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Brief Paper

A Derivation of the Glover–Doyle Algorithms for General H^∞ Control Problems*

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Key Words—H^∞ control; control algorithms; coprime factorization.

Abstract—We show that the Glover–Doyle algorithm can be formulated simply by using the (J, J')-lossless factorization method and chain scattering matrix description. This algorithm was first stated by Glover and Doyle in 1988. Because the corresponding diagonal block of the (J, J')-lossless matrix in the general 4-block H^∞ control problem of the Glover–Doyle algorithm is not square, a new type of chain scattering matrix description is developed. With this description in hand, we obtain two types of state-space solution, which are similar to each other. Thus a similarity transformation between these solutions in the 4-block H^∞ control problem can also be obtained. The main idea of the solution is illustrated by means of block diagrams.

1. *Introduction*

Since Zames (1981) proposed the concept of sensitivity minimization in the H^∞ domain, many researchers have made valuable contributions to the study of the H^∞ domain. Transparent controllers for the standard 4-block H^∞ problem were not obtained until Glover and Doyle (1988, 1989) developed their well-known dual GD algorithms.

After Glover and Doyle (1989), Green *et al.* (1990) and Kimura (1991a) offered alternative developments using a J-spectral factorization, a characteristic of a (J, J')-lossless matrix. These methods are all based on the model-matching problem. Green (1992) combined an analytic system with J-lossless factorization to solve the H^∞ control problem, which gradually yielded a problem in the form of the model-matching problem. Using (J, J')-lossless factorization and a chain-scattering matrix description, Kimura (1991b) and Ball *et al.* (1991) gave a fictitious signal method for solving the 4-block case of the problem. Furthermore, Kondo and Hara (1990) and Tsai and Tsai (1993) obtained results similar to those of Green (1992).

However, in these papers the (1,1) block or the (2,2) block of the (J, J')-lossless matrix is required to be square or to need additional fictitious signals. Consequently, the results obtained by using the (J, J')-lossless factorization method to solve the H^∞ control problem were not the same as those obtained by the Glover–Doyle algorithms. In this paper we combine a normalized coprime factorization of the plant and (J, J')-factorization of one of the coprime factors, together with an alternative type of chain matrix description to recover precisely the results of Glover and Doyle (1988) (by using a left-coprime factorization) and Glover and Doyle (1989) (by using a right-copime factorization).

Despite the specific features of the two cases, the transfer functions for the resulting compensators turn out to be the same. We also obtain an explicit state-space similarity between the realizations for the two compensators thus obtained.

In Section 2 we briefly state the standard H^∞ control problem. The (J, J')-lossless, conjugate (J, J')-lossless and conjugate (J, J')-expansive matrices are also discussed. In Section 3 we develop alternative chain-scattering matrix descriptions, and discuss their chain properties. In Section 4 the relationship between the H^∞ control problem and the chain scattering matrix description is stated. The main results and the solution are presented in Section 5.

2. *Notation and preliminaries*

Throughout this paper $\mathbb{R}$ denotes the real numbers, RL^∞ denotes the set of proper real rational function matrices with no pole on the jw axis, and RH^∞ denotes the RL^∞ subspace with no poles in the right half-plane. Furthermore, $\mathcal{G}H^\infty$ denotes the units of RH^∞ (i.e. if $\Phi \in \mathcal{G}H^\infty$ then $\Phi \in RH^\infty$ and $\Phi^{-1} \in RH^\infty$) and $BH^\infty := \{\Phi \in RH^\infty \mid \|\Phi\|_\infty < 1\}$, $\gamma BH^\infty := \{\Phi \in RH^\infty \mid \|\Phi\|_\infty < \gamma\}$. dom (Ric) denotes the set of Hamiltonian matrices with no pure imaginary eigenvalues, and Ric (H) is the unique solution of the corresponding ARE of the Hamiltonian matrix H. $G^\sim(s)$ denotes $G^{\mathrm{T}}(-s)$ and $G^*(s)$ denotes $G^{\mathrm{T}}(\bar{s})$. As usual, the packed form $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is equivalent to $C(sI - A)^{-1}B + D$.

2.1. *The standard 4-block H^∞ control problem.* In the standard H^∞ framework, the transfer functions from $\begin{bmatrix} w \\ u \end{bmatrix}$ to $\begin{bmatrix} z \\ y \end{bmatrix}$ are denoted by

$$P \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \overset{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right].$$

where $z(t) \in \mathbb{R}^{p_1}$, $y(t) \in \mathbb{R}^{p_2}$, $w(t) \in \mathbb{R}^{m_1}$, and $u(t) \in \mathbb{R}^{m_2}$ are the error, observation, disturbance and control input respectively.

The suboptimal H^∞ control problem is then modeled so as to choose a controller K, connecting the observation vector y to u, such that K internally stabilizes the closed-loop system. Furthermore, the closed-loop transfer function, denoted by

$$F_l(P, K) \triangleq P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21},$$

satisfies the H^∞ norm bound

$$\|F_l(P, K)\|_\infty < \gamma, \quad \gamma \in \mathbb{R}^+.$$

For simplicity and without loss of generality of the derivations in subsequent sections, we let $\|F_l(P_\gamma, K)\|_\infty < 1$ instead of $\|F_l(P, K)\|_\infty < \gamma$, i.e.

$$F_l(P_\gamma, K) = \frac{1}{\gamma} F_l(P, K) = \frac{1}{\gamma} P_{11} + \frac{1}{\gamma} P_{12}K(I - P_{22}K)^{-1}P_{21}.$$

Figure 1 shows a general set-up for linear fractional transformation (LFT).

The assumption of the standard 4-block H^∞ control problem are as follows.

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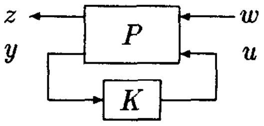

Fig. 1. The general set-up for linear fractional transformation (LFT).

Assumptions.

A1. (A, B_2) is stabilizable and (C_2, A) is detectable.

A2. rank $D_{12} = m_2$ and rank $D_{21} = p_2$.

A3. (a) $D_{12}[0 \quad I]^{\mathrm{T}}$, $D_{21} = [0 \quad I]$;

(b) $$D_{11} = \begin{array}{cc} \begin{bmatrix} D_{1111} & D_{1112} \\ D_{1121} & D_{1122} \end{bmatrix} & \begin{array}{l} \updownarrow\ p_1 - m_2 \\ \updownarrow\ m_2 \end{array} \\ \begin{array}{cc} \leftrightarrow & \leftrightarrow \\ m_1 - p_2 & p_2 \end{array} & \end{array}$$

A4. $D_{22} = 0$ (i.e. P_{22} is strictly proper).

A5. rank $\begin{bmatrix} A - \mathrm{j}\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + m_2 \quad \forall \omega \in \mathbb{R}$;

A6. rank $\begin{bmatrix} A - \mathrm{j}\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + p_2 \quad \forall \omega \in \mathbb{R}$.

In the above assumptions, as in the general 4-block H^{∞} control problem, the inequalities $m_1 > p_2$ and $p_1 > m_2$ must hold.

Under the conditions stated above, the main results of the Glover–Doyle algorithm are stated in Theorem 1 in Glover and Doyle (1988) and Theorem 4.1 in Glover and Doyle (1989).

2.2. *(J, J')-lossless, conjugate (J, J')-lossless and conjugate (J, J')-expansive matrices.* A partitioned matrix $\Theta(s) \in RL^{\infty}_{(m+r)\times(p+q)}$ is said to be a (J, J')-lossless or (J_{mr}, J_{pq})-lossless matrix if $m \geq p$, $r \geq q$ and

$$\begin{aligned} \Theta(s)^{\sim} J_{mr} \Theta(s) &= J_{pq} \quad \text{for each } s \in \mathrm{j}w, \\ \Theta(s)^{*} J_{mr} \Theta(s) &\leq J_{pq} \quad \text{for each } \mathrm{Re}\,[s] \geq 0, \end{aligned} \tag{1}$$

where

$$J_{mr} = \mathrm{diag}\,\{I_m, -I_r\}, \quad J_{pq} = \mathrm{diag}\,\{I_p, -I_q\}.$$

Also, $\Theta(s)$ is called conjugate (J_{mr}, J_{pq})-lossless if (1) holds and $\Theta J_{pq} \Theta^{*} \leq J_{mr}$ for each $\mathrm{Re}\,[s] \geq 0$. Finally, $\Theta(s)$ is called conjugate (J_{mr}, J_{pq})-expansive if (1) is satisfied and $\Theta J_{pq} \Theta^{*} \geq J_{mr}$ for each $\mathrm{Re}\,[s] \geq 0$.

Their relative properties are stated below. Here Lemma 1 is quoted from Kumura (1991a), and Lemmas 2 and 3 are extensions of Lemma 1 to stabilizable and observable realizations.

Lemma 1. Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in RL^{\infty}_{(m+r)\times(p+q)}$, with (A, B) controllable and (C, A) detectable. Then G is (J_{mr}, J_{pq})-lossless iff

(i) $A^{\mathrm{T}}X + XA + C^{\mathrm{T}}J_{mr}C = 0$;

(ii) $XB + C^{\mathrm{T}}J_{mr}D = 0$;

(iii) $D^{\mathrm{T}}J_{mr}D = J_{pq}$;

(iv) $X \geq 0$.

Lemma 2. Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in RL^{\infty}_{(m+r)\times(p+q)}$, with (C, A) observable and (A, B) stabilizable. Then G is conjugate (J_{mr}, J_{pq})-expansive iff

(i) $-AY - YA^{\mathrm{T}} + BJ_{pq}B^{\mathrm{T}} = 0$;

(ii) $DJ_{pq}B^{\mathrm{T}} - CY = 0$;

(iii) $DJ_{pq}D^{\mathrm{T}} = J_{mr}$;

(iv) $Y \geq 0$.

Lemma 3. Let $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in RL^{\infty}_{(m+r)\times(p+q)}$, with (C, A) observable and (A, B) stabilizable. Then G is conjugate (J_{mr}, J_{pq})-lossless iff

(i) $AY + YA^{\mathrm{T}} + BJ_{pq}B^{\mathrm{T}} = 0$;

(ii) $DJ_{pq}B^{\mathrm{T}} + CY = 0$;

(iii) $DJ_{pq}D^{\mathrm{T}} = J_{mr}$;

(iv) $Y \geq 0$.

2.3. *The (J, J')-lossless factorization.* Since any real rational proper matrix $G(s)$ has a right- and a left-coprime factorization, we have $G = \Theta\Pi^{-1} = \tilde{\Pi}^{-1}\tilde{\Theta}$, where $\Theta, \Pi, \tilde{\Theta}, \tilde{\Pi} \in RH^{\infty}$, and $\Pi(\infty)$ and $\tilde{\Pi}(\infty)$ are nonsingular.

What we shall investigate below is how to choose a particular state feedback gain matrix F, an observer gain matrix H, a scalar matrix W_v and a scalar matrix W_z such that Θ is (J, J')-lossless, $\tilde{\Theta}$ is conjugate (J, J')-expansive and $\Pi, \tilde{\Pi} \in \mathscr{G}H^{\infty}$. We can indeed find such matrices by the following lemmas, which link (J, J')-lossless factorization to the solution of Riccati equations. Here Lemma 4 is essentially from Tsai and Tsai (1993).

Lemma 4. Let $G \in RH^{\infty}_{(m+r)\times(p+q)}$. Then there exists a right-coprime factorization (r.c.f.) $G = \Theta\Pi^{-1}$ such that Θ is (J_{mr}, J_{pq})-lossless and $\Pi \in \mathscr{G}H^{\infty}_{p+q}$ iff

(i) there exists a nonsingular matrix W_v such that

$$W_v^{\mathrm{T}}D^{\mathrm{T}}J_{mr}DW_v = J_{pq};$$

(ii) $A_{H_v} \in \mathrm{dom}\,(\mathrm{Ric})$ and $V = \mathrm{Ric}\,(A_{H_v}) \geq 0$, where $R_v = D^{\mathrm{T}}J_{mr}D$ and

$$A_{H_v} =$$

$$\begin{bmatrix} A - BR_v^{-1}D^{\mathrm{T}}J_{mr}C & -BR_v^{-1}B^{\mathrm{T}} \\ -C^{\mathrm{T}}(J_{mr} - J_{mr}DR_v^{-1}D^{\mathrm{T}}J_{mr})C & -(A - BR_v^{-1}D^{\mathrm{T}}J_{mr}C)^{\mathrm{T}} \end{bmatrix},$$

$$F = -(D^{\mathrm{T}}J_{mr}D)^{-1}(B^{\mathrm{T}}V + D^{\mathrm{T}}J_{mr}C).$$

Lemma 5. Let $G \in RH^{\infty}_{(m+r)\times(p+q)}$. Then there exists a left-coprime factorization (l.c.f.) $G = \tilde{\Pi}^{-1}\tilde{\Theta}$ such that $\tilde{\Theta}$ is conjugate (J_{mr}, J_{pq})-expansive and $\tilde{\Pi} \in \mathscr{G}H^{\infty}_{m+r}$ iff

(i) there exists a nonsingular matrix W_z such that

$$W_zDJ_{pq}D^{\mathrm{T}}W_z^{\mathrm{T}} = J_{mr};$$

(ii) $A_{H_z} \in \mathrm{dom}\,(\mathrm{Ric})$ and $\hat{Z} = \mathrm{Ric}\,(A_{H_z}) \geq 0$, where $R_z = DJ_{pq}D^{\mathrm{T}}$ and

$$A_{H_z} = \begin{bmatrix} (A - BJ_{pq}D^{\mathrm{T}}R_z^{-1}C)^{\mathrm{T}} & C^{\mathrm{T}}R_z^{-1}C \\ B(J_{pq} - J_{pq}D^{\mathrm{T}}R_z^{-1}DJ_{pq})B^{\mathrm{T}} & -(A - BJ_{pq}D^{\mathrm{T}}R_z^{-1}C) \end{bmatrix},$$

$$H = (\hat{Z}C^{\mathrm{T}} - BJ_{pq}D^{\mathrm{T}})(DJ_{pq}D^{\mathrm{T}})^{-1}.$$

Proof. This lemma can be obtained directly from Lemma 2. □

3. *Alternative chain-scattering matrix description (CSMD)*

Much of this section is concerned with developing various types of chain scattering matrices for the cases where the square matrix is on the off-diagonal block. These types of chain scattering matrices combined with the (J, J')-lossless factorization method play a central role in our derivation of the GD algorithm. Since the location of the square matrix changes when we combine these types of CSMD with the (J, J')-lossless property, the characteristics of these CSMDs are quite different from those of traditional CSMDs.

Furthermore, the following various linear fractional transformations are defined only for those Ks such that the inverse appearing in the formula exists.

Type I. If

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \boxed{\Theta_{21}} & \Theta_{22} \end{bmatrix},$$

where Θ_{21} is square,

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}, \quad u = Ky,$$

then

$$z = (\Theta_{11} + \Theta_{12}K)(\Theta_{21} + \Theta_{22}K)^{-1}w \triangleq F_R^{(2,1)}(\Theta, K)w.$$

We use a subscript R here to label the right chain-scattering matrix description and a superscript $(2, 1)$ to indicate that $F_R(\Theta, K)$ refers to the location of the square matrix.

Type II. If

$$\Theta = \begin{bmatrix} \Theta_{11} & \boxed{\Theta_{12}} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

where Θ_{12} is square,

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}, \quad u = Ky,$$

then

$$w = (\Theta_{21}K + \Theta_{22})(\Theta_{11}K + \Theta_{12})^{-1}z \triangleq F_R^{(1,2)}(\Theta, K)z.$$

The superscript $(1, 2)$ indicates the location of the square matrix.

Type III. If Θ_{12} is square, and

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \boxed{\Theta_{12}} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}, \quad u = Ky$$

then

$$w = (\Theta_{12} - K\theta_{22})^{-1}(K\Theta_{21} - \Theta_{11})z \triangleq F_L^{(1,2)}(\Theta, K)z,$$

where L labels the left chain-scattering matrix description.

Type IV. If Θ_{12} is square, and

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \boxed{\Theta_{12}} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}, \quad u = Ky$$

then

$$z = (\Theta_{12} - K\Theta_{22})^{-1}(K\Theta_{21} - \Theta_{11})w \triangleq F_L^{(1,2)}(\Theta, K)w.$$

In the following lemmas, some properties of the above CSMDs (Types I–IV) are represented by the concept of an analytic system due to Green (1992), and are different from the traditional CSMDs. These properties are used to prove the sufficient condition of our main theorem.

Lemma 6. (*Type I.*) Assume that Θ is a (J_{mr}, J_{rq})-lossless matrix, in which Θ_{21} is square, and define

$$F_R^{(2,1)}(\Theta, \Phi) \triangleq (\Theta_{11} + \Theta_{12}\Phi)(\Theta_{21} + \Theta_{22}\Phi)^{-1}.$$

Then

$$\|\Phi\|_\infty > 1 \Rightarrow \|F_R^{(2,1)}(\Theta, \Phi)\|_\infty < 1.$$

Proof. Let

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \begin{bmatrix} I \\ \Phi \end{bmatrix}.$$

Then $F_R^{(2,1)}(\Theta, \Phi) = XY^{-1}$. Since Θ is (J_{mr}, J_{rq})-lossless, i.e. $\Theta^*J\Theta \le J$, and

$$[X^* \quad Y^*]J\begin{bmatrix} X \\ Y \end{bmatrix} = [I \quad \Phi^*]\Theta^*J_{mr}\Theta\begin{bmatrix} I \\ \Phi \end{bmatrix} \le [I \quad \Phi^*]J_{rq}\begin{bmatrix} I \\ \Phi \end{bmatrix},$$

we have

$$X^*X - Y^*Y \le I - \Phi^*\Phi$$

$$\Rightarrow Y^*[(Y^*)^{-1}X^*XY^{-1} - I]Y \le I - \Phi^*\Phi.$$

Thus

$$\|\Phi\|_\infty > 1 \Rightarrow \|F_R^{(2,1)}(\Theta, \Phi)\|_\infty < 1. \qquad \square$$

According to the same analytic method as that in Lemma 6, we can obtain Lemmas 7–9 as follows.

Lemma 7. (*Type III.*) Assume that Θ is a conjugate (J_{mr}, J_{pm})-expansive matrix, in which Θ_{12} is square. Define

$$F_L^{(1,2)}(\Theta, \Phi) \triangleq (\Theta_{12} - \Phi\Theta_{22})^{-1}(\Phi\Theta_{21} - \Theta_{11}).$$

Then

$$\|\Phi\|_\infty < 1 \Rightarrow \|F_L^{(1,2)}(\Theta, \Phi)\|_\infty > 1.$$

Lemma 8. (*Type IV.*) Assume that Θ is a conjugate (J_{mr}, J_{pm})-lossless matrix, in which Θ_{12} is square. Define $F_L^{(1,2)}(\Theta, \Phi)$ as in Lemma 7. Then

$$\|\Phi\|_\infty > 1 \Rightarrow \|F_L^{(1,2)}(\Theta, \Phi)\|_\infty < 1.$$

Lemma 9. (*Type II*). Assume that Θ is a (J_{mr}, J_{pm})-lossless matrix, in which Θ_{12} is square. Define

$$F_R^{(1,2)}(\Theta, \Phi) \triangleq (\Theta_{21}\Phi + \Theta_{22})(\Theta_{11}\Phi + \Theta_{12})^{-1}.$$

Then

$$\|\Phi\|_\infty < 1 \Rightarrow \|F_R^{(1,2)}(\Theta, \Phi)\|_\infty > 1.$$

The following lemma proposed by Walker (1990) states the relationship between the solutions of two algebraic Riccati equations (ARE) whose Hamiltonian matrices are related by a similarity transformation. There is thus also a similarity transformation property between these solutions. Since this property enables us to simplify the derivation and gives us the similarity transformation of H^∞ controllers, we rewrite this lemma below.

Lemma 10. Let

$$A_{H_y} = \begin{bmatrix} A_y & -R_y \\ -Q_y & -A_y^{\mathrm{T}} \end{bmatrix} \in \mathrm{dom}\,(\mathrm{Ric}),$$

and suppose that

$$A_{H_z} = \begin{bmatrix} A_z & -R_z \\ -Q_z & -A_z^{\mathrm{T}} \end{bmatrix}$$

is a Hamiltonian matrix given by

$$A_{H_z} = TA_{H_y}T^{-1},$$

where

$$T = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}, \quad X = X^{\mathrm{T}}. \tag{2}$$

If $I - XY > 0$ then $A_{H_z} \in \mathrm{dom}\,(\mathrm{Ric})$ and $\hat{Z} = Y(I - XY)^{-1}$ and $I + \hat{Z}X = (I - YX)^{-1}$, where $\hat{Z} = \mathrm{Ric}\,(A_{H_z})$ and $Y = \mathrm{Ric}\,(A_{H_y})$.

The results of the following derivation, which are related to the above lemma, will be used in our main results in Section 5. For explicitness we state this derivation below. By definition of $Y = \mathrm{Ric}\,(A_{H_y})$, we obtain

$$A_{H_y} = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \begin{bmatrix} A_y - R_yY & -R_y \\ 0 & -(A_y - R_yY)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Y & I \end{bmatrix}.$$

Thus, from (2), we have

$$\begin{bmatrix} I & 0 \\ \hat{Z} & I \end{bmatrix} \begin{bmatrix} A_z - R_z\hat{Z} & -R_z \\ 0 & -(A_z - R_z\hat{Z})^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I & 0 \\ -\hat{Z} & I \end{bmatrix} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix}$$
$$\times \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \begin{bmatrix} A_y - R_yY & -R_y \\ 0 & -(A_y - R_yY)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} I & 0 \\ -Y & I \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}.$$

This implies

$$(A_z - R_z\hat{Z})(I - XY) = (I - XY)(A_y - R_yY)$$

$$\Rightarrow (I + X\hat{Z})(A_z - R_z\hat{Z}) = (A_y - R_yY)(I + X\hat{Z}), \qquad (3)$$

$$(A_z - R_z\hat{Z})^{\mathrm{T}}(I + \hat{Z}X) = (I + \hat{Z}X)(A_y - R_yY)^{\mathrm{T}}. \qquad (4)$$

Furthermore, the above lemma entails that, if $X \geq 0$, $Y \geq 0$, $I - XY > 0$, $I - YX > 0$, and if there exists another Hamiltonian matrix A_{H_v} given by

$$A_{H_v} = \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} A_{H_x} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix},$$

with $V = \mathrm{Ric}\,(A_{H_v})$, A_{H_x} is a Hamiltonian matrix and $A_{H_x} \in \mathrm{dom}\,(\mathrm{Ric})$, $X = \mathrm{Ric}\,(A_{H_x})$, then

$$\hat{Z} = Y(I - XY)^{-1} \geq 0, \qquad (5)$$

$$V = X(I - YX)^{-1} \geq 0, \qquad (6)$$

$$I + \hat{Z}X = I + YV = (I - YX)^{-1}, \qquad (7)$$

$$(I + YV)(A_v - R_vV) = (A_x - R_xX)(I + YV). \qquad (8)$$

As we shall state in Section 5, (7) is the similarity transformation in the 4-block H^∞ controllers.

4. *The relationship between GD algorithms and CSMD*

Since our final objective in this paper is to derive the solutions in the distinct GD algorithms (Glover and Doyle 1988, 1989) simultaneously, and these solutions are related to the right- and left-coprime factorization of the augmented plant, we shall discuss both the right- and left-coprime cases.

4.1. *Case I: right-coprime case.* From Fig. 1, $P = NM^{-1}$, and

$$N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}, \quad M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad N, M \in RH^\infty.$$

We can obtain the structure shown in Fig. 2, with

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\begin{bmatrix} z' \\ w' \end{bmatrix}, \quad \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} z' \\ w' \end{bmatrix}.$$

For convenience, we define G_1 and G_2 as

$$\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix}\begin{bmatrix} z' \\ w' \end{bmatrix} \triangleq G_1\begin{bmatrix} z' \\ w' \end{bmatrix},$$

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} M_{21} & M_{22} \\ N_{21} & N_{22} \end{bmatrix}\begin{bmatrix} z' \\ w' \end{bmatrix} \triangleq G_2\begin{bmatrix} z' \\ w' \end{bmatrix},$$

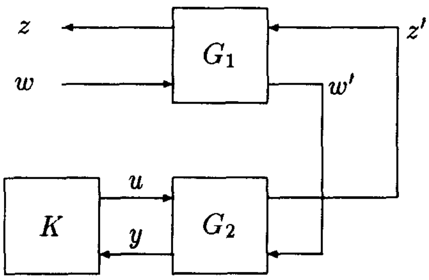

Fig. 3. The chain-scattering matrix description (CSMD) of the right-coprime case.

and depict this in Fig. 3. Thus the state-space form of N, M, G_1 and G_2 is

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A + BF & BW_a \\ \hline F & W_a \\ C + dF & DW_a \end{array}\right],$$

where

$$W_a = \begin{bmatrix} W_{a_{11}} & W_{a_{12}} \\ W_{a_{21}} & W_{a_{22}} \end{bmatrix}$$

is any nonsingular matrix. More precisely, we partition the matrices M and N in the forms

$$M = \left[\begin{array}{c|cc} A + bF & B_1W_{a_{11}} + B_2W_{a_{21}} & B_1W_{a_{12}} + B_2W_{a_{22}} \\ \hline F_1 & W_{a_{11}} & W_{a_{12}} \\ F_2 & W_{a_{21}} & W_{a_{22}} \end{array}\right],$$

$$N = \left[\begin{array}{c|cc} A + BF & B_1W_{a_{11}} + B_2W_{a_{21}} & B_1W_{a_{12}} + B_2W_{a_{22}} \\ \hline C_1 + D_{11}F_1 + D_{12}F_2 & D_{11}W_{a_{11}} + D_{12}W_{a_{21}} & D_{11}W_{a_{12}} + D_{12}W_{a_{22}} \\ C_2 + D_{21}F_1 & D_{21}W_{a_{11}} & D_{21}W_{a_{12}} \end{array}\right].$$

Then, from the preceding computations, we obtain G_1 and G_2 as

$$G_1 = \left[\begin{array}{c|cc} A + BF & B_1W_{a_{11}} + B_2W_{a_{21}} & B_1W_{a_{12}} + B_2W_{a_{22}} \\ \hline \begin{bmatrix} C_1 \\ O \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ I & O \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} & \begin{matrix} D_{11}W_{a_{11}} + D_{12}W_{a_{21}} \\ W_{a_{11}} \end{matrix} & \begin{matrix} D_{11}W_{a_{12}} + D_{12}W_{a_{22}} \\ W_{a_{12}} \end{matrix} \end{array}\right], \qquad (9)$$

$$G_2 = \begin{bmatrix} M_{21} & M_{22} \\ N_{21} & N_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A + BF & B_1W_{a_{11}} + B_2W_{a_{21}} & B_1W_{a_{12}} + B_2W_{a_{22}} \\ \hline F_2 & W_{a_{21}} & W_{a_{22}} \\ C_2 + D_{21}F_1 & D_{21}W_{a_{11}} & D_{21}W_{a_{12}} \end{array}\right]. \qquad (10)$$

where $G_1 \in RH^\infty_{(p_1+m_1)\times(m_1+m_2)}$ and $G_2 \in RH^\infty_{(m_2+p_2)\times(m_1+m_2)}$.

Remark 1. From Assumptions A1–A6, if we choose W_a as

$$W_a = \begin{bmatrix} (I - D_{11}^{\mathrm{T}}D_{11})^{-1}D_{11}^{\mathrm{T}}D_{12}[D_{12}^{\mathrm{T}}(I - D_{11}D_{11}^{\mathrm{T}})^{-1}D_{12}]^{-1/2} & (I - D_{11}^{\mathrm{T}}D_{11})^{-1/2} \\ [D_{12}^{\mathrm{T}}(I - D_{11}D_{11}^{\mathrm{T}})^{-1}D_{12}]^{-1/2} & 0 \end{bmatrix}$$

then G_1 (9), will be $(J_{p_1m_1}, J_{m_1m_2})$-lossless. Furthermore, if we rewrite G_1 as

$$G_1(s) = \left[\begin{array}{c|c} \hat{A} & BW_a \\ \hline \hat{C} & DW_a \end{array}\right],$$

where $\hat{A} = A + BF$ and $\hat{C} = C + DF$, then, from Lemma 1,

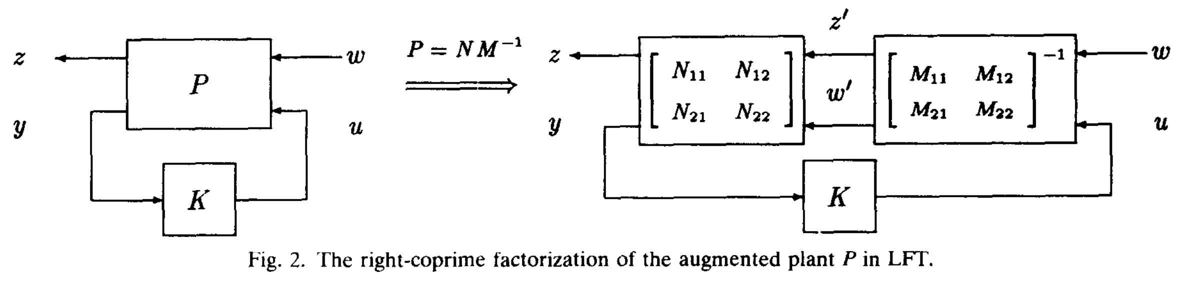

Fig. 2. The right-coprime factorization of the augmented plant P in LFT.

we have the following properties (as in Glover *et al.*, 1988, 1989):

$$\text{(i)}\ R = D^{\mathrm{T}}J_{p_1m_1}D = \begin{bmatrix} D_{11}^{\mathrm{T}} & I \\ D_{12}^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} I_{p_1} & 0 \\ O & -I_{m_1} \end{bmatrix}\begin{bmatrix} D_{11} & D_{12} \\ I & 0 \end{bmatrix}$$
$$= D_{1\cdot}^{\mathrm{T}}D_{1\cdot} - \begin{bmatrix} I_{m_1} & 0 \\ 0 & 0 \end{bmatrix};$$

$$\text{(ii)}\ XB + \hat{C}^{\mathrm{T}}JD = 0 \Rightarrow B^{\mathrm{T}}X + RF + D^{\mathrm{T}}JC = 0$$
$$\Rightarrow F = -R^{-1}(B^*X + D_{1\cdot}^*C_1), \quad \text{where } D_{1\cdot} = [D_{11} \quad D_{12}];$$

$$\text{(iii)}\ \hat{A}^{\mathrm{T}}X + X\hat{A} + \hat{C}^{\mathrm{T}}J\hat{C} = 0$$
$$\Rightarrow X(A - BR^{-1}D^{\mathrm{T}}JC) + (A - BR^{-1}D^{\mathrm{T}}JC)^{\mathrm{T}}X$$
$$- XBR^{-1}B^{\mathrm{T}}X + C^{\mathrm{T}}(J - JDR^{-1}D^{\mathrm{T}}J)C = 0.$$

From Section 1.4 in Glover and Doyle (1989), Assumptions A1 and A5 guarantee that the Hamiltonian matrix belongs to dom (Ric). So, from Lemma 4 in Doyle *et al.* (1989), if a Hamiltonian matrix H belongs to dom (Ric) then its solution $\mathrm{Ric}(H) \geq 0$, and thus the following solution exists:

$$X = \mathrm{Ric}\left(\begin{bmatrix} A - BR^{-1}D^{\mathrm{T}}JC & -BR^{-1}B^{\mathrm{T}} \\ -C^{\mathrm{T}}(J - JDR^{-1}D^{\mathrm{T}}J)C & -(A - BR^{-1}D^{\mathrm{T}}JC)^{\mathrm{T}} \end{bmatrix}\right)$$
$$\geq 0 \tag{11}$$

$$\Rightarrow X = \mathrm{Ric}(H_\infty) \geq 0.$$

where

$$H_\infty = \begin{bmatrix} A & 0 \\ -C_1^{\mathrm{T}}C_1 & -A^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} B \\ -C_1^{\mathrm{T}}D_{1\cdot} \end{bmatrix}R^{-1}[D_{1\cdot}^{T}C_1 \quad B^{\mathrm{T}}],$$

which is obtained by replacing (11) with (9).

4.2. *Case II*: *left-coprime case*. From Fig. 1 and $P = \hat{M}^{-1}\tilde{N}$, with

$$\tilde{M} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix}, \quad \tilde{N} = \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix},$$

and using a similar procedure as in Fig. 2, we obtain

$$\begin{bmatrix} z' \\ w' \end{bmatrix} = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix}\begin{bmatrix} z \\ y \end{bmatrix}, \quad \begin{bmatrix} z' \\ w' \end{bmatrix} = \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix}.$$

So

$$\left.\begin{aligned} z' &= \tilde{M}_{11}z + \tilde{M}_{12}y = \tilde{N}_{11}w + \tilde{N}_{12}u \\ w' &= \tilde{M}_{21}z + \tilde{M}_{22}y = \tilde{N}_{21}w + \tilde{N}_{22}u \end{aligned}\right\}$$

$$\Rightarrow \begin{cases} \tilde{N}_{11}w - \tilde{M}_{11}z = \tilde{M}_{12}y - \tilde{N}_{12}u \\ \tilde{M}_{21}z - \tilde{N}_{21}w = \tilde{N}_{22}u - \tilde{M}_{22}y \end{cases}.$$

Here we use two new variables w'' and z'' such that

$$\begin{aligned} w'' &= \tilde{N}_{11}w - \tilde{M}_{11}z = \tilde{M}_{12}y - \tilde{N}_{12}u, \\ z'' &= \tilde{M}_{21}z - \tilde{N}_{21}w = \tilde{N}_{22}u - \tilde{M}_{22}y. \end{aligned} \tag{12}$$

Thus we have

$$\begin{bmatrix} w'' \\ z'' \end{bmatrix} = \begin{bmatrix} \tilde{N}_{11} & -\tilde{M}_{11} \\ -\tilde{N}_{21} & \tilde{M}_{21} \end{bmatrix}\begin{bmatrix} w \\ z \end{bmatrix} \triangleq J_{p_1p_2}\tilde{G}_1J_{m_1p_1}\begin{bmatrix} w \\ z \end{bmatrix},$$

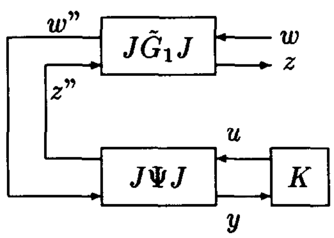

Fig. 4. The chain-scattering matrix description of the left-coprime case.

where

$$\tilde{G}_1 = \begin{bmatrix} \tilde{N}_{11} & \tilde{M}_{11} \\ \tilde{N}_{21} & \tilde{M}_{21} \end{bmatrix},$$

$$\begin{bmatrix} z'' \\ w'' \end{bmatrix} = \begin{bmatrix} \tilde{N}_{22} & -\tilde{M}_{22} \\ -\tilde{N}_{12} & \tilde{M}_{12} \end{bmatrix}\begin{bmatrix} u \\ y \end{bmatrix} \triangleq J_{p_2p_1}\Psi J_{m_2p_2}\begin{bmatrix} u \\ y \end{bmatrix},$$

where

$$\Psi = \begin{bmatrix} \tilde{N}_{22} & \tilde{M}_{22} \\ \tilde{N}_{12} & \tilde{M}_{12} \end{bmatrix}. \tag{13}$$

The block diagram for this left-coprime case is plotted in Fig. 4.

In precisely the same fashion as in Section 4.1, the left-coprime case $P = \tilde{M}^{-1}\tilde{N}$ implies

$$[\tilde{M} \quad \tilde{N}] = \left[\begin{array}{c|cc} A + HC & H & B + HD \\ \hline W_bC & W_b & W_bD \end{array}\right],$$

where

$$W_b = \begin{bmatrix} W_{b_{11}} & W_{b_{12}} \\ W_{b_{21}} & W_{b_{22}} \end{bmatrix}.$$

Thus $\tilde{G}_1$ and Ψ become

$$\tilde{G}_1 = \begin{bmatrix} \tilde{N}_{11} & \tilde{M}_{11} \\ \tilde{N}_{21} & \tilde{M}_{21} \end{bmatrix}$$
$$= \left[\begin{array}{c|c} A + HC & [B_1 \quad 0] + [H_1 \quad H_2]\begin{bmatrix} D_{11} & I \\ D_{21} & 0 \end{bmatrix} \\ \hline W_{b_{11}}C_1 + W_{b_{12}}C_2 & W_{b_{11}}D_{11} + W_{b_{12}}D_{21} \qquad W_{b_{11}} \\ W_{b_{21}}C_1 + W_{b_{22}}C_2 & W_{b_{21}}D_{11} + W_{b_{22}}D_{21} \qquad W_{b_{21}} \end{array}\right], \tag{14}$$

$$\Psi = \begin{bmatrix} \tilde{N}_{22} & \tilde{M}_{22} \\ \tilde{N}_{12} & \tilde{M}_{12} \end{bmatrix}$$
$$= \left[\begin{array}{c|cc} A + HC & B_2 + H_1D_{12} & H_2 \\ \hline W_{b_{21}}C_1 + W_{b_{22}}C_2 & W_{b_{21}}D_{12} & W_{b_{22}} \\ W_{b_{11}}C_1 + W_{b_{12}}C_2 & W_{b_{11}}D_{12} & W_{b_{12}} \end{array}\right], \tag{15}$$

where $\tilde{G}_1 \in RH^\infty_{(p_1+p_2)\times(m_1+p_1)}$ and $\Psi \in RH^\infty_{(p_2+p_1)\times(m_2+p_2)}$.

Remark 2. From Assumptions A1–A6, if we choose W_b as

$$W_b = \begin{bmatrix} [D_{21}(I - D_{11}^{\mathrm{T}}D_{11})^{-1}D_{21}^{\mathrm{T}}]^{-1/2}D_{21}D_{11}^{\mathrm{T}}(I - D_{11}D_{11}^{\mathrm{T}})^{-1} & [D_{21}(I - D_{11}^{\mathrm{T}}D_{11})^{-1}D_{21}^{\mathrm{T}}]^{-1/2} \\ (I - D_{11}D_{11}^{\mathrm{T}})^{-1/2} & 0 \end{bmatrix}$$

then $\tilde{G}_1$, (14), is conjugate $(J_{p_1p_2}, J_{m_1p_1})$-lossless. So, if we rewrite $\tilde{G}_1$ as

$$\tilde{G}_1(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline W_bC & W_bD \end{array}\right],$$

where $\hat{A} = A + HC$ and $\hat{B} = B + HD$, then from Lemma 3, we have the following properties:

$$\text{(i)}\ \tilde{R} = DJ_{m_1p_1}D^{\mathrm{T}} = \begin{bmatrix} D_{11} & I \\ D_{21} & 0 \end{bmatrix}\begin{bmatrix} I_{m_1} & 0 \\ O & -I_{p_1} \end{bmatrix}\begin{bmatrix} D_{11}^{\mathrm{T}} & D_{21}^{\mathrm{T}} \\ I & 0 \end{bmatrix}$$
$$= D_{\cdot 1}D_{\cdot 1}^{\mathrm{T}} - \begin{bmatrix} I_{p_1} & 0 \\ 0 & 0 \end{bmatrix};$$

$$\text{(ii)}\ CY + DJ\hat{B}^{\mathrm{T}} = 0 \Rightarrow YC^{\mathrm{T}} + H\tilde{R} + BJD^{\mathrm{T}} = 0$$
$$\Rightarrow H = -(B_1D_{\cdot 1}^* + YC^*)\tilde{R}^{-1}, \quad \text{where } D_{\cdot 1} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix};$$

$$\text{(iii)}\ \hat{A}Y + Y\hat{A}^{\mathrm{T}} + \hat{B}J\hat{B}^{\mathrm{T}} = 0$$
$$\Rightarrow (A - BJD^{\mathrm{T}}\tilde{R}^{-1}C)Y + Y(A - BJD^{\mathrm{T}}\tilde{R}^{-1}C)^{\mathrm{T}}$$
$$- YC^{\mathrm{T}}\tilde{R}^{-1}CY + B(J - JD^{\mathrm{T}}\tilde{R}^{-1}DJ)B^{\mathrm{T}} = 0.$$

As stated in Remark 1, Assumptions A1 and A6 guarantee that the Hamiltonian matrix belongs to Dom (Ric). Thus the following equation exists:

$$Y = \text{Ric}\left(\begin{bmatrix} (A - BJD^{\text{T}}\tilde{R}^{-1}C)^{\text{T}} & -C^{\text{T}}\tilde{R}^{-1}C \\ -B(J - JD^{\text{T}}\tilde{R}^{-1}DJ)B^{\text{T}} & -(A - BJD^{\text{T}}\tilde{R}^{-1}C) \end{bmatrix}\right) \geq 0 \qquad (16)$$

$$\Rightarrow Y = \text{Ric}\,(J_\infty) \geq 0.$$

where

$$J_\infty = \begin{bmatrix} A^{\text{T}} & 0 \\ -B_1B_1^{\text{T}} & -A \end{bmatrix} - \begin{bmatrix} C^{\text{T}} \\ -B_1D_{\cdot 1}^{\text{T}} \end{bmatrix}\tilde{R}^{-1}[D_{\cdot 1}B_1^{\text{T}} \quad C],$$

which is obtained by substituting (14) into (16).

5. *Main results*

As a summary of the discussion so far, we state the following important theorems, which are the main tools we use to derive the Glover–Doyle algorithm. The two theorems both describe the results of the GD algorithm, but from different points of view.

Case I: the right-coprime case.

Theorem 1. Under Assumptions A1–A6. Suppose that $P \in RL^\infty_{(p_1+p_2)\times(m_1+m_2)}$ has the specific right-coprime factorization

$$P = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}^{-1}$$

satisfying Remark 1. Then

(A) there exists an internally stabilizing controller K such that $\|F_l(P, K)\|_\infty < 1$ iff

(i) $G_1 := \begin{bmatrix} N_{11} & N_{12} \\ M_{11} & M_{12} \end{bmatrix}$ is $(J_{p_1m_1}, J_{m_1m_2})$-lossless,

(ii) $G_2 := \begin{bmatrix} M_{21} & M_{22} \\ N_{21} & N_{22} \end{bmatrix}$ has a left-coprime factorization $G_2 = \tilde{\Pi}^{-1}\tilde{\Theta}$ such that $\tilde{\Theta}$ is conjugate $(J_{m_2p_2}, J_{m_1m_2})$-expansive and $\tilde{\Pi} \in \mathcal{G}H^\infty_{m_2+p_2}$.

(B) if the conditions of (A) are satisfied then all real rational internally stabilizing controllers K such that $\|F_l(P, K)\|_\infty < 1$ are given by $K = F_L(\tilde{\Pi}, \Phi)$ $\forall\Phi \in BH^\infty$.

Proof of necessity. From Section 4.1 (the selection of G_1 and G_2) and Section 3 (Types I and III), we know that

$$\begin{aligned} F_l(P, K) &= F_l(NM^{-1}, K) \\ &= F_R^{(2,1)}(G_1, F_L^{(1,2)}(G_2, K)). \end{aligned}$$

So, from Lemma 6, we have $\|F_l(P, K)\|_\infty < 1$ if G_1 is $(J_{p_1m_1}, J_{m_1m_2})$-lossless and $\|F_L^{(1,2)}(G_2, K)\|_\infty > 1$. Since, from Remark 1, we have already obtained that G_1 is $(J_{p_1m_1}, J_{m_1m2})$-lossless, it remains to show that $\|F_L^{(1,2)}(G_2, K)\|_\infty > 1$.

By direct computation, we can verify that

$$A_{H_z} = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} J_\infty \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}, \qquad (17)$$

which implies that A_{H_z} is similar to J_∞, where X and J_∞ are as shown in Remarks 1 and 2 and A_{H_z} is obtained by the following computation. Rewrite G_2 in (10) as

$$G_2 = \left[\begin{array}{c|c} A_{G_2} & B_{G_2} \\ \hline C_{G_2} & D_{G_2} \end{array}\right]$$

and factorize it as $G_2 = \tilde{\Pi}^{-1}\tilde{\Theta}$ such that

$$[\tilde{\Pi} \quad \tilde{\Theta}] = \left[\begin{array}{c|cc} A_{G_2} + H_zC_{G_2} & H_z & B_{G_2} + H_zD_{G_2} \\ \hline W_zC_{G_2} & W_z & W_zD_{G_2} \end{array}\right] \qquad (18)$$

and

(i) $W_zD_{G_2}JD_{G_2}^{\text{T}}W_z^{\text{T}} = J,$ (19)

(ii) $R_z = D_{G_2}JD_{G_2}^{\text{T}},$

(iii) $A_{H_z} \in \text{dom (Ric)}$ and $\hat{Z} = \text{Ric}\,(A_{H_z}),$

$$A_{H_z} = \begin{bmatrix} (A_{G_2} - B_{G_2}JD_{G_2}^{\text{T}}R_z^{-1}C_{G_2})^{\text{T}} & C_{G_2}^{\text{T}}R_z^{-1}C_{G_2} \\ B_{G_2}(J - JD_{G_2}^{\text{T}}R_z^{-1}D_{G_2}J)B_{G_2}^{\text{T}} & -(A_{G_2} - B_{G_2}JD_{G_2}^{\text{T}}R_z^{-1}C_{G_2}) \end{bmatrix} \qquad (20)$$

(iv) $H_z = (\hat{Z}C_{G_2}^{\text{T}} - B_{G_2}JD_{G_2}^{\text{T}})R_z^{-1}.$

Thus, from (5) in Lemma 10, with $I - XY > 0$, we can also have the solution of A_{H_z}, i.e., $\hat{Z} = \text{Ric}\,(A_{H_z}) \geq 0.$

The above conditions show that $G_2 = \tilde{\Pi}^{-1}\tilde{\Theta}$ satisfies Lemma 5, i.e. $\tilde{\Theta}$ is conjugate $(J_{m_2p_2}, J_{m_1m_2})$-expansive and $\tilde{\Pi} \in \mathcal{G}H^\infty_{m_2+p_2}$. Now, if we choose K as $F_L(\tilde{\Pi}, \Phi)$ $\forall\Phi \in BH^\infty$ then

$$\begin{aligned} F_L^{(1,2)}(G_2, K) &= F_L^{(1,2)}(\tilde{\Pi}^{-1}\tilde{\Theta}, F_L(\tilde{\Pi}, \Phi)) \quad \forall\Phi \in BH^\infty \\ &= F_L^{(1,2)}(\tilde{\Theta}, \Phi). \end{aligned}$$

Therefore, from Lemma 7, we have $\|F_L^{(1,2)}(G_2, K)\|_\infty > 1$. This completes the proof of necessity of (A).

Proof of sufficiency. Since

$$\begin{aligned} F_l(P, K) &= F_R^{(2,1)}(G_1, F_L^{(1,2)}(G_2, K)) \\ &= F_R^{(2,1)}(G_1, F_L^{(1,2)}(\tilde{\Pi}^{-1}\tilde{\Theta}, F_L(\tilde{\Pi}, \Phi))), \quad \Phi \in BH^\infty \\ &= F_R^{(2,1)}(G_1, F_L^{(1,2)}(\tilde{\Theta}, \Phi)), \quad \Phi \in BH^\infty \end{aligned}$$

and G_1 is $(J_{p_1m_1}, J_{m_1m_2})$-lossless, $\tilde{\Theta}$ is conjugate $(J_{m_2p_2}, J_{m_1m_2})$-expansive. Thus, from Lemma 7, we have $\|\Phi\|_\infty < 1 \Rightarrow \|F_L^{(1,2)}(\tilde{\Theta}, \Phi)\|_\infty > 1$. Furthermore, from Lemma 6, $\|F_L^{(1,2)}(\tilde{\Theta}, \Phi)\|_\infty > 1 \Rightarrow \|F_R^{(2,1)}(G_1, F_L^{(1,2)}(\tilde{\Theta}, \Phi))\|_\infty < 1$, so $\|F_l(P, K)\|_\infty < 1$.

The reason that K is an internally stabilizing controller is as follows. Let P_{22} have a doubly coprime factorization as

$$P_{22} = \mathcal{N}\mathcal{M}^{-1} = \tilde{\mathcal{M}}^{-1}\tilde{\mathcal{N}}, \quad \begin{bmatrix} \tilde{U} & -\tilde{V} \\ -\tilde{\mathcal{N}} & \tilde{\mathcal{M}} \end{bmatrix}\begin{bmatrix} \mathcal{M} & \hat{V} \\ \mathcal{N} & \hat{U} \end{bmatrix} = I.$$

To see that the controller K is an internally stabilizing controller, let us consider the following computations. Redrawing Fig. 2 in Kondo and Hara (1990), we obtain Fig. 5. Rewriting (6.18) and (6.19) in Kondo and Hara (1990), we get

$$\begin{bmatrix} T_2 & T_1 \\ 0 & I \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ I & 0 \end{bmatrix}\begin{bmatrix} 0 & I \\ \mathcal{M} & \mathcal{M}\tilde{V}P_{21} \end{bmatrix}, \qquad (21)$$

$$\begin{bmatrix} I & 0 \\ 0 & T_3 \end{bmatrix} = \begin{bmatrix} \tilde{U} & -\tilde{V} \\ -\tilde{\mathcal{N}} & \tilde{\mathcal{M}} \end{bmatrix}\begin{bmatrix} 0 & I \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & I \\ \mathcal{M} & \mathcal{M}\tilde{V}P_{21} \end{bmatrix}. \qquad (22)$$

Figure 5 shows that the last term of each of the above equations will be cancelled in the closed-loop system. Hence, from Fig. 6, we see that the overall closed-loop system is consituted by the augmented plant and $F_L\left(\begin{bmatrix} \tilde{U} & -\tilde{V} \\ -\tilde{\mathcal{N}} & \tilde{\mathcal{M}} \end{bmatrix}, Q\right)$. Since we have not changed the structure of the augmented plant in our computation, $K = F_L(\tilde{\Pi}, \Phi)$ is thus equal to $K = F_L\left(\begin{bmatrix} \tilde{U} & -\tilde{V} \\ -\tilde{\mathcal{N}} & \tilde{\mathcal{M}} \end{bmatrix}, Q\right),$

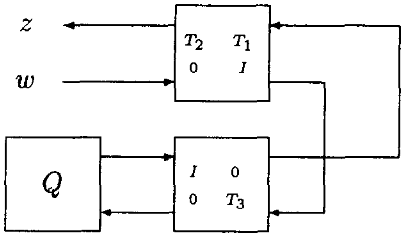

Fig. 5. The CSMD of model-matching problem for the 4-block H^∞ control problem.

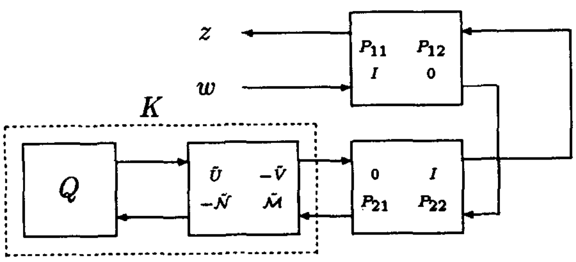

Fig. 6. The internally stabilizing controller of the CSMD for the 4-block H^∞ control problem.

$Q \in RH^\infty$. We conclude that K is an internally stabilizing controller, as in Doyle (1984). This completes the proof of sufficiency of (A).

Proof of (*B*). From the Youla parametrization, we know that all the internally stabilizing controllers can be represented in the form $K = F_L(\tilde{\Pi}, \Phi)\ \forall\Phi \in \mathrm{BH}^\infty$. □

Theorem 1 can be described graphically as in Fig. 7.

In Section 5.1, we shall show that Case I leads to the same result as in Glover and Doyle (1989).

Case II: *the left-coprime case.*

Theorem 2. Under Assumptions A1–A6, suppose that $P \in RL^\infty_{(p_1+p_2)\times(m_1+m_2)}$ has the specific left-coprime factorization

$$P = \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{21} & \tilde{M}_{22} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{N}_{11} & \tilde{N}_{12} \\ \tilde{N}_{21} & \tilde{N}_{22} \end{bmatrix}$$

satisfying Remark 2. Then

(A) there exists an internally stabilizing controller K such that $\|F_l(P, K)\|_\infty < 1$, iff

(i) $\tilde{G}_1 := \begin{bmatrix} \tilde{N}_{11} & \tilde{M}_{11} \\ \tilde{N}_{21} & \tilde{M}_{21} \end{bmatrix}$

is conjugate $(J_{p_1p_2}1, J_{m_1p_2})$-lossless,

(ii) $\Psi := \begin{bmatrix} \tilde{N}_{22} & \tilde{M}_{22} \\ \tilde{N}_{12} & \tilde{M}_{12} \end{bmatrix}$

has a right-coprime factorization $\Psi = \Theta\Pi^{-1}$ such that Θ is $(J_{p_2p_1}, J_{m_2p_2})$-lossless and $\Pi \in \mathcal{G}H^\infty_{m_2+p_2}$;

(B) if the conditions of (A) are satisfied then all the real rational internally stabilizing controllers K such that $\|F_l(P, K)\|_\infty < 1$ are given by $K = F_R(J\Pi, \Phi)\ \forall\Phi \in BH^\infty$.

Proof. This follows by the same lines as for the right-coprime case. □

Note that, by a similar computation to that in Theorem 1, we have the following properties:

$$A_{H_v} = \begin{bmatrix} I & -Y \\ 0 & I \end{bmatrix} H_\infty \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}, \tag{23}$$

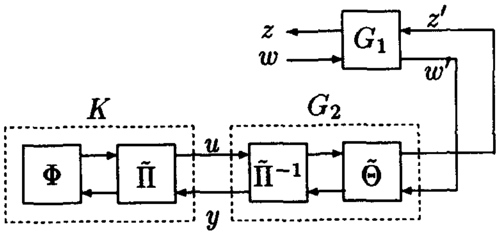

Fig. 7. The overall system of the CSMD for the right-coprime case.

where H_∞ and Y are shown in Remarks 1 and 2, and A_{H_v} is obtained as follows. Rewrite Ψ in (15) as

$$\Psi = \left[\begin{array}{c|c} A_\Psi & B_\Psi \\ \hline C_\Psi & D_\Psi \end{array}\right]$$

and let $\Psi = \Theta\Pi^{-1}$, where

$$\begin{bmatrix} \Pi \\ \Theta \end{bmatrix} = \left[\begin{array}{c|c} A_\Psi + B_\Psi F_v & B_\Psi W_v \\ \hline F_v & W_v \\ C_\Psi + D_\Psi F_v & D_\Psi W_v \end{array}\right], \tag{24}$$

and

(i) $W_v^{\mathrm{T}} D_\Psi^{\mathrm{T}} J D_\Psi W_v = J$; (25)

(ii) $R_v = D_\Psi^{\mathrm{T}} J D_\Psi$;

(iii) $A_{H_v} \in \mathrm{dom}\,(\mathrm{Ric})$ and $V = \mathrm{Ric}\,(A_{H_v}) \geq 0$, as we obtain in (6),

$$A_{H_v} = \begin{bmatrix} A_\Psi - B_\Psi R_v^{-1} J D_\Psi^{\mathrm{T}} J C_\Psi & -B_\Psi R_v^{-1} B_\Psi^{\mathrm{T}} \\ -C_\Psi^{\mathrm{T}}(J - J D_\Psi R_v^{-1} D_\Psi^{\mathrm{T}} J) C_\Psi & -(A_\Psi - B_\Psi R_v^{-1} J D_\Psi^{\mathrm{T}} J C_\Psi)^{\mathrm{T}} \end{bmatrix}; \tag{26}$$

(iv) $F_v = -R_v^{-1}(B_\Psi^{\mathrm{T}} V + D_\Psi^{\mathrm{T}} J C_\Psi)$.

Theorem 2 can be illustrated as shown in Fig. 8.

In Section 5.1, we shall show that Case II leads to the same result as in Glover and Doyle (1988).

5.1. *The derivation of the controller K_a.* In this subsection we show how to derive the controllers K_a of the GD algorithm by the relationship between $\tilde{\Pi}$ and $J\Pi$. Furthermore, we state the similarity transformation of these solutions; this also implies that the K_as in Glover and Doyle (1988, 1989) are the same. As we shall discuss, the controllers K_a can be found directly from the relationship between the structure of CSMD and the linear fractional transformation (LFT).

First, in the right-coprime case, where, from (10) and (18), we have

$$A_{G_2} = A + BF, \quad H_z = [H_{z_1} \quad H_{z_2}], \quad C_{G_2} = \begin{bmatrix} F_2 \\ C_2 + D_{21}F_1 \end{bmatrix},$$

$$\tilde{\Pi} = \left[\begin{array}{c|c} A + BF + H_{z_1}F_2 + H_{z_2}(C_2 + D_{21}F_1) & H_{z_1} \quad H_{z_2} \\ \hline W_z \begin{bmatrix} F_2 \\ C_2 + D_{21}F_1 \end{bmatrix} & W_z \end{array}\right]. \tag{27}$$

Similarly, in the left-corpime case, from (15) and (24), because

$$A_\Psi = A + HC, \quad B_\Psi = [B_2 + H_1D_{12} \quad H_2], \quad F_v = \begin{bmatrix} F_{v_1} \\ F_{v_2} \end{bmatrix},$$

we have

$$J\Pi = \left[\begin{array}{c|c} A + HC + (B_2 + H_1D_{12})F_{v_1} + H_2F_{v_2} & [B_2 + H_1D_{12} \quad H_2]Wv \\ \hline \begin{matrix} F_{v_1} \\ -F_{v_2} \end{matrix} & JW_v \end{array}\right]. \tag{28}$$

If we use Lemma 10 and substitute (17) and (20) into (4), we obtain

$$\begin{aligned}(I + \hat{Z}X)(A + HC) &= (A_{G_2} + H_zC_{G_2})(I + \hat{Z}X) \\ &= \left(A + BF + H_z\begin{bmatrix} F_2 \\ C_2 + D_{21}F_1 \end{bmatrix}\right)(I + \hat{Z}X).\end{aligned} \tag{29}$$

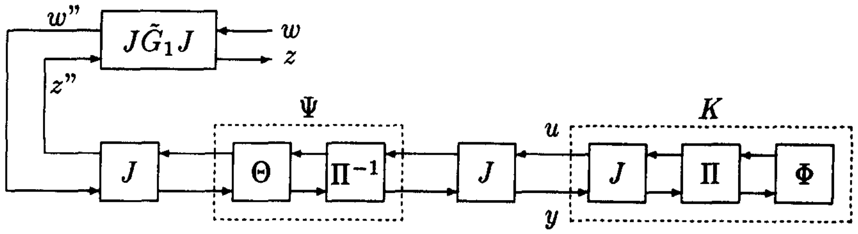

Fig. 8. The overall system of the CSMD for the left-coprime case.

Substituting (23) and (26) into (8), we obtain

$$\begin{aligned}(I+YV)(A_\Psi+B_\Psi F_v)&=(A+BF)(I+YV)\\ \Rightarrow(I+YV)(A+HC+[B_2+H_1D_{12}\quad H_2]F_v)&\\ &=(A+BF)(I+YV).\end{aligned}\tag{30}$$

Note that in (7), $I+\hat{Z}X=I+YV=(I-YX)^{-1}$. Thus, if we let $Z=I+\hat{Z}X=I+YV=(I-YX)^{-1}$ and substitute (29) into (30), we obtain

$$H_z\begin{bmatrix}F_2\\ C_2+D_{21}F_1\end{bmatrix}Z=-Z[B_2+H_1D_{12}\quad H_2]F_v.\tag{31}$$

One of the solutions of (31) is

$$\begin{aligned}[H_{z_1}\quad H_{z_2}]&=-Z[B_2+H_1D_{12}\quad H_2]J\\ &=[-Z(B_2+H_1D_{12})\quad ZH_2],\end{aligned}\tag{32}$$

$$\begin{bmatrix}F_{v_1}\\ F_{v_2}\end{bmatrix}=J\begin{bmatrix}F_2\\ C_2+D_{21}F_1\end{bmatrix}Z=\begin{bmatrix}F_2Z\\ -(C_2+D_{21}F_1)Z\end{bmatrix},\tag{33}$$

where $J=\operatorname{diag}\{I,-I\}$. Therefore, for the right-coprime case, if we let

$$W_z=\begin{bmatrix}\hat{D}_{12}^{-1} & -\hat{D}_{12}^{-1}\hat{D}_{11}\\ 0 & \hat{D}_{21}\end{bmatrix},$$

satisfying (19), where W_z can be obtained by properly choosing $\hat{D}_{11}$, $\hat{D}_{12}$ and $\hat{D}_{21}$ such that

$$\hat{D}_{12}\hat{D}_{12}^{\mathrm{T}}-\hat{D}_{11}\hat{D}_{11}^{\mathrm{T}}=[D_{12}^{\mathrm{T}}(I-D_{11}D_{11}^{\mathrm{T}})^{-1}D_{12}]^{-1},\quad \hat{D}_{21}\hat{D}_{21}^{\mathrm{T}}=I,$$

then (27) becomes

$$\tilde{\Pi}=\left[\begin{array}{c|cc}A+BF+H_{z_1}F_2+H_{z_2}(C_2+D_{21}F_1) & H_{z_1} & H_{z_2}\\ \hline \hat{D}_{12}^{-1}F_2-\hat{D}_{12}^{-1}\hat{D}_{11}(C_2+D_{21}F_1) & \hat{D}_{12}^{-1} & -\hat{D}_{12}^{-1}\hat{D}_{11}\\ \hat{D}_{21}(C_2+D_{21}F_1) & 0 & \hat{D}_{21}\end{array}\right].\tag{34}$$

The internally stabilizing controller in CSMD form is $F_L(\tilde{\Pi},\Phi)$. This needs to be transformed into LFT form as $F_l(K_a,\Phi)$ as shown in Fig. 9. If we rewrite $\tilde{\Pi}$ as

$$\tilde{\Pi}=\begin{bmatrix}\tilde{\Pi}_{11} & \tilde{\Pi}_{12}\\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22}\end{bmatrix}=\left[\begin{array}{c|cc}A_{\tilde{\Pi}} & B_{\tilde{\Pi}_1} & B_{\tilde{\Pi}_2}\\ \hline C_{\tilde{\Pi}_1} & D_{\tilde{\Pi}_{11}} & D_{\tilde{\Pi}_{12}}\\ C_{\tilde{\Pi}_2} & D_{\tilde{\Pi}_{21}} & D_{\tilde{\Pi}_{22}}\end{array}\right]$$

then, from Fig. 9, we can see that the CSMD form is

$$\begin{bmatrix}v\\ \sigma\end{bmatrix}=\begin{bmatrix}\tilde{\Pi}_{11} & \tilde{\Pi}_{12}\\ \tilde{\Pi}_{21} & \tilde{\Pi}_{22}\end{bmatrix}\begin{bmatrix}u\\ y\end{bmatrix}$$

and the LFT form is

$$\begin{bmatrix}u\\ \sigma\end{bmatrix}=\begin{bmatrix}-\tilde{\Pi}_{11}^{-1}\tilde{\Pi}_{12} & \tilde{\Pi}_{11}^{-1}\\ \tilde{\Pi}_{22}-\tilde{\Pi}_{21}\tilde{\Pi}_{11}^{-1}\tilde{\Pi}_{12} & \tilde{\Pi}_{21}\tilde{\Pi}_{11}^{-1}\end{bmatrix}\begin{bmatrix}y\\ v\end{bmatrix}=K_a*\begin{bmatrix}y\\ v\end{bmatrix}.\tag{35}$$

Therefore, we have the following state-space representation of K_a:

$$K_a=\left[\begin{array}{c|cc}A_{\tilde{\Pi}}-B_{\tilde{\Pi}_1}D_{\tilde{\Pi}_{11}}^{-1}C_{\tilde{\Pi}_1} & B_{\tilde{\Pi}_2}-B_{\tilde{\Pi}_1}D_{\tilde{\Pi}_{11}}^{-1}D_{\tilde{\Pi}_{12}} & B_{\tilde{\Pi}_1}D_{\tilde{\Pi}_{11}}^{-1}\\ \hline -D_{\tilde{\Pi}_{11}}^{-1}C_{\tilde{\Pi}_1} & -D_{\tilde{\Pi}_{11}}^{-1}D_{\tilde{\Pi}_{12}} & D_{\tilde{\Pi}_{11}}^{-1}\\ C_{\tilde{\Pi}_2}-D_{\tilde{\Pi}_{21}}D_{\tilde{\Pi}_{11}}^{-1}C_{\tilde{\Pi}_1} & D_{\tilde{\Pi}_{22}}-D_{\tilde{\Pi}_{21}}D_{\tilde{\Pi}_{11}}^{-1}D_{\tilde{\Pi}_{12}} & D_{\tilde{\Pi}_{21}}D_{\tilde{\Pi}_{11}}^{-1}\end{array}\right].\tag{36}$$

Now we use the same notation as in the Glover–Doyle algorithm, i.e.

$$K_a=\left[\begin{array}{c|cc}\hat{A} & \hat{B}_1 & \hat{B}_2\\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12}\\ \hat{C}_1 & \hat{D}_{21} & \hat{D}_{22}\end{array}\right],$$

and, replacing (36) by (34), we obtain

$$\begin{aligned}\hat{A}&=A+BF+\hat{B}_1\hat{D}_{21}^{-1}\hat{C}_2,\\ \hat{B}_1&=-ZH_2+\hat{B}_2\hat{D}_{21}^{-1}\hat{D}_{11},\\ \hat{B}_2&=Z(B_2+H_{12})\hat{D}_{12},\\ \hat{C}_1&=F_2+\hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2,\\ \hat{C}_2&=-\hat{D}_{21}(C_2+F_{12}),\end{aligned}\tag{37}$$

which are the same as in Theorem 4.1 in Glover and Doyle (1989).

Furthermore, for the left-coprime case, if

$$W_v=\begin{bmatrix}\hat{D}_{12} & \hat{D}_{11}\hat{D}_{21}^{-1}\\ 0 & -\hat{D}_{21}^{-1}\end{bmatrix}$$

satisfies

$$\hat{D}_{21}^{\mathrm{T}}\hat{D}_{21}-\hat{D}_{11}^{\mathrm{T}}\hat{D}_{11}=[D_{21}(I-D_{11}^{\mathrm{T}}D_{11})^{-1}D_{21}^{\mathrm{T}}]^{-1},\quad \hat{D}_{12}^{\mathrm{T}}\hat{D}_{12}=I,$$

then W_v satisfies (25). Thus from (28), we have

$$J\Pi=\left[\begin{array}{c|cc}A+HC+(B_2+H_1D_{12})F_{v_1}+H_2F_{v_2} & (B_2+H_1D_{12})\hat{D}_{12} & (B_2+H_1D_{12})\hat{D}_{11}\hat{D}_{21}^{-1}-H_2\hat{D}_{21}^{-1}\\ \hline F_{v_1} & \hat{D}_{12} & \hat{D}_{11}\hat{D}_{21}^{-1}\\ -F_{v_2} & 0 & \hat{D}_{21}^{-1}\end{array}\right].\tag{38}$$

We also have $K(s)=F_R(J\Pi,\Phi)$. Transforming this to LFT form, graphically, we obtain the result shown in Fig. 10, and

$$K_a=\left[\begin{array}{c|cc}A_\Pi-B_{\Pi_2}D_{\Pi_{22}}^{-1}C_{\Pi_2} & -B_{\Pi_2}D_{\Pi_{22}}^{-1} & B_{\Pi_1}-B_{\Pi_2}D_{\Pi_{22}}^{-1}D_{\Pi_{21}}\\ \hline C_{\Pi_1}-D_{\Pi_{12}}D_{\Pi_{22}}^{-1}C_{\Pi_2} & -D_{\Pi_{12}}D_{\Pi_{22}}^{-1} & D_{\Pi_{11}}-D_{\Pi_{12}}D_{\Pi_{22}}^{-1}D_{\Pi_{21}}\\ -D_{\Pi_{22}}^{-1}C_{\Pi_2} & -D_{\Pi_{22}}^{-1} & -D_{\Pi_{22}}^{-1}D_{\Pi_{21}}\end{array}\right].\tag{39}$$

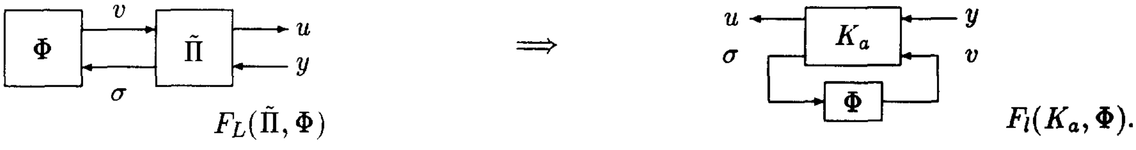

Fig. 9. The transformation of the internally stabilizing controller from left CSMD to LFT.

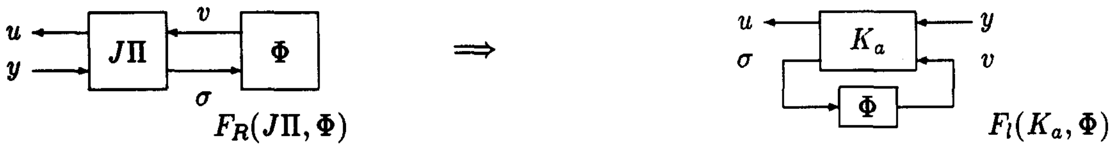

Fig. 10. The transformation of the internally stabilizing controller from right CSMD to LFT.

Using the same notation as in the GD algorithm and substituting (38) into (39), we obtain

$$\begin{aligned}\hat{A} &= A + HC + \hat{B}_2\hat{D}_{12}^{-1}\hat{C}_1,\\ \hat{B}_1 &= -H_2 + \hat{B}_2\hat{D}_{12}^{-1}\hat{D}_{11},\\ \hat{B}_2 &= (B_2 + H_{12})\hat{D}_{12},\\ \hat{C}_1 &= F_2Z + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2,\\ \hat{C}_2 &= -\hat{D}_{21}(C_2 + F_{12})Z,\end{aligned} \tag{40}$$

which is equivalent to Theorem 1 in Glover and Doyle (1988).

5.2. *The similarity transformation of the dual solutions.* We know that the transfer function of the dual solutions are equivalent. Thus there must exist a similarity transformation between these dual state-space solutions.

If we substitute (32) into (29), we have

$$\begin{aligned}Z(A + HC) &= (A_{G_2} + H_zC_{G_2})Z\\ &= \left(A + BF + H_z\begin{bmatrix} F_2 \\ C_2 + D_{21}F\end{bmatrix}\right)Z\\ &= (A + BF)Z - Z(B_2 + H_1D_{12})F_2Z\\ &\quad + ZH_2(C_2 + D_{21}F_1)Z.\end{aligned} \tag{41}$$

Substituting (33) into (30), we obtain

$$\begin{aligned}(A + BF)Z &= Z(A_\Psi + B_\Psi F_v)\\ &= Z(A + HC + [B_2 + H_1D_{12} \quad H_2]F_v)\\ &= Z(A + HC) + Z(B_2 + H_1D_{12})F_2Z\\ &\quad - ZH_2(C_2 + D_{21}F_1)Z.\end{aligned} \tag{42}$$

Thus, if we compare (37) and (40) with (41) and (42), we find that the similarity transformation between the controllers of the dual case is $Z = (I - YX)^{-1}$. Using subscripts 1988 and 1989 to denote the results in Glover and Doyle (1988, 1989), we have

$$\left[\begin{array}{c|c}\hat{A}_{1988} & \hat{B}_{1988}\\ \hline \hat{C}_{1988} & \hat{D}\end{array}\right] = \left[\begin{array}{c|c}Z\hat{A}_{1988}Z^{-1} & Z\hat{B}_{1988}\\ \hline \hat{C}_{1988}Z^{-1} & \hat{D}\end{array}\right] = \left[\begin{array}{c|c}\tilde{A}_{1989} & \tilde{B}_{1989}\\ \hline \tilde{C}_{1989} & \hat{D}\end{array}\right].$$

This relationship also means that the K_as in Glover and Doyle (1988, 1989) are the same.

6. *Conclusions*

We have combined coprime factorization and (J, J')-lossless factorization to derive the two distinct Glover–Doyle algorithms of Glover and Doyle (1988, 1989). We have also stated sufficient and necessary conditions for the existence of all controllers $K(s)$. Because the corresponding square matrix of the (J, J')-lossless matrix in the Glover–Doyle algorithms is not on the diagonal block, some alternative chain scattering matrix descriptions have been proposed. Furthermore, a similarity transformation between these standard 4-block H^∞ controllers has been given.

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