \mid 2 \leq i \leq 4\}$. On page 122 of [3], conditions (1–2) show that $b_1 \cup b_2 \cup b_3 \subset \mathscr{B}_1$. Also in [3], the second last paragraph on page 121 shows that $B_1 \subset \mathscr{B}_1$ is a maximum 4-cycle packing of $K(4, 4, 1)$ with partition we can name $\{p_1 \cup p_4, p_2 \cup p_3, \{z_1\}\}$. Also, let $(V, \mathscr{B}_2)$ be a 4-cycle packing of $2G$ with leave $L_2 = \{\{y_4, y_8\}, \{y_4, y_8\}\}$.

If $B$ is a set of 4-cycles, then let $E(B)$ denote the multiset of all edges occuring in the 4-cycles in $B$. Then one can check that, using the graph $H(1, \ldots, 12)$ defined in 3.3, $E(H(z_2, z_1, y_1, y_2, \ldots, y_{10})) = E(B_1 \cup b_1 \cup b_2 \cup b_3) \cup L_1 \cup L_2$ (see Fig. 2). Lemma 3.3 can now be used to form a 4-cycle packing $T$ of $H(z_2, z_1, y_1, y_2, \ldots, y_{10})$ with leave consisting of 5 independent edges.

Therefore, we can replace $B_1 \cup b_1 \cup b_2 \cup b_3$ in $\mathscr{B}_1$ with $T$ to produce a 4-cycle packing of $3G$ with leave $L_3$, where $|L_3| = \eta/2$.

Finally, suppose $|S| = 3$. By our assumptions, there are two pairs in $V_1$ and a third pair in another part. Instead of applying Lemma 5.6 to the graph formed from $K_{11}$ by deleting the two disjoint edges joining vertices in $p_1$ and $p_2$, repeat the process described at the start of this subsection, when $|S| = 2$ or $|S| \geq 4$. □

## 4. The Cases $\lambda \geq 4$

We first deal with one exceptional case.

If $G = K(1, 1, n)$, then let $V_1 = \{u_1\}$, $V_2 = \{u_2\}$, and $V_3 = \{u_i \mid 3 \leq i \leq n + 2\}$. Clearly the edges joining $u_1$ to $u_2$ are in no 4-cycle in $\lambda G$, so these edges must occur in $L_\lambda$. Simply take the union of $\lfloor \lambda/2 \rfloor$ copies of a maximum 4-cycle packing of $2G$ with leave $\{\{u_1, u_2\}, \{u_1, u_2\}\}$ together with, if $\lambda$ is odd, a maximum 4-cycle

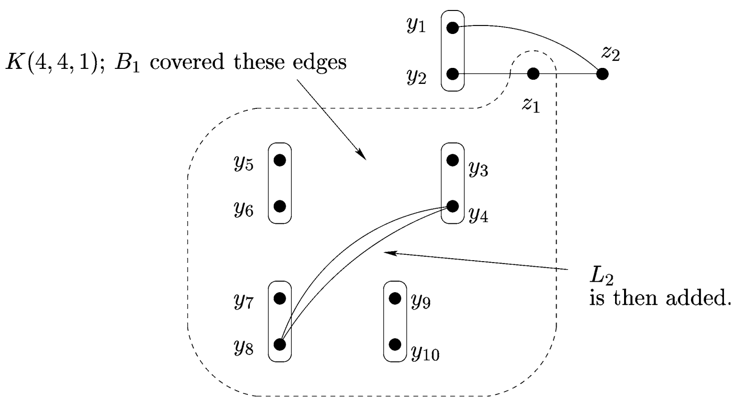


**Fig. 2.**

packing of $G$ with leave either $\{u_1, u_2\}$ (if $n$ is even) or $\{\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_1\}\}$ (if $n$ is odd).

Next consider the case $\lambda = 4$, for $G \neq K(1, 1, n)$ or $K(1, n)$. We shall show that there is a maximum packing of $4G$ with 4-cycles having leave $\emptyset$. Despite the fact that the case $\lambda = 2$ (in all but two exceptional cases) has leave either $2K_2$ or $\emptyset$, in order to obtain an empty leave when $\lambda = 4$, we have some work to do!

First, note that if $G$ contains no parts of size 1, then by [14], since $\lambda = 4$, we can take all pairs of parts and pack each bipartite subgraph with 4-cycles with empty leave.

So now suppose that $G$ contains at least one part of size 1.

If $G$ contains one part of size 1, since $G \neq K(1, n)$, it must contain at least three parts. If $G$ contains precisely three parts altogether, the other two parts are both of size greater than 1 (since $G \neq K(1, 1, n)$). We deal first with some small cases, in the following seven lemmas. These small cases have three or four parts, and at least one part of size 1.

**Lemma 4.1.** *There is a* 4-*cycle system of* $4K(1, 2, 2)$.

*Proof.* Let $V_1 = \{1\}$, $V_2 = \{2, 3\}$ and $V_3 = \{4, 5\}$. Take each of the following 4-cycles twice:

$$(1, 2, 4, 3),\ (1, 2, 5, 3),\ (1, 4, 3, 5),\ (1, 4, 2, 5).$$

□

**Lemma 4.2.** *There is a* 4-*cycle system of* $4K(1, 2, 3)$.

*Proof.* Let $V_1 = \{1\}$, $V_2 = \{2, 3\}$ and $V_3 = \{4, 5, 6\}$. Then $(V_1, V_2, V_3; B)$ is a 4-cycle system, where

$$B = \{(1, 2, 5, 3), (1, 2, 4, 3), (1, 2, 4, 3), (1, 2, 6, 3), (1, 5, 2, 6), (1, 5, 3, 6), \\ (1, 4, 2, 5), (1, 4, 2, 6), (1, 4, 3, 5), (1, 4, 3, 6), (2, 5, 3, 6)\}.$$

□

**Lemma 4.3.** *There is a* 4-*cycle system of* $4K(1, 3, 3)$.

*Proof.* Let $V_1 = \{\infty\}$, $V_2 = \{(0, 0), (1, 0), (2, 0)\}$ and $V_3 = \{(0, 1), (1, 1), (2, 1)\}$. For each $i \in \{0, 1, 2\}$ let $B = \{\{\infty, (i, 0), (i + 2, 1), (i + 1, 0)\}, \{\infty, (i, 0), (i, 1), (i + 1, 0)\}, \{\infty, (i, 1), (i, 0), (i + 1, 1)\}, \{\infty, (i, 1), (i + 1, 0), (i + 2, 1)\}, \{(i, 0), (i, 1), (i + 1, 0), (i + 1, 1)\}\}$, reducing each sum modulo 3. Then $(V_1, V_2, V_3; B)$ is the required 4-cycle system.

□

**Lemma 4.4.** *There is a* 4-*cycle system of* $4\ K(1,1,1,1)$.

*Proof.* Since $K(1, 1, 1, 1) = K_4$, this is trivial. □

**Lemma 4.5.** *There is a* 4-*cycle system of* $4K(1,1,1,2)$.

*Proof.* Let $V_1 = \{1\}$, $V_2 = \{2\}$, $V_3 = \{3\}$ and $V_4 = \{4,5\}$. Then $((V_1, V_2, V_3, V_4), B)$ is a 4-cycle system, where $B = \{(1,2,3,4), (1,2,4,3), (1,2,3,5), (1,2,5,3), (1,4,2,3), (1,4,2,5), (1,4,3,5), (1,5,2,3), (3,4,2,5).\}$ □

**Lemma 4.6.** *There is a* 4-*cycle system of* $4K(1,1,2,2)$.

*Proof.* Let $V_1 = \{1\}$, $V_2 = \{2\}$, $V_3 = \{3,4\}$ and $V_4 = \{5,6\}$. Then $((V_1, V_2, V_3, V_4), B)$ is a 4-cycle system, where

$$B = \{(1,2,5,3), (1,2,3,5), (1,2,3,6), (1,2,4,5), (1,3,6,4), (1,3,2,5), (1,3,6,4), (1,4,2,6), (1,4,2,6), (1,5,2,6), (2,3,5,4), (2,5,4,6), (3,5,4,6)\}.$$

□

**Lemma 4.7.** *There is a* 4-*cycle system of* $4K(1,2,2,2)$.

*Proof.* Let $|V_1| = 1$ and $|V_2| = |V_3| = |V_4| = 2$. Take 4-cycle systems of $2K(1,2,2)$ on: $\{V_1, V_2, V_3\}$, $\{V_1, V_2, V_4\}$, $\{V_1, V_3, V_4\}$, and 4-cycle systems of $2K(2,2)$ on: $\{V_2, V_3\}$, $\{V_2, V_4\}$,$\{V_3, V_4\}$. □

We can now deal with 4-fold complete multipartite graphs.

**Theorem 4.8.** *Let G be a complete multipartite graph. There exists a maximum* 4-*cycle packing of* 4G *with leave L where*

*(a)* *if* $G = K(1,n)$ *or* $K(1,1,1)$, *then* $L = E(4G)$,
*(b)* *if* $n > 1$ *and* $G = K(1,1,n)$, *then* $L = 4K_2$, *and*
*(c)* $L = \emptyset$ *otherwise.*

*Proof.* Clearly $K(1,n)$ and $K(1,1,1)$ have no 4-cycles. Also, if $G = K(1,1,n)$ where $n > 1$ then the result follows by taking 2 copies of a 2-fold maximum 4-cycle packing.

Otherwise, first suppose that $G$ has $n \geq 4$ parts. For $1 \leq i \leq 4$ let $W_i \subseteq V_i$ with $|W_i| = 1$ or 2 if $|V_i| \neq 2$ or $|V_i| = 2$ respectively. Let $((W_1, W_2, W_3, W_4), T)$ be a 4-cycle system of $4K(|W_1|, |W_2|, |W_3|, |W_4|)$ (see Lemmas 4.4–4.7). Then $(V, T \cup (\bigcup_{i=1}^{n} 4B(V_i \setminus W_i, (\bigcup_{j=1}^{i-1} W_j) \cup (\bigcup_{j=i+1}^{n} V_j))))$ is a 4-cycle system of $4G$.

Suppose that $G$ has three parts. Unless $G = K(1,2,3)$ or $K(1,3,3)$, for $1 \leq i \leq 3$ we can choose $W_i \subseteq V_i$ such that

(a) two of $W_1, W_2$ and $W_3$ have size 2 and one has size 1, and
(b) $|V_i \setminus W_i| \neq 2$.

A 4-cycle system of $4G$ is provided in these two exceptional cases in Lemmas 4.2 and 4.3. So in each other case, let $((W_1, W_2, W_3), T)$ be a 4-cycle system of $K(|W_1|, |W_2|, |W_3|)$ (see Lemma 4.1). Then $(V, T \cup (\bigcup_{i=1}^{3} 4B(V_i \setminus W_i, (\bigcup_{j=1}^{i-1} W_j) \cup (\bigcup_{j=i+1}^{3} V_j))))$ is a 4-cycle system of $4G$. □

**Corollary 4.9.** *Let G be a complete multipartite graph. There exists a* 4*-cycle system of* 4G if and only if $G \neq K(1,n)$ *and* $G \neq K(1,1,n)$.

For higher values of $\lambda = 4a + b$ where $0 \leq b < 4$, we may simply combine a maximum packing with $\lambda = b$ (see Theorems 1.1, 2.7 and 3.4) together with $a$ copies when $\lambda = 4$ (see Theorem 4.8).

We may summarise this as follows.

**Theorem 4.10.** *Let* $\lambda \geq 4$ *and let G be a complete multipartite graph. Let* $\eta(\lambda)$ *be the number of vertices of odd degree in* $\lambda G$*, and let* $v(\lambda)$ *be the number of vertices in the largest part of* $\lambda G$ *containing vertices of odd degree. There exists a maximum* 4*-cycle packing of* $\lambda G$ *with some leave* $L_\lambda$ *satisfying* $|L_\lambda| \leq max\{\eta(\lambda)/2 + 3, v(\lambda) + 3\}$*, except if*

*(i)* $G = K(1,n)$ *or* $K(1,1,1)$*, in which case* $L_\lambda = E(\lambda G)$;

*(ii)* $G = K(1,1,n)$ *with* $n > 1$*, in which case*

$$L_\lambda = \begin{cases} \lambda K_2 \vee K_1 & \text{if } n \text{ and } \lambda \text{ are odd, and} \\ \lambda K_2 & \text{otherwise;} \end{cases}$$

*(iii)* $\eta(\lambda) = 0$, $|E(\lambda G)| \equiv 1 \pmod 4$, *and* $G \neq K(1,1,n)$*, in which case* $|L_\lambda| = 5$.

## 5. Conclusion

We now can summarise our work as follows.

**Main Theorem.** *Let G be a complete multipartite graph. Let* $\eta(\lambda)$ *be the number of vertices of odd degree in* $\lambda G$*, and let* $v(\lambda)$ *be the number of vertices in the largest part of* $\lambda G$ *containing vertices of odd degree. There exists a maximum* 4*-cycle packing of* $\lambda G$ *with some leave* $L_\lambda$ *satisfying* $|L_\lambda| = l$ *if and only if*

*(i)* *if* $G = (1,n)$ *or* $K(1,1,1)$, then $L_\lambda = E(\lambda G)$,

*(ii)* *if* $G = K(1,1,n)$ *and* $n > 1$ *then*

$$L_\lambda = \begin{cases} \lambda K_2 \vee K_1 & \text{if } n \text{ and } \lambda \text{ are odd, and} \\ \lambda K_2 & \text{otherwise,} \end{cases}$$

*(iii)* *if* $\eta(\lambda) = 0$, $|E(\lambda G)| \equiv 1 \pmod 4$*, and* $G \neq K(1,1,n)$*, then* $|L_\lambda| = 5$,

*(iv)* *if* $\eta(\lambda) = 0$, $|E(\lambda G)| \equiv 2 \pmod 4$*, and* $\lambda = 1$*, then* $|L_\lambda| = 6$*, and otherwise*

*(v)* *l is the unique integer satisfying*
(1) $max\{\eta(\lambda)/2, v(\lambda)\} \leq l \leq max\{\eta(\lambda)/2 + 3, v(\lambda) + 3\}$*, and*
(2) 4 *divides* $|E(\lambda G)| - l$.

*Proof.* The necessity of conditions *(i)–(v)* was dealt with in Lemma 1.2. The sufficiency follows from Theorem 1.1 for $\lambda = 1$, Theorem 2.7 for $\lambda = 2$, Theorem 3.4 for $\lambda = 3$, and Theorem 4.10 for $\lambda \geq 4$. □

Clearly this research raises several interesting related questions. The method of attack used here suggests that finding all possible leaves of maximum packings may well be possible, although it is probably a lot of work, and perhaps not so easy to display. Moreover, the the minimum covering problem is a natural followup; it is likely that such problems can be attacked by using the results in this paper.

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