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Pattern formations and spatial entropy for spatially discrete diffusion equations

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Abstract

Formation of mosaic patterns for spatially discrete diffusion equations with cubic nonlinearity is investigated. We construct feasible basic patterns in each parameter region and combine these basic patterns into large patterns on one- and two-dimensional lattices. The basic patterns are characterized and constructed through formulating parameter conditions based on a geometrical setting. Spatial entropy associated with these patterns are computed or estimated. We also consider three typical boundary conditions and investigate their influences on pattern formations and spatial entropy. Several numerical computations are performed to illustrate such a formation of patterns.

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Keywords: Pattern formation; Spatial entropy; Spatial chaos; Lattice systems

1. Introduction

In this presentation, we investigate spatial patterns of the following spatially discrete diffusion equations:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \beta\Delta u_i + \alpha f(u_i), \quad \Delta u_i := u_{i+1} + u_{i-1} - 2u_i, \tag{1.1}$$

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where $i \in \Lambda_1 \subseteq \mathbb{Z}^1$, or

$$\frac{du_{i,j}}{dt} = \beta^+ \Delta^+ u_{i,j} + \beta^\times \Delta^\times u_{i,j} + \alpha f(u_{i,j}), \quad \Delta^+ u_{i,j} := u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j},$$
$$\Delta^\times u_{i,j} := u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1} - 4u_{i,j}, \tag{1.2}$$

where $(i, j) \in \Lambda_2 \subseteq \mathbb{Z}^2$, and Λ_1 and Λ_2 are connected subsets of $\mathbb{Z}^1$ and $\mathbb{Z}^2$, respectively. Herein, we consider a typical cubic nonlinearity

$$f(\xi) = \xi^3 - \xi. \tag{1.3}$$

The present approach can be extended to (1.1) and (1.2) with other nonlinearity and other lattice dynamical system, continuous-time or discrete-time.

Stationary solutions (patterns) constitute fundamental structure for differential equations. This presentation attempts to extend previous studies on lattice dynamical systems to further generality. Moreover, it is hoped to contribute toward treating the problems of allocating the parameters with which the considered system exhibits desirable patterns or some specific behaviors. Such problems are a kind of inverse problems and have been attracting much scientific interests. In this work, we are especially interested in a class of stationary patterns called mosaic patterns. We shall present a methodology for constructing mosaic patterns of the above systems. These patterns are characterized and constructed through formulating parameter conditions based on a geometrical setting. Stability of these patterns can also be investigated through estimating their basins of attraction, under further parameter conditions.

Formation of mosaic patterns and their spatial entropy for systems (1.1) and (1.2) have been investigated in [1–3], with the double-obstacle nonlinearity:

$$f(\xi) = \begin{cases} (-\infty, -\gamma] & \text{if } \xi = -1, \\ \gamma\xi & \text{if } |\xi| < 1, \\ [\gamma, \infty) & \text{if } \xi = 1, \\ \emptyset & \text{if } |\xi| > 1, \end{cases} \tag{1.4}$$

which is a set-valued function. The mosaic patterns and solutions therein take the value u_i or $u_{i,j} = -1, 0, 1$. Same considerations were adopted on Cahn–Hilliard equation in [4,5]. In this work, we employ the basic pattern formulation to discuss formation of mosaic patterns and spatial entropy for (1.1) and (1.2), with cubic nonlinearity (1.3). Our treatments are motivated by numerical spirit as well as the sense from real-world pattern formations. We consider the component of the stationary solutions to lie within small ranges, instead of being some single exact value, namely

$$u_i \text{ or } u_{i,j} \in [-1-\sigma, -1+\sigma] \cup [-\sigma, \sigma] \cup [1-\sigma, 1+\sigma], \tag{1.5}$$

where σ is a small number. Indeed, if a pattern in nature is represented by or is a presentation of certain quantities, these quantities are likely lying in small ranges, under a tolerance of error. The approach employed here is an extension from the work [6] on mosaic patterns of cellular neural networks. One first explores feasible basic patterns under various parameter conditions. These basic patterns are then combined through an attaching process to form patterns of larger sizes. The component y_i or $y_{i,j}$ of mosaic patterns (output patterns) in [6] takes the value $-1, 1$. Herein, the attaching process needs to be modified since components of the basic patterns to be overlapped may take different values, although they lie in the same interval in one of (1.5). We propose a fixed-point argument to assure the validity of such an attaching process. The performance of this fixed-point argument is based on our geometric formulation on the parameter conditions.

If Λ_1 or Λ_2 is finite, boundary conditions need to be imposed to have a well-defined system. An interested problem for systems (1.1) and (1.2) has been raised in [7]:

$$h = h_{\mathrm{N}} = h_{\mathrm{P}} = h_{\mathrm{D}}?$$

Herein, h denotes the spatial entropy, and h_{N}, h_{P} and h_{D}, respectively, represents the spatial entropy for the same type of patterns satisfying Neumann, periodic and Dirichlet boundary conditions. Such a problem has been investigated in [8] with examples from cellular neural networks. With the present approach, the effect of boundary conditions upon pattern formations and spatial entropy for (1.1) and (1.2) can be analogously investigated. Notably, only infinite lattices $\mathbb{Z}^1$, $\mathbb{Z}^2$, and thus no boundary effects, were considered in [1–3].

Other frequently considered nonlinearities for (1.1) and (1.2) include the cubic polynomial $f(\xi) = \gamma\xi + \xi^3$, $f(\xi) = (\xi^2 - 1)(\xi - a)$, and the logarithmic nonlinearity $f(\xi) = \gamma\xi + \ln[(1 + \xi)/(1 - \xi)]$ which restricts the range of its argument to $-1 < \xi < 1$. Our results can be adapted to (1.1) and (1.2) with these nonlinearities. It actually can be generalized to constructing stationary states of other lattice systems with components near finite number of specific values.

Lattice dynamical systems have been attracting great scientific interests, especially in chemical reactions [9], image processing and patterns recognition [10,11], material science [12,13], and biology [14,15,19].

As β in (1.1) or β^+, $\beta^\times$ in (1.2) is large, our results can be compared to the PDE case, namely the Allen–Cahn or the Nagumo equation:

$$\frac{\partial u}{\partial t} = d_\mu \Delta u + f(u), \tag{1.6}$$

on a one-dimensional interval domain with the Laplacian $\Delta u = \partial^2 u/\partial x^2$ or on a two-dimensional square domain with $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$, and with certain boundary conditions. In addition, discretization of partial differential equations and systems of partial differential equations can be regarded as lattice systems. Thus, the approach herein is also related to numerical solutions of the corresponding partial differential equations. There have been circuit implementations for simulating nonlinear PDEs via autonomous cellular neural networks [11]. Those PDEs include wave equations and reaction-diffusion equations. This study also provides a theoretical basis for pattern formation in these circuit implementations.

In the following, we write the spatially discrete diffusion equations as (sd-DE) as an abbreviation. The rest of this paper is organized as follows. In Section 2, we introduce a geometric formulation to partition the parameter space. Corresponding to each partitioned parameter region, there exists a collection of basic patterns. In Section 3, the basic patterns established in Section 2 are confirmed to be feasible basic patterns for (sd-DE), by applying a fixed-point theorem. One can then combine these basic patterns through an attaching process into mosaic patterns. We investigate stability of the mosaic patterns in Section 4. In Section 5, for mosaic patterns on one-dimensional lattice, transition matrices are formulated to describe the formation of patterns and compute the spatial entropy. In addition, the entropy for patterns on two-dimensional lattice is estimated. In Section 6, we investigate the influence of boundary conditions upon pattern formation as well as the problem: $h = h_{\mathrm{N}} = h_{\mathrm{P}} = h_{\mathrm{D}}$? We provide some numerical illustrations for two-dimensional patterns in Section 6.

2. Partitioning parameter space and basic patterns

In this section, we shall introduce the mosaic solutions and mosaic patterns for (1.1) and (1.2). The mosaic patterns are piled up through an attaching process on the so-called basic patterns. We propose a geometrical formulation to characterize the existence of basic patterns and derive the parameter conditions for such an existence. The methodology we propose is valid for systems (1.1) and (1.2) on both finite lattices and infinite lattices. The infinite lattices we consider herein is the whole $\mathbb{Z}^1$ or $\mathbb{Z}^2$. As a representative of finite lattices, we consider the rectangular ones:

$$\Lambda_1 = T_k = \{i \in \mathbb{Z}^1 | 1 \leq i \leq k\}, \tag{2.1}$$

$$\Lambda_2 = T_{\mathbf{k}} = \{(i, j) \in \mathbb{Z}^2 | 1 \leq i \leq k_1, 1 \leq j \leq k_2\}, \tag{2.2}$$

for cases $d = 1$ and $d = 2$, respectively, where k, k_1, k_2 are positive integers. The results herein can be extended to other lattices and lattices of higher dimensions.

For (1.1) on T_k or (1.2) on $T_{\mathbf{k}}$, boundary conditions need to be imposed so that the equations at boundary sites are well defined. There are three typical types of boundary conditions:

(i) Neumann boundary condition:

$$u_0 = u_1, \qquad u_{k+1} = u_k,$$

for $d = 1$. For $d = 2$, $0 \leq i \leq k_1 + 1$ and $0 \leq j \leq k_2 + 1$:

$$u_{0,j} = u_{1,j}, \qquad u_{k_1+1,j} = u_{k_1,j},$$
$$u_{i,0} = u_{i,1}, \qquad u_{i,k_2+1} = u_{i,k_2}.$$

(ii) Periodic boundary condition:

$$u_0 = u_k, \qquad u_{k+1} = u_1,$$

for $d = 1$. For $d = 2$, $0 \leq i \leq k_1 + 1$ and $0 \leq j \leq k_2 + 1$:

$$u_{0,j} = u_{k_1,j} \qquad u_{k_1+1,j} = u_{1,j},$$
$$u_{i,0} = u_{i,k_2} \qquad u_{i,k_2+1} = u_{i,1}.$$

(iii) Dirichlet boundary condition:

$$u_{\mathbf{i}} = \tilde{u}_{\mathbf{i}},$$

for $\mathbf{i}$ in the exterior neighbors $\mathbf{b}$ of the boundary sites, where $\tilde{u}_{\mathbf{i}}$ are prescribed data and $\mathbf{b} := \{0, k + 1\}$ if $d = 1$ and $\mathbf{b} := \{(i, 0), (0, j), (k_1 + 1, j), (i, k_2 + 1) \mid 0 \leq i \leq k_1 + 1, 0 \leq j \leq k_2 + 1\}$ if $d = 2$.

For convenience of discussion, the prescribed boundary data $\tilde{u}_{\mathbf{i}}$ also take the values as in (1.5). Systems (1.1) on T_k or (1.2) on $T_{\mathbf{k}}$ with the Neumann, periodic, and Dirichlet boundary conditions are denoted by $(\text{sd-DE})_{\text{N}}$, $(\text{sd-DE})_{\text{P}}$, and $(\text{sd-DE})_{\text{D}}$, respectively. These systems are regular ordinary differential equations on Euclidean spaces. Notably, (1.1) on infinite lattice $\mathbb{Z}^1$ or (1.2) on $\mathbb{Z}^2$ is a system of differential equations on infinite-dimensional vector space. Fundamental theory on existence and uniqueness of solutions for such systems can be found in [16]. Let $0 < \sigma < 1/11$ be a fixed number. The reason for requiring $\sigma < 1/11$ will be clear later.

Definition 2.1. We say that a stationary solution $\mathbf{u} = \{u_{\mathbf{i}}\}_{\mathbf{i} \in \Lambda_d}$ of (1.1) or (1.2) is a mosaic solution if

$$u_{\mathbf{i}} \in [-1 - \sigma, -1 + \sigma] \cup [-\sigma, \sigma] \cup [1 - \sigma, 1 + \sigma],$$

for all $\mathbf{i} \in \Lambda_d$. We denote by $\mathcal{M}_1^\sigma(\alpha, \beta)$ and $\mathcal{M}_2^\sigma(\alpha, \beta^+, \beta^\times)$ the set of all mosaic solutions for (1.1) with parameters α, β and (1.2) with parameters $\alpha, \beta^+, \beta^\times$, respectively.

We employ the symbols $\oplus, \ominus, \otimes$ to characterize such mosaic solutions. Restated, we call $\{s_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d}$ the corresponding *mosaic pattern* of a mosaic solution $\{u_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d}$, where

$$\begin{aligned} s_{\mathbf{i}} &= \oplus, \quad \text{if } 1-\sigma \le u_{\mathbf{i}} \le 1+\sigma. \\ s_{\mathbf{i}} &= \otimes, \quad \text{if } -\sigma \le u_{\mathbf{i}} \le \sigma, \\ s_{\mathbf{i}} &= \ominus, \quad \text{if } -1-\sigma \le u_{\mathbf{i}} \le -1+\sigma. \end{aligned} \tag{2.3}$$

We call a 1×3 (respectively, 3×3) array of $\oplus, \otimes, \ominus$, in the case $d=1$ (respectively, $d=2$), a *basic pattern*. There are totally 3^3 possible basic patterns in the case $d=1$ and 3^9 possible basic patterns in the case $d=2$, namely

$$\bullet\bullet\bullet, \quad \begin{matrix}\bullet\bullet\bullet\\ \bullet\bullet\bullet\\ \bullet\bullet\bullet\end{matrix}, \quad \bullet = \oplus, \otimes, \ominus.$$

We denote by $N_1(i)=\{i-1,i,i+1\}$, $N_2(i,j)=\{(i+1,j),(i-1,j),(i,j+1),(i,j-1),(i,j),(i+1,j+1),(i+1,j-1),(i-1,j+1),(i-1,j-1)\}$ the nearest neighbors of i and (i,j), respectively. Let $\mathbf{u}=\{u_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d}$ be a mosaic solution according to the above definition and let $\{s_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d}$ be the corresponding mosaic pattern. We call the projection (or restriction) of $\{s_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d}$ onto the nearest neighbors $N_1(i)$ for the case of $d=1$, and $N_2(i,j)$ for the case of $d=2$, a *feasible basic pattern*, for any interior sites i of Λ_1 and (i,j) of Λ_2, respectively.

A scheme for constructing mosaic patterns may go the other way around. If one can find out the feasible basic patterns for (1.1) and (1.2), then attaching these basic patterns compatibly produces patterns of larger sizes. Mosaic patterns can be obtained through such an attaching successively. This is basically the approach in [6] for constructing mosaic patterns of cellular neural networks. In cellular neural networks, a stationary solution $\mathbf{x}=\{x_{\mathbf{i}}\}$ is called mosaic if the output of $x_{\mathbf{i}}$ is either exactly 1 or -1. A successful attaching yields a corresponding solution automatically. The situation is different herein, as the component of $u_{\mathbf{i}}$ is only required to lie in a range as indicated in (2.3). We will discuss the attaching process and justify how such a process yields a solution in Section 3. We shall call those feasible basic patterns that can be confirmed by our theory in Section 3 "*affirmatively*" *feasible basic pattern*.

The crucial part in the above-mentioned pattern formation scheme is allocating the parameters in (1.1) or (1.2) to identify the existence of basic patterns. We take the case $d=1$ to illustrate the idea. The stationary equation for (1.1) is

$$\beta(u_{i+1}+u_{i-1}-2u_i)+\alpha f(u_i)=0. \tag{2.4}$$

We assume $\alpha\neq 0$ and set $b=\beta/\alpha$. For a fixed i, given u_{i-1} and u_{i+1}, u_i^* satisfies (2.4) if and only if there is an intersection (u_i^*, y^*) for curves

$$y=b[2u_i-u_{i-1}-u_{i+1}], \tag{2.5}$$

$$y=f(u_i), \tag{2.6}$$

cf. Fig. 1. Therefore, the configurations for the graphs of these two functions determine the existence of the feasible basic patterns.

Let us use the following example to illustrate the construction of basic patterns. Given $\tilde{u}_{i-1}, \tilde{u}_{i+1}\in[-\sigma,\sigma]$, if there is an intersection for (2.5) and (2.6) with $u_{i-1}=\tilde{u}_{i-1}$, $u_{i+1}=\tilde{u}_{i+1}$ at $u_i^*\in[1-\sigma,1+\sigma]$, then we have a candidate for feasible basic pattern $\overline{\underline{\otimes\oplus\otimes}}$ corresponding to the three tuple $(\tilde{u}_{i-1}, u_i^*, \tilde{u}_{i+1})$. In order to guarantee such an intersection, we need to restrict the value of b such that the graph of f between $L_1: y=2bx+2b\sigma$ and $L_2: y=2bx-2b\sigma$ lies entirely in the shadow region R which is bounded by $x=1+\sigma$ and $x=1-\sigma$, as indicated in Fig. 1. It can be computed that such an intersection always holds if $0\le b\le \frac{f(1+\sigma)}{2+4\sigma}$ or $-\frac{f(-1+\sigma)}{2}\le b\le 0$. With

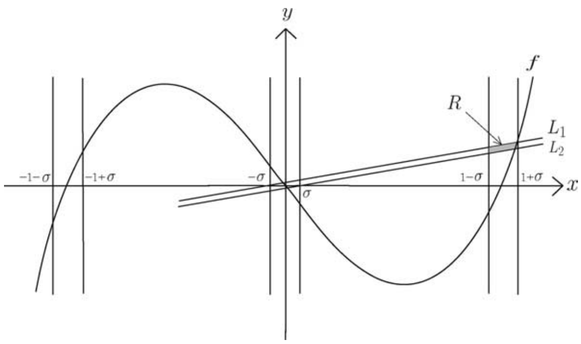

Fig. 1. Configuration of intersection for Eqs. (2.5) and (2.6).

our formulation, it will be shown in the next section that such candidates of feasible basic patterns will turn out to be real feasible basic patterns.

Through analyzing these geometrical configurations, we can characterize and classify the existence of all 27 basic patterns. The parameter space $\mathcal{P}_1 = \{b : b \in \mathbb{R}\}$ can be partitioned into finitely many regions so that (1.1) has the same collection of affirmatively feasible basic patterns for parameters in each region. Through computations, it is found that some feasible basic patterns exist in groups. We thus introduce the following notations:

$$B^{\bullet}_{\{m_1,m_2,\ldots,m_k\}} = \bigcup_{l=m_1,\ldots,m_k} B^{\bullet}_l,$$

where $B^{\bullet}_l$, $l = 0, \pm 1, \pm 2$, "$\bullet$" $= \oplus, \otimes, \ominus$, are described in Table 1. The superscript bullet "$\bullet$" herein means the symbol at the center of a basic pattern and the integer in the subscript indicates the states in its neighbor. Thorough computations yield the following classification for the existence of feasible basic patterns.

Theorem 2.2. *Suppose that $0 < \sigma < \frac{1}{11}$ is fixed. The parameter space $\mathcal{P}_1 = \{b : b \in \mathbb{R}\}$ can be partitioned so that the set of feasible basic patterns for (1.1) with (1.3) and parameters in each region contains the ones described in Table 2.*

The reason for considering $0 < \sigma < \frac{1}{11}$ is to avoid overlap of the partitioned intervals in Fig. 2. Confirmations for the feasibility of basic patterns in Theorem 2.2 are in fact completed in Section 3, in respecting our definition of feasible basic pattern. We remark that there may be other intersections for Eqs. (2.5) and (2.6) and thus other possibilities for the existence of feasible basic patterns for each set of parameters. Further partitioning of parameter space can be carried out to capture these possible intersections. The feasible basic patterns we list in Table 2 are the ones which can be confirmed by the theory in Section 3. We display, in the left half of Fig. 2, in each parameter

Table 1
Notations for collections of basic patterns, $\bullet = \oplus, \otimes, \ominus$

Notation	Basic patterns
$B^{\bullet}_2$	$\oplus \bullet \oplus$
$B^{\bullet}_1$	$\oplus \bullet \otimes$, $\otimes \bullet \oplus$
$B^{\bullet}_0$	$\oplus \bullet \ominus$, $\otimes \bullet \otimes$, $\ominus \bullet \oplus$
$B^{\bullet}_{-1}$	$\otimes \bullet \ominus$, $\ominus \bullet \otimes$
$B^{\bullet}_{-2}$	$\ominus \bullet \ominus$

Table 2
Affirmatively feasible basic patterns corresponding to each parameter region in the case $d = 1$

Parameter region	Affirmatively feasible basic patterns
$I_7 = \left[\frac{f(-1+\sigma)}{4\sigma}, \infty\right]$	$B^{\otimes}_{\{0\}}$
$I_6 = \left[\frac{f(1+\sigma)}{1+4\sigma}, \frac{f(-1+\sigma)}{4\sigma}\right]$	$B^{\oplus}_{\{2\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-2\}}$
$I_5 = \left[f(-\sigma), \frac{f(1+\sigma)}{1+4\sigma}\right]$	$B^{\oplus}_{\{2,1\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-1,-2\}}$
$I_4 = \left[\frac{f(1+\sigma)}{2+4\sigma}, f(-\sigma)\right]$	$B^{\oplus}_{\{2,1\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{-1,-2\}}$
$I_3 = \left[\frac{f(1+\sigma)}{3+4\sigma}, \frac{f(1+\sigma)}{2+4\sigma}\right]$	$B^{\oplus}_{\{2,1,0\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{0,-1,-2\}}$
$I_2 = \left[\frac{f(1+\sigma)}{4+4\sigma}, \frac{f(1+\sigma)}{3+4\sigma}\right]$	$B^{\oplus}_{\{2,1,0,-1,\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{1,0,-1,-2\}}$
$I_1 = \left[\frac{f(-\sigma)}{2}, \frac{f(1+\sigma)}{4+4\sigma}\right]$	$B^{\oplus}_{\{2,1,0,-1,-2\}}, B^{\otimes}_{\{2,1,0,-1,-2\}}, B^{\ominus}_{\{1,0,-1\}}$
$I_0 = \left[-\frac{f(-\sigma)}{2+4\sigma}, \frac{f(-\sigma)}{2}\right]$	$B^{\oplus}_{\{2,1,0,-1,-2\}}, B^{\otimes}_{\{2,1,0,-1,-2\}}, B^{\ominus}_{\{2,1,0,-1,-2\}}$
$I_{-1} = \left[-\frac{f(-1+\sigma)}{4}, -\frac{f(-\sigma)}{2+4\sigma}\right]$	$B^{\oplus}_{\{2,1,0,-1,-2\}}, B^{\otimes}_{\{2,1,0,-1,-2\}}, B^{\ominus}_{\{1,0,-1\}}$
$I_{-2} = \left[-\frac{f(-1+\sigma)}{3}, -\frac{f(-1+\sigma)}{4}\right]$	$B^{\oplus}_{\{2,1,0,-1,\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{1,0,-1,-2\}}$
$I_{-3} = \left[-\frac{f(-\sigma)}{1+4\sigma}, -\frac{f(-1+\sigma)}{3}\right]$	$B^{\oplus}_{\{2,1,0\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{0,-1,-2\}}$
$I_{-4} = \left[-\frac{f(-1+\sigma)}{2}, -\frac{f(-\sigma)}{1+4\sigma}\right]$	$B^{\oplus}_{\{2,1,0\}}, B^{\otimes}_{\{0,-1,-2\}}, B^{\ominus}_{\{0\}}$
$I_{-5} = \left[-f(-1+\sigma), -\frac{f(-1+\sigma)}{2}\right]$	$B^{\oplus}_{\{2,1\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-1,-2\}}$
$I_{-6} = \left[-\frac{f(-\sigma)}{4\sigma}, -f(-1+\sigma)\right]$	$B^{\oplus}_{\{2\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-2\}}$
$I_{-7} = \left[-\infty, -\frac{f(-\sigma)}{4\sigma}\right]$	$B^{\oplus}_{\{2\}}, B^{\ominus}_{\{-2\}}$

region, the existence of feasible basic patterns which can be confirmed by our treatment, and in the right half of Fig. 2, with further partitioning on the parameter space, the existence of all other possible basic patterns. We will address more on that as we estimate the entropy of the system in Section 5.

Let us also describe the partitioning of parameters and corresponding existence of basic patterns for the case of two-dimensional lattice, i.e., for Eq. (1.2). The formulation is analogous to the one-dimensional case. The stationary equation for (1.2) is

$$\beta^{+}\Delta^{+}u_{i,j} + \beta^{\times}\Delta^{\times}u_{i,j} + \alpha f(u_{i,j}) = 0, \tag{2.7}$$

for $(i, j) \in \Lambda_2 \subseteq \mathbb{Z}^2$. We assume $\alpha \neq 0$ and set $b_1 = \beta^{+}/\alpha$, $b_2 = \beta^{\times}/\alpha$. Then, for fixed (i, j), (2.7) holds if and only if there is an intersection for curves

$$y = -b_1\Delta^{+}u_{i,j} - b_2\Delta^{\times}u_{i,j}, \tag{2.8}$$

$$y = f(u_{i,j}). \tag{2.9}$$

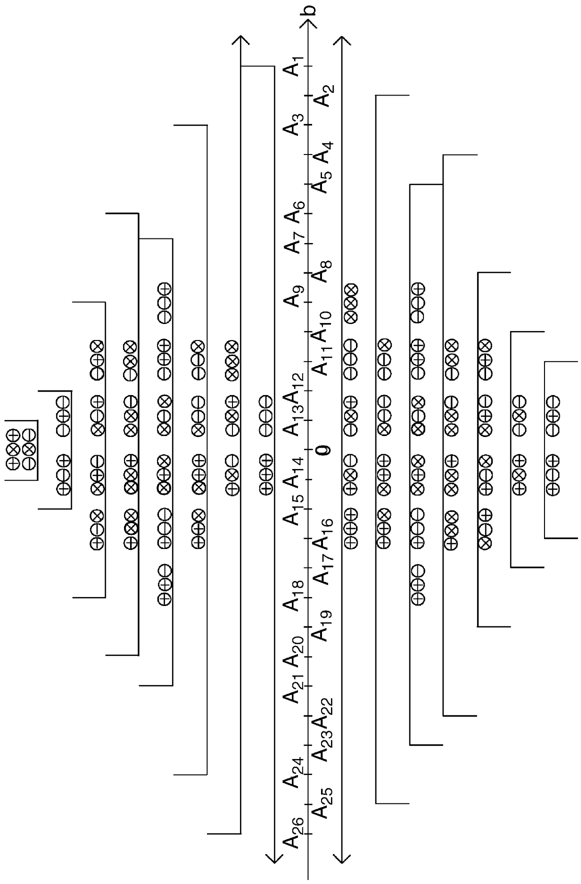

Fig. 2. Partition of parameter space and feasible basic patterns. $A_1 = \frac{f(-1+\sigma)}{4\sigma}$, $A_2 = f(1+\sigma)$, $A_3 = \frac{f(1+\sigma)}{1+4\sigma}$, $A_4 = \frac{f(-\sigma)}{1-4\sigma}$, $A_5 = \frac{f(1+\sigma)}{2}$, $A_6 = f(-\sigma)$, $A_7 = \frac{f(1+\sigma)}{2+4\sigma}$, $A_8 = \frac{f(1+\sigma)}{3}$, $A_9 = \frac{f(1+\sigma)}{3+4\sigma}$, $A_{10} = \frac{f(-\sigma)}{2-4\sigma}$, $A_{11} = \frac{f(1+\sigma)}{4}$, $A_{12} = \frac{f(1+\sigma)}{4+4\sigma}$, $A_{13} = \frac{f(-\sigma)}{2}$, $A_{14} = -\frac{f(-\sigma)}{2+4\sigma}$, $A_{15} = -\frac{f(-1+\sigma)}{4}$, $A_{16} = -\frac{f(-1+\sigma)}{4-4\sigma}$, $A_{17} = -\frac{f(-\sigma)}{2}$, $A_{18} = -\frac{f(-1+\sigma)}{3}$, $A_{19} = -\frac{f(-1+\sigma)}{3-4\sigma\sigma}$, $A_{20} = -\frac{f(-\sigma)}{1+4\sigma}$, $A_{21} = -\frac{f(-1+\sigma)}{2}$, $A_{22} = -f(-\sigma)$, $A_{23} = -\frac{f(-1+\sigma)}{2-4\sigma}$, $A_{24} = -f(-1+\sigma)$, $A_{25} = -\frac{f(-1+\sigma)}{1-4\sigma}$, $A_{26} = -\frac{f(-\sigma)}{4\sigma}$.

The parameter space can be partitioned so that the set of feasible basic patterns for (1.2) with (1.3) and parameters in each partitioned region are the same as the case $d = 1$.

3. From basic patterns to mosaic patterns

Let us describe the attaching process on the basic patterns and justify that the process indeed produces corresponding solutions for (1.1) and (1.2), under our setting and formulations in Section 2. Consider two basic patterns $\mathbf{s}_p = \overline{\underline{\bullet\ p_1\ p_2}}$ and $\mathbf{s}_q = \overline{\underline{q_1\ q_2\ \bullet}}$, "$\bullet$", $p_1, p_2, q_1, q_2 = \oplus, \otimes, \ominus$. We say that the basic pattern $\mathbf{s}_q$ can be attached, with two sites overlapped, to the right of basic pattern $\mathbf{s}_p$, if $q_1 = p_1, q_2 = p_2$. For example, attaching $\mathbf{s}_q = \overline{\underline{\oplus \ominus \ominus}}$ to the right of $\mathbf{s}_p = \overline{\underline{\ominus \oplus \ominus}}$ with two sites overlapped, yields $\overline{\underline{\ominus \oplus \ominus\ominus}}$. Continuing the attaching process produces mosaic patterns of any size. However, such a construction for patterns of larger sizes from patterns of smaller sizes through attaching does not automatically produce mosaic solutions to (1.1) and (1.2). Indeed, the value corresponding to symbol p_1 (respectively, p_2) is only known to lie in an interval of length 2σ; thus, it is not assured a priori whether if this value is exactly equal to the value corresponding to symbol q_1 (respectively, q_2) . Nevertheless, such a construction of mosaic patterns can be confirmed through a fixed-point theorem and our formulation on the existence of feasible basic patterns described in Section 2.

Theorem 3.1. *Assume that* $0 < \sigma < \frac{1}{11}$ *is fixed. Let* $\{s_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d}$ *be an array of symbols* $\oplus, \otimes, \ominus$ *(i.e.,* $s_{\mathbf{i}} = \oplus, \otimes, \ominus$*), obtained from the above attaching process on a collection of basic patterns corresponding to a single partitioned parameter region. Then, there exists a mosaic solution* $\mathbf{u} = \{u_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d}$ *to (1.1) or (1.2). Moreover, in terms of symbols,* $\mathbf{u}$ *is exactly represented by* $\{s_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d}$ *so that* $\{s_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d}$ *is indeed a mosaic pattern for (1.1) or (1.2).*

Proof. We present the case $d = 1$. Assume that Λ_1 is a finite lattice. Let $\{s_i\}_{i\in\Lambda_1}$ be an array of $\oplus, \otimes, \ominus$, obtained from the attaching process on the collection of basic patterns corresponding to a partitioned parameter region. Let $\{\tilde{u}_i\}_{i\in\Lambda_1}$ be an array of real numbers with

$$\begin{aligned} &\tilde{u}_i \in [-1-\sigma, -1+\sigma], && \text{if } s_i = \ominus,\\ &\tilde{u}_i \in [-\sigma, \sigma], && \text{if } s_i = \otimes,\\ &\tilde{u}_i \in [1-\sigma, 1+\sigma], && \text{if } s_i = \oplus. \end{aligned} \tag{3.1}$$

According to our previous formulations, there always exists an intersection for line (2.5) and curve (2.6). Restated,

$$\begin{aligned} y_i &= b[2u_i - \tilde{u}_{i-1} - \tilde{u}_{i+1}],\\ y_i &= f(u_i), \end{aligned}$$

always have an intersection (u_i^*, y_i^*) for each $i \in \Lambda_1$ with

$$\begin{aligned} &u_i^* \in [-1-\sigma, -1+\sigma], && \text{if } s_i = \ominus,\\ &u_i^* \in [-\sigma, \sigma], && \text{if } s_i = \otimes,\\ &u_i^* \in [1-\sigma, 1+\sigma], && \text{if } s_i = \oplus. \end{aligned} \tag{3.2}$$

Notably, if $i \in \Lambda_1$ with $i+1$ or $i-1 \notin \Lambda_1$, then $\tilde{u}_{i+1}$ or $\tilde{u}_{i-1}$ should be interpreted from boundary condition. Set

$$\begin{aligned} \mathcal{V} = \{&\{v_i\}_{i\in\Lambda_1} : -1-\sigma \le v_i \le -1+\sigma, \text{if } s_i = \ominus, \quad -\sigma \le v_i \le \sigma, \text{if } s_i = \otimes,\\ &1-\sigma \le v_i \le 1+\sigma, \text{if } s_i = \oplus\}. \end{aligned} \tag{3.3}$$

Define a mapping $G : \mathcal{V} \to \mathcal{V}$ which maps the given $\{\tilde{u}_i\}_{i\in\Lambda_1}$ in (3.1) to $\{u_i^*\}_{i\in\Lambda_1}$ in (3.2). G is obviously continuous. It follows from the Brouwer's fixed-point theorem that there exists a fixed point $\bar{\mathbf{u}} = \{\bar{u}_i\}_{i\in\Lambda_1}$ for G. This fixed point $\bar{\mathbf{u}}$ is exactly a stationary solution to (1.1). Moreover, $\bar{\mathbf{u}}$ is represented by the array of symbols $\{s_i\}_{i\in\Lambda_1}$ and thus $\{s_i\}_{i\in\Lambda_1}$ is exactly a mosaic pattern for (1.1). If Λ_1 is an infinite lattice, for example, $\Lambda_1 = \mathbb{Z}^1$, then the phase space for (1.1) is

$$\mathcal{X} = \{\mathbf{u} = \{u_i\}_{i\in\mathbb{Z}^1}, \|\mathbf{u}\| < \infty\}.$$

Under the circumstances, the existence of fixed point for G can be confirmed by the Schauder fixed-point theorem with a suitable topology (norm) on $\mathcal{X}$. □

4. Stability of mosaic patterns

In this section, we study the stability of the mosaic solutions obtained in Section 3. Let $\bar{\mathbf{u}} = \{u_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d} \in \mathcal{M}_d^\sigma$, $\Lambda_d \subseteq \mathbb{Z}^d$ (i.e., $\bar{\mathbf{u}}$ is a mosaic solution of (sd-DE) on lattice Λ_d), which is represented by pattern $\{s_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d}$, $s_{\mathbf{i}} =$

$\oplus, \ominus, \otimes$. We consider its neighborhood

$$\mathcal{N}(\bar{\mathbf{u}}, \theta, \delta) = \{\mathbf{v} = \{v_{\mathbf{i}}\}_{\mathbf{i}\in\Lambda_d} | |v_{\mathbf{i}} - \bar{u}_{\mathbf{i}}| \le \theta, \text{ if } s_{\mathbf{i}} = \otimes \quad \text{and} \quad |v_{\mathbf{i}} - \bar{u}_{\mathbf{i}}| \le \delta, \text{ if } s_{\mathbf{i}} = \oplus \text{ or } \ominus\}. \tag{4.1}$$

We will show that the mosaic solution of system (1.1) or (1.2) with nonlinearity (1.3) is stable, by proving the positive invariance of the set $\mathcal{N}(\bar{\mathbf{u}}, \theta, \delta)$ for appropriate $\theta > 0$ and $\delta > 0$, under some conditions. Moreover, the asymptotic stability of $\bar{\mathbf{u}}$ will also be established.

We introduce some notations concerning the states in the neighborhood of each $\mathbf{i} \in \Lambda_d$. For the one-dimensional case, $d = 1$, set

$$\begin{aligned} p_i &= \text{card}\{k \in \{i-1, i+1\} | s_k = \oplus\}, \\ n_i &= \text{card}\{k \in \{i-1, i+1\} | s_k = \ominus\}, \\ q_i &= \text{card}\{k \in \{i-1, i+1\} | s_k = \otimes\}. \end{aligned}$$

For the two-dimensional case, let "•" represent + (square-cross) or "×" (diagonal-cross). We denote that

$$\begin{aligned} p^{\bullet}_{i,j} &= \text{card}\{(k,\ell) \in N^{\bullet}_{i,j} | s_{k,\ell} = \oplus\}, \\ n^{\bullet}_{i,j} &= \text{card}\{(k,\ell) \in N^{\bullet}_{i,j} | s_{k,\ell} = \ominus\}, \\ q^{\bullet}_{i,j} &= \text{card}\{(k,\ell) \in N^{\bullet}_{i,j} | s_{k,\ell} = \otimes\}, \end{aligned}$$

where $N^{+}_{i,j} = \{(i+1, j), (i-1, j), (i, j+1), (i, j-1)\}$ and $N^{\times}_{i,j} = \{(i+1, j+1), (i+1, j-1), (i-1, j+1), (i-1, j-1)\}$. We present the following theorem for the stability of mosaic solutions on finite lattice Λ_d, with $d = 1$ in part (I), $d = 2$ in part (II). The case of $\Lambda_d = \mathbb{Z}^d$, an infinite lattice, will be remarked after the proof of the theorems.

Theorem 4.1. *(I) Let $\bar{\mathbf{u}} \in \mathcal{M}^{\sigma}_1(\alpha, \beta)$, which is represented by the patterns $\{s_i\}$. Then, the set $\mathcal{N}(\bar{\mathbf{u}}, \theta, \delta)$ is positively invariant for (1.1), if $\theta > 0, \delta > 0$ satisfy*

$$2\beta\delta + \alpha[f(\bar{u}_i - \delta) - f(\bar{u}_i)] - (p_i + n_i)|\beta|\delta > q_i|\beta|\theta, \tag{4.2}$$

$$2\beta\delta + \alpha[f(\bar{u}_i) - f(\bar{u}_i + \delta)] - (p_i + n_i)|\beta|\delta > q_i|\beta|\theta, \tag{4.3}$$

whenever $s_i = \oplus$ or $s_i = \ominus$, and

$$2\beta\theta + \alpha[f(\bar{u}_i - \theta) - f(\bar{u}_i)] - q_i|\beta|\theta > (p_i + n_i)|\beta|\delta, \tag{4.4}$$

$$2\beta\theta + \alpha[f(\bar{u}_i) - f(\bar{u}_i + \theta)] - q_i|\beta|\theta > (p_i + n_i)|\beta|\delta, \tag{4.5}$$

whenever $s_i = \otimes$. (II) Let $\bar{\mathbf{u}} \in \mathcal{M}^{\sigma}_2(\alpha, \beta^+, \beta^\times)$, which is represented by the patterns $\{s_{i,j}\}$. Then, the set $\mathcal{N}(\bar{\mathbf{u}}, \theta, \delta)$ is positively invariant for (1.2), if $\theta > 0, \delta > 0$ satisfy

$$(4\beta^+ + 4\beta^\times)\delta + \alpha[f(\bar{u}_{i,j} - \delta) - f(\bar{u}_{i,j})] - [(p^+_{i,j} + n^+_{i,j})|\beta^+| + (p^\times_{i,j} + n^\times_{i,j})|\beta^\times|]\delta > (q^+_{i,j}|\beta^+| + q^\times_{i,j}|\beta^\times|)\theta, \tag{4.6}$$

$$(4\beta^+ + 4\beta^\times)\delta + \alpha[f(\bar{u}_{i,j}) - f(\bar{u}_{i,j} + \delta)] - [(p^+_{i,j} + n^+_{i,j})|\beta^+| + (p^\times_{i,j} + n^\times_{i,j})|\beta^\times|]\delta > (q^+_{i,j}|\beta^+| + q^\times_{i,j}|\beta^\times|)\theta, \tag{4.7}$$

whenever $s_{i,j} = \oplus$ or $s_{i,j} = \ominus$, and

$$(4\beta^+ + 4\beta^\times)\theta + \alpha[f(\bar{u}_{i,j} - \theta) - f(\bar{u}_{i,j})] - (q^+_{i,j}|\beta^+| + q^\times_{i,j}|\beta^\times|)\theta > [(p^+_{i,j} + n^+_{i,j})|\beta^+| + (p^\times_{i,j} + n^\times_{i,j})|\beta^\times|]\delta, \tag{4.8}$$

$$(4\beta^{+}+4\beta^{\times})\theta+\alpha[f(\bar{u}_{i,j})-f(\bar{u}_{i,j}+\theta)]-(q^{+}_{i,j}|\beta^{+}|+q^{\times}_{i,j}|\beta^{\times}|)\theta>[(p^{+}_{i,j}+n^{+}_{i,j})|\beta^{+}|+(p^{\times}_{i,j}+n^{\times}_{i,j})|\beta^{\times}|]\delta, \tag{4.9}$$

whenever $s_{i,j}=\otimes$.

Asymptotic stability for the mosaic solutions can further be established in the following theorem. The situations are rather different between the cases $\alpha<0$ and $\alpha>0$.

Theorem 4.2. *Let* $\bar{\mathbf{u}}\in\mathcal{M}^{\sigma}_{1}$ *or* $\mathcal{M}^{\sigma}_{2}$, *which is represented by the patterns* $\{s_{\mathbf{i}}\}$. (*i*) *For* $\alpha<0$, *if* (4.2) and (4.3) (*respectively*, (4.6) and (4.7)) *hold for* $\mathbf{i}$ *with* $s_{\mathbf{i}}=\oplus$ *and* $s_{\mathbf{i}}=\ominus$ *respectively*, *as well as* (4.4) and (4.5) (*respectively*, (4.8) and (4.9)) *hold for* $\mathbf{i}$ *with* $\bar{u}_{\mathbf{i}}\in[0,\sigma]$ *and* $\bar{u}_{\mathbf{i}}\in[-\sigma,0]$ *respectively*, *then* $\bar{\mathbf{u}}$ *is asymptotically stable for* (1.1) (*respectively*, (1.2)). (*ii*) *For* $\alpha>0$, *if* $s_{\mathbf{i}}=\otimes$ *for all* $\mathbf{i}$, (4.4) and (4.5) (*respectively*, (4.8) and (4.9)) *hold for* $\mathbf{i}$ *with* $\bar{u}_{\mathbf{i}}\in[-\sigma,0]$ *and* $\bar{u}_{\mathbf{i}}\in[0,\sigma]$, *respectively*, *then* $\bar{\mathbf{u}}$ *is asymptotically stable for* (1.1) (respectively, (1.2)).

Although inequalities (4.6)–(4.9) seem complicated, for practical application, writing a computer program to examine these inequalities is straightforward. We make a few observations and arrange them in the following remarks, before we prove the theorems.

Remark 1. Notably, $f(\bar{u}_i-\delta)-f(\bar{u}_i)$, $f(\bar{u}_i)-f(\bar{u}_i+\delta)$ are both negative whenever $s_i=\oplus$ or $\ominus$, and $f(\bar{u}_i-\theta)-f(\bar{u}_i)$, $f(\bar{u}_i)-f(\bar{u}_i+\theta)$ are both positive whenever $s_i=\otimes$. The assumptions (4.2) and (4.3) (respectively, (4.4) and (4.5)) are more likely to hold if α is negative (respectively, α is positive). Similar observations are valid for (4.6)–(4.9).

Remark 2. Recall that we have taken $\sigma<\frac{1}{11}$. With the characteristics of the nonlinearity f defined in (1.3), we can derive the following:

(a) Case $\alpha<0$.
  (i) If $s_{\mathbf{i}}=\oplus$, then (4.2) (respectively, (4.6)) implies (4.3) (respectively, (4.7)).
  If $s_{\mathbf{i}}=\ominus$, then (4.3) (respectively, (4.7)) implies (4.2) (respectively, (4.6)).
  If $s_{\mathbf{i}}=\otimes$ with $\bar{u}_{\mathbf{i}}\in[0,\sigma]$, then (4.4) (respectively, (4.8)) implies (4.5) (respectively, (4.9)).
  If $s_{\mathbf{i}}=\otimes$ with $\bar{u}_{\mathbf{i}}\in[-\sigma,0]$, then (4.5) (respectively, (4.9)) implies (4.4) (respectively, (4.8)).
  (ii) Moreover, if $s_{\mathbf{i}}=\oplus$, and (4.2) (respectively, (4.6)) holds for some θ and δ, then it also holds with θ and δ replaced by $\nu\theta$ and $\nu\delta$, respectively, where $0<\nu<1$. Same conclusions hold for (4.3) (respectively, (4.7)) if $s_{\mathbf{i}}=\ominus$. If $s_{\mathbf{i}}=\otimes$ with $\bar{u}_{\mathbf{i}}\in[0,\sigma]$, and (4.4) (respectively, (4.8)) holds for some θ and δ, then it also holds with θ and δ replaced by $\nu\theta$ and $\nu\delta$, respectively, where $0<\nu<1$. Same conclusions hold for (4.5) (respectively, (4.9)) if $s_{\mathbf{i}}=\otimes$ with $\bar{u}_{\mathbf{i}}\in[-\sigma,0]$.

(b) Case $\alpha>0$.
  (i) If $s_{\mathbf{i}}=\oplus$, then (4.3) (respectively, (4.7)) implies (4.2) (respectively, (4.6)).
  If $s_{\mathbf{i}}=\ominus$, then (4.2) (respectively, (4.6)) implies (4.3) (respectively, (4.7)).
  If $s_{\mathbf{i}}=\otimes$ with $\bar{u}_{\mathbf{i}}\in[0,\sigma]$, then (4.5) (respectively, (4.9)) implies (4.4) (respectively, (4.8)).
  If $s_{\mathbf{i}}=\otimes$ with $\bar{u}_{\mathbf{i}}\in[-\sigma,0]$, then (4.4) (respectively, (4.8)) implies (4.5) (respectively, (4.9)).
  (ii) If $s_{\mathbf{i}}=\otimes$ with $\bar{u}_{\mathbf{i}}\in[-\sigma,0]$ and (4.4) (respectively, (4.8)) holds for some θ and δ, then it also holds with θ and δ replaced by $\nu\theta$ and $\nu\delta$, respectively, where $0<\nu<1$. Same conclusions hold for (4.5) (respectively, (4.9)) if $s_{\mathbf{i}}=\otimes$ and $\bar{u}_{\mathbf{i}}\in[0,\sigma]$.

Notably, we have utilized the concavity of f in deriving the results (a)(ii) and (b)(ii).

Proof of Theorem 4.1. We only prove part (I), the one-dimensional case. The two-dimensional case is similar. If $\mathbf{v} = \{v_i\} \in \mathcal{N}(\overline{\mathbf{u}}, \theta, \delta)$ for some $\theta, \delta > 0$, then from the definitions of p_i, n_i, q_i and Δv_i, we have

$$\Delta v_i \le \overline{u}_{i+1} + \overline{u}_{i-1} + (p_i + n_i)\delta + q_i\theta - 2v_i,$$
$$\Delta v_i \ge \overline{u}_{i+1} + \overline{u}_{i-1} - (p_i + n_i)\delta - q_i\theta - 2v_i.$$

Hence, we have a lower bound for $\beta\Delta v_i$:

$$\beta\Delta v_i \ge \beta(\overline{u}_{i+1} + \overline{u}_{i-1}) - |\beta|(p_i + n_i)\delta - |\beta|q_i\theta - 2\beta v_i$$

for any $\beta \in \mathbb{R}$. Since $\overline{\mathbf{u}}$ is an equilibrium solution of (1.1), it follows that

$$\beta\Delta v_i \ge 2\beta(\overline{u}_i - v_i) - \alpha f(\overline{u}_i) - |\beta|(p_i + n_i)\delta - |\beta|q_i\theta. \tag{4.10}$$

Similarly, we obtain a upper bound for $\beta\Delta v_i$ as

$$\beta\Delta v_i \le 2\beta(\overline{u}_i - v_i) - \alpha f(\overline{u}_i) + |\beta|(p_i + n_i)\delta + |\beta|q_i\theta. \tag{4.11}$$

Let $\mathbf{v} = \mathbf{v}(t)$ be a solution to (1.1) lying in $\mathcal{N}(\overline{\mathbf{u}}, \theta, \delta)$, (4.10) and (4.11) imply

$$\dot{v}_i(t) = \beta\Delta u_i + \alpha f(u_i) \ge 2\beta(\overline{u}_i - v_i) + \alpha[f(v_i) - f(\overline{u}_i)] - |\beta|(p_i + n_i)\delta - |\beta|q_i\theta =: L_i(v_i, \theta, \delta), \tag{4.12}$$

and

$$\dot{v}_i(t) \le 2\beta(\overline{u}_i - v_i) + \alpha[f(v_i) - f(\overline{u}_i)] + |\beta|(p_i + n_i)\delta + |\beta|q_i\theta =: U_i(v_i, \theta, \delta). \tag{4.13}$$

Now, let us prove that with $\mathbf{v}(0) \in \mathcal{N}(\overline{\mathbf{u}}, \theta, \delta)$, the solution $\mathbf{v}(t)$ to (1.1) remains in the set $\mathcal{N}(\overline{\mathbf{u}}, \theta, \delta)$ for all $t \ge 0$. Notably, the inequalities (4.2) and (4.3) are equivalent to $L_i(\overline{u}_i - \delta, \theta, \delta) > 0$ and $U_i(\overline{u}_i + \delta, \theta, \delta) < 0$, respectively. Since L_i and U_i are continuous functions of their arguments, there exist μ with $0 < \mu < 1$, and $C_1 > 0$, $C_2 > 0$, such that

$$L_i(v_i, \theta', \delta') \ge C_1, \quad \text{if} \quad \frac{v_i}{\overline{u}_i - \delta}, \frac{\theta'}{\theta}, \frac{\delta'}{\delta} \in (\mu, \mu^{-1}), \tag{4.14}$$

$$U_i(v_i, \theta', \delta') \le -C_2, \quad \text{if} \quad \frac{v_i}{\overline{u}_i + \delta}, \frac{\theta'}{\theta}, \frac{\delta'}{\delta} \in (\mu, \mu^{-1}). \tag{4.15}$$

On the other hand, the inequalities (4.4) and (4.5) are equivalent to $L_i(\overline{u}_i - \theta, \theta, \delta) > 0$ and $U_i(\overline{u}_i + \theta, \theta, \delta) < 0$, respectively. For this case, we also have

$$L_i(v_i, \theta', \delta') \ge C_3, \quad \text{if} \quad \frac{v_i}{\overline{u}_i - \theta}, \frac{\theta'}{\theta}, \frac{\delta'}{\delta} \in (\mu, \mu^{-1}),$$

$$U_i(v_i, \theta', \delta') \le -C_4, \quad \text{if} \quad \frac{v_i}{\overline{u}_i + \theta}, \frac{\theta'}{\theta}, \frac{\delta'}{\delta} \in (\mu, \mu^{-1}),$$

for some $C_3 > 0$, $C_4 > 0$. We note that μ and C_1, C_2, C_3, C_4 can be chosen independent of i. For small $\varepsilon > 0$, there exists $K > 0$ such that $|\beta\Delta v_i + f(v_i)| \le K$ for all i, for all $\mathbf{v} \in \mathcal{N}(\overline{\mathbf{u}}, \theta + \varepsilon, \delta + \varepsilon)$. Hence, for any solution $\mathbf{v}(t)$ with $\mathbf{v}(0) \in \mathcal{N}(\overline{\mathbf{u}}, \theta, \delta)$, we have $\mathbf{v}(t) \in \mathcal{N}(\overline{\mathbf{u}}, \theta + \varepsilon, \delta + \varepsilon)$, for $0 \le t \le T := \frac{\varepsilon}{K}$. Herein, we choose $\varepsilon = \min\{(\frac{1}{\mu} -$

$1)\theta, (\frac{1}{\mu}-1)\delta\}$, and claim that $\overline{u}_i-\delta \le v_i(t) \le \overline{u}_i+\delta$, for all $t \in [0, T]$, whenever $s_i = \oplus$ or $\ominus$. Suppose, on the contrary, that $\overline{u}_i-\delta-\varepsilon \le v_i(t) \le \overline{u}_i-\delta$, for some $t \in [0, T]$ and some i with $s_i = \oplus$ or $\ominus$. Form (4.14),

$$\dot{v}_i(t) \ge L_i(v_i, \theta, \delta) \ge L_i(v_i, \theta+\varepsilon, \delta+\varepsilon) \ge C_1 > 0.$$

Hence, if $\mathbf{v}(0) \in \mathcal{N}(\overline{\mathbf{u}}, \theta, \delta)$, $v_i(t) \ge \overline{u}_i-\delta$, for all $t \in [0, T]$, for all i. In addition, by (4.15), $v_i(t) \le \overline{u}_i+\delta$, for all $t \in [0, T]$. Similarly, if $s_i = \otimes$, it can be shown that $|v_i(t)-\overline{u}_i| \le \theta$, for all $t \in [0, T]$. Thus, we have that $\mathbf{v}(t) \in \mathcal{N}(\overline{\mathbf{u}}, \theta, \delta)$, for all $t \in [0, T]$. Note that we only require $\mathbf{v}(0) \in \mathcal{N}(\overline{\mathbf{u}}, \theta, \delta)$ to derive this result. Therefore, we conclude that $\mathcal{N}(\overline{\mathbf{u}}, \theta, \delta)$ is positively invariant. □

Proof of Theorem 4.2. We only prove the one-dimensional case. Consider a solution $\mathbf{v}(t)$ to (1.1) with $\mathbf{v}(0) \in \mathcal{N}(\overline{\mathbf{u}}, \theta, \delta)$. Recall that if i is such that $s_i = \oplus$, then $\dot{v}_i(t) \ge C_1$, whenever $\overline{u}_i-\delta \le v_i(t) \le \overline{u}_i-\mu\delta$, and $\dot{v}_i(t) \le -C_2$, whenever $\overline{u}_i+\mu\delta \le v_i(t) \le \overline{u}_i+\delta$. Thus, for $\mathbf{v}(0) \in \mathcal{N}(\overline{\mathbf{u}}, \theta, \delta)$, $|v_i(t)-\overline{u}_i| \le \mu\delta$ for all $t \ge (1-\mu)\delta/C$, where $C = \min\{C_1, C_2, C_3, C_4\}$. Similarly, we have that $|v_i(t)-\overline{u}_i| \le \mu\theta$, for all $t \ge (1-\mu)\theta/C$. Therefore, we conclude that $\mathbf{v}(t) \in \mathcal{N}(\overline{\mathbf{u}}, \mu\theta, \mu\delta)$, for all $t \ge T$, where $T = \max\{(1-\mu)\delta/C, (1-\mu)\theta/C\}$. Using the observations in Remark 2(a)(ii) and (b)(ii), there is a sequence of positive time $T_1 < T_2 < T_3 < \cdots$, which converge to infinity, such that $\mathbf{v}(t) \in \mathcal{N}(\overline{\mathbf{u}}, \mu^n\theta, \mu^n\delta)$, for all $t \ge T_n$. Notice that the choice of T_n is independent of the solution $\mathbf{v}(t)$. Therefore, we conclude that $|\mathbf{v}(t)-\overline{\mathbf{u}}| \to 0$ as $t \to \infty$, i.e., $\overline{\mathbf{u}}$ is asymptotically stable. □

Remark 3. We can replace (4.2)–(4.9) by stronger conditions which do not depend on the exact values of $\overline{u}_i$. For example, if $\alpha < 0$, we replace (4.2) and (4.3) by

$$2\beta\delta+\alpha[f(-1+\sigma-\delta)-f(-1+\sigma)]-(p_i+n_i)|\beta|\delta > q_i|\beta|\theta, \tag{4.16}$$

$$2\beta\delta+\alpha[f(1-\sigma)-f(1-\sigma+\delta)]-(p_i+n_i)|\beta|\delta > q_i|\beta|\theta, \tag{4.17}$$

if $s_i = \ominus$ and $s_i = \oplus$, respectively, as $f(-1+\sigma-\delta)-f(-1+\sigma) > f(\overline{u}_i-\delta)-f(\overline{u}_i)$ and $f(1-\sigma)-f(1-\sigma+\delta) > f(\overline{u}_i)-f(\overline{u}_i+\delta)$ for the respective case. For the case of infinite lattice $\Lambda_d = \mathbb{Z}^d$, one can derive similar results as Theorems 4.1 and 4.2 by replacing (4.2)–(4.9) with stronger ones as (4.16) and (4.17) in the spirit mentioned herein.

A simple way to construct stable mosaic patterns is to consider the case $\alpha < 0$ and the mosaic solution $\overline{\mathbf{u}}$ represented by $\{s_i\}$ with $s_i = \oplus, \ominus$ for all i. In this situation, we only need to verify (4.2) and (4.3) for the case of $d = 1$ and (4.6) and (4.7) for the case of $d = 2$ for the stability of $\overline{\mathbf{u}}$. If $d = 1$, and β is fixed, one can always choose negative α with large magnitude to satisfy (4.2) and (4.3). Regarding the existence of these patterns with $s_i = \oplus, \ominus$, we note that as $|b|$ is small enough ($b = \beta/\alpha$), all the basic patterns $\overline{\underline{\bullet\bullet\bullet}}$ with $\bullet = \oplus, \ominus$, exist. More precisely, if (α, β) satisfying $|b| < \frac{f(-\sigma)}{2+4\sigma}$, all mosaic patterns $\{s_i\}$, $s_i = \oplus, \ominus$, exist. These patterns are asymptotically stable if β is fixed, $\alpha < 0$ and $|\alpha|$ is large. It is also straightforward to find parameters for the existence of stable pattern $\{s_{\mathbf{i}}\}$, with $s_{\mathbf{i}} = \otimes$ for all $\mathbf{i} \in \Lambda_d$. Notably, the patterns $\{s_{\mathbf{i}}\}$ are regarded as spatially uniform ones if $s_{\mathbf{i}} = \otimes$ (or $\oplus$, or $\ominus$) for all $\mathbf{i} \in \Lambda_d$. The existence of the above-mentioned stable patterns can be extended to the system on other lattices of higher dimension. We give a concrete example.

Example 1. Consider the case $d = 1$. Let $\Lambda_1 = T_k$, where

$$T_k = \{i \in \mathbb{Z}^1 | 1 \le i \le k\}.$$

We impose the Neumann boundary condition on T_k, i.e.,

$$u_{k+1} = u_k, \qquad u_0 = u_1.$$

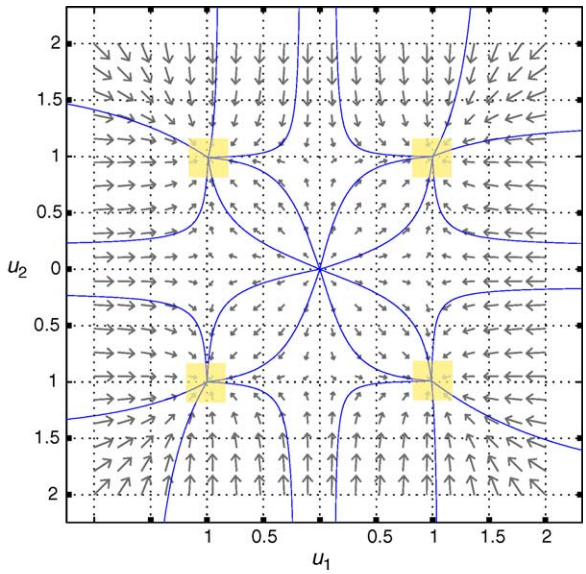

Fig. 3. Phase portrait for (1.1) with $\beta = 1, \alpha = -100$, on T_2. The shadow regions depicted from the estimates $\delta = \theta = 0.2$ indicate subsets of the basins around the stable equilibrium points.

Since the conditions (4.2)–(4.9) in Theorems 4.1 and 4.2 concern themselves with the states at each ith site and its adjacent sites, we could also examine these conditions for the boundary sites $i = 1, i = k$. We illustrate the numerics by the following instance with $k = 2$:

$$\dot{u}_i = \beta(u_{i+1} + u_{i-1} - 2u_i) + \alpha f(u_i), i = 1, 2, \tag{4.18}$$

where $u_3 = u_2, u_0 = u_1$. If we take $\beta = 1, \alpha = -100$, then $\delta = \theta = 0.2$ satisfy (4.2) and (4.3). The phase portrait for such a system is illustrated in Fig. 3.

5. Spatial entropy

Let us review the notion of spatial entropy for lattice dynamical systems [2,6]. Let $\mathcal{A}$ be a finite set of elements (symbols) which are used to represent the patterns at each site on the lattice. In the case herein, $\mathcal{A} = \{\oplus, \otimes, \ominus\}$. Let $\mathcal{A}^{\mathbb{Z}^d} = \{\mathbf{s} | \mathbf{s} : \mathbb{Z}^d \to \mathcal{A}\}$. Consider the natural projection

$$\pi_{\mathbf{k}} : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{T_{\mathbf{k}}}, \tag{5.1}$$

given by restricting any $\mathbf{s} \in \mathcal{A}^{\mathbb{Z}^d}$ to finite subset $T_{\mathbf{k}}$ (defined in (2.1) for $d = 1$, (2.2) for $d = 2$). Let $\mathcal{S}$ be a translation invariant subset of the feasible global patterns (corresponding to stationary solutions) of (1.1) and (1.2) on $\mathbb{Z}^d$, with certain parameters. Set

$$\Gamma_{\mathbf{k}}^{\infty} = \Gamma_{\mathbf{k}}(\mathcal{S}) := \text{card}(\pi_{\mathbf{k}}(\mathcal{S})), \tag{5.2}$$

where $\Gamma_{\mathbf{k}}^{\infty}$ denotes the number of distinct feasible mosaic patterns projected from elements in $\mathcal{S}$ onto $T_{\mathbf{k}}$. The spatial entropy $h(\mathcal{S})$ of the set $\mathcal{S}$ is defined as

$$h(\mathcal{S}) := \lim_{\mathbf{k}\to\infty} \frac{1}{k_1 k_2 \ldots k_d} \ln \Gamma_{\mathbf{k}}(\mathcal{S}). \tag{5.3}$$

There are other considerations for spatial entropy; in particular, if boundary condition is taken into account, then definition (5.3) should be modified. We arrange such a consideration in Section 6. According to our formulation, the partitioning of parameters in Section 2 allows us to discover the major portion of feasible basic patterns corresponding to each parameter region. There are some other possible basic patterns that are not included in these collections. They arise from other possible intersections for the graphs of (2.5) and (2.6). Their existence as feasible basic patterns cannot be justified from our fixed-point arguments. As a subsequence of this formulation, we further introduce the following notations. Under the same parameters for $\mathcal{S}$, let $\underline{\mathcal{S}}$ be the translation invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$, which is formed from attaching the affirmatively feasible basic patterns established in Section 2, and let $\overline{\mathcal{S}}$ be the one formed from attaching both the affirmatively feasible basic patterns as well as possible basic patterns. Set

$$h(\underline{\mathcal{S}}) := \lim_{\mathbf{k}\to\infty} \frac{1}{k_1 k_2, \ldots, k_d} \ln \Gamma_{\mathbf{k}}(\underline{\mathcal{S}}), \qquad h(\overline{\mathcal{S}}) := \lim_{\mathbf{k}\to\infty} \frac{1}{k_1 k_2, \ldots, k_d} \ln \Gamma_{\mathbf{k}}(\overline{\mathcal{S}}). \tag{5.4}$$

Obviously, $h(\underline{\mathcal{S}}) \le h(\mathcal{S}) \le h(\overline{\mathcal{S}})$. We recall the following definition in [2].

Definition 5.1. The system (1.1) or (1.2) is said to exhibit *spatial chaos* at parameters (α, β) or $(\alpha, \beta^+, \beta^\times)$, if the spatial entropy is positive. The system (1.1) or (1.2) is said to exhibit *pattern formation* at parameters (α, β) or $(\alpha, \beta^+, \beta^\times)$, if the spatial entropy is zero.

The notion of spatial entropy resembles the one of topological entropy for Markov shift [17]. In the case of one-dimensional lattice $d = 1$, a transition matrix can be formulated to depict the attaching process of basic patterns. Accordingly, total number of mosaic patterns obtained from the attaching can be calculated and the spatial entropy can be computed exactly. Let us introduce this formulation. We employ the following identification between the indices $\{1, 2, 3, \ldots, 9\}$ and the nine 1×2 patterns $\{\oplus\oplus, \oplus\otimes, \oplus\ominus, \otimes\oplus, \otimes\otimes, \otimes\ominus, \ominus\oplus, \ominus\otimes, \ominus\ominus\}$:

$$\begin{array}{lll} 1 \longleftrightarrow \oplus\oplus, & 2 \longleftrightarrow \oplus\otimes, & 3 \longleftrightarrow \oplus\ominus, \\ 4 \longleftrightarrow \otimes\oplus, & 5 \longleftrightarrow \otimes\otimes, & 6 \longleftrightarrow \otimes\ominus, \\ 7 \longleftrightarrow \ominus\oplus, & 8 \longleftrightarrow \ominus\otimes, & 9 \longleftrightarrow \ominus\,\ominus\,. \end{array} \tag{5.5}$$

Consider the 9×9 matrix M:

$$M = M(\mathbf{r}) := \begin{pmatrix} r_1 & r_2 & r_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_4 & r_5 & r_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_7 & r_8 & r_9 \\ r_{10} & r_{11} & r_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{13} & r_{14} & r_{15} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_{16} & r_{17} & r_{18} \\ r_{19} & r_{20} & r_{21} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r_{22} & r_{23} & r_{24} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r_{25} & r_{26} & r_{27} \end{pmatrix}, \tag{5.6}$$

where $r = \{r_j\}_{j=1}^{27}$, $r_j = 0$ or 1, $j \in \{1, 2, \ldots, 27\}$. The formation of feasible mosaic patterns depicted by the transition matrix can be described as follows: the (i, j)-entry of M is one if and only if the jth 1×2 pattern in (5.5) can be attached, with one site overlapped, to the right of the ith 1×2 pattern in (5.5) to form a 1×3 feasible pattern. For example, if $b \in I_5 = [f(-\sigma), \frac{f(1+\sigma)}{1+4\sigma}]$, the set of affirmatively feasible basic patterns are

$$\{\overline{\underline{\oplus \oplus \oplus}}, \overline{\underline{\oplus \oplus \otimes}}, \overline{\underline{\oplus \otimes \ominus}}, \overline{\underline{\otimes \oplus \oplus}}, \overline{\underline{\otimes \otimes \otimes}}, \overline{\underline{\otimes \ominus \ominus}}, \overline{\underline{\ominus \otimes \oplus}}, \overline{\underline{\ominus \ominus \otimes}}, \overline{\underline{\ominus \ominus \ominus}}\},$$

and the corresponding transition matrix is (5.6) with $r_1 = r_2 = r_6 = r_{10} = r_{14} = r_{18} = r_{22} = r_{26} = r_{27} = 1$ and $r_j = 0$ for all other j.

Moreover, the total number of mosaic patterns on the lattice T_k obtained from such a formulation is

$$\sum_{1 \le i, j \le 9} M_{ij}^{k-2}.$$

The spatial entropy can be computed from the largest eigenvalue λ_1 of M, namely

$$h(\mathcal{S}) = \ln \lambda_1.$$

Recall Fig. 1, where we illustrate the partitioning of parameter space. Therein, we have determined a collection of basic patterns corresponding to each parameter region. These basic patterns are confirmed to be feasible later in Section 3. In fact, in our geometric formulation, there may exist more possible basic patterns if the graph of f between the lines L_1 and L_2 has intersection with the shadow region R (even a point). In Fig. 2, we display the existence of feasible basic patterns(left half) which can be confirmed by the treatment in Section 3 and the existence of all other possible basic patterns (right half) in each respective parameter region. In order to achieve this, further partitioning of the parameters needs to be performed. In Table 3, we have further partitioned regions I_i into I_i^ℓ so that all possible basic patterns are identified for parameters in subregion I_i^ℓ, in addition to those feasible basic patterns already confirmed in region I_i. We summarize our computations in Table 4 and the following theorem.

Theorem 5.2. *System (1.1) exhibits spatial chaos in parameter regions I_i, $-5 \le i \le 5$, and exhibits pattern formation in parameter regions $I_{\pm 7}$ and $I_{\pm 6}^2$.*

Theorem 5.2 and Table 4 are completely obtained from computing the eigenvalues of the transition matrix corresponding to each parameter region. $\underline{\lambda}_1$ is the largest eigenvalue of the transition matrix corresponding to attaching affirmatively feasible basic patterns, while $\overline{\lambda}_1$ is the largest eigenvalue of the transition matrix corresponding to attaching both affirmatively feasible basic patterns and possible basic patterns.

In the two dimension case $d = 2$, one no longer has a transition matrix to describe the formation of patterns, except some special situations [18]. Therefore, in most cases, we can only estimate the spatial entropy. By employing the methodology in [2,6], i.e., constructing adjoinable building blocks from feasible basic patterns, we can compute the lower bound of the spatial entropy. For example, if we can find three 2×2 patterns so that any one of them can be joined (without overlapping) from the left, the right, upward, and downward directions to any one of themselves, then on a $2k \times 2k$ square lattice, there are at least 3^{k^2} distinct mosaic patterns. It follows that a lower bound for the entropy is

$$\lim_{k \to \infty} \frac{\ln 3^{k^2}}{k^2} = \ln 3.$$

If there exist only very few feasible basic patterns in a parameter region, then it can be easily seen that the spatial entropy is zero. We summarize our computations in Table 5.

Table 3
Additional possible basic patterns in further partitioned parameter regions

Parameter space	Possibe basic patterns
$I_7=\left[\frac{f(-1+\sigma)}{4\sigma},\infty\right]$	$B^{\oplus}_{\{2\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-2\}}$
I_6	
$I_6^2=\left[f(1+\sigma),\frac{f(-1+\sigma)}{4\sigma}\right]$	$B^{\oplus}_{\{2\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-2\}}$
$I_6^1=\left[\frac{f(1+\sigma)}{1+4\sigma},f(1+\sigma)\right]$	$B^{\oplus}_{\{2,1\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-1,-2\}}$
I_5	
$I_5^3=\left[\frac{f(-\sigma)}{1-4\sigma},\frac{f(1+\sigma)}{1+4\sigma}\right]$	$B^{\oplus}_{\{2,1\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-1,-2\}}$
$I_5^2=\left[\frac{f(1+\sigma)}{2},\frac{f(-\sigma)}{1-4\sigma}\right]$	$B^{\oplus}_{\{2,1\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{-1,-2\}}$
$I_5^1=\left[f(-\sigma),\frac{f(1+\sigma)}{2}\right]$	$B^{\oplus}_{\{2,1,0\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{0,-1,-2\}}$
$I_4=\left[\frac{f(1+\sigma)}{2+4\sigma},f(-\sigma)\right]$	$B^{\oplus}_{\{2,1,0\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{0,-1,-2\}}$
I_3	
$I_3^2=\left[\frac{f(1+\sigma)}{3},\frac{f(1+\sigma)}{2+4\sigma}\right]$	$B^{\oplus}_{\{2,1,0\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{0,-1,-2\}}$
$I_3^1=\left[\frac{f(1+\sigma)}{3+4\sigma},\frac{f(1+\sigma)}{3}\right]$	$B^{\oplus}_{\{2,1,0,-1\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{1,0,-1,-2\}}$
I_2	
$I_2^3=\left[\frac{f(-\sigma)}{2-4\sigma},\frac{f(1+\sigma)}{3+4\sigma}\right]$	$B^{\oplus}_{\{2,1,0,-1\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{1,0,-1,-2\}}$
$I_2^2=\left[\frac{f(1+\sigma)}{4},\frac{f(-\sigma)}{2-4\sigma}\right]$	$B^{\oplus}_{\{2,1,0,-1\}}, B^{\otimes}_{\{2,1,0,-1,-2\}}, B^{\ominus}_{\{1,0,-1,-2\}}$
$I_2^1=\left[\frac{f(1+\sigma)}{4+4\sigma},\frac{f(1+\sigma)}{4}\right]$	$B^{\oplus}_{\{2,1,0,-1,-2\}}, B^{\otimes}_{\{2,1,0,-1,-2\}}, B^{\ominus}_{\{2,1,0,-1,-2\}}$
$I_1=\left[\frac{f(-\sigma)}{2},\frac{f(1+\sigma)}{4+4\sigma}\right]$	$B^{\oplus}_{\{2,1,0,-1,-2\}}, B^{\otimes}_{\{2,1,0,-1,-2\}}, B^{\ominus}_{\{2,1,0,-1,-2\}}$
$I_0=\left[-\frac{f(-\sigma)}{2+4\sigma},\frac{f(-\sigma)}{2}\right]$	$B^{\oplus}_{\{2,1,0,-1,-2\}}, B^{\otimes}_{\{2,1,0,-1,-2\}}, B^{\ominus}_{\{2,1,0,-1,-2\}}$
$I_{-1}=\left[-\frac{f(-1+\sigma)}{4},-\frac{f(-\sigma)}{2+4\sigma}\right]$	$B^{\oplus}_{\{2,1,0,-1,-2\}}, B^{\otimes}_{\{2,1,0,-1,-2\}}, B^{\ominus}_{\{2,1,0,-1,-2\}}$
I_{-2}	
$I_{-2}^1=\left[-\frac{f(-1+\sigma)}{4-4\sigma},-\frac{f(-1+\sigma)}{4}\right]$	$B^{\oplus}_{\{2,1,0,-1,-2\}}, B^{\otimes}_{\{2,1,0,-1,-2\}}, B^{\ominus}_{\{2,1,0,-1,-2\}}$
$I_{-2}^2=\left[-\frac{f(-\sigma)}{2},-\frac{f(-1+\sigma)}{4-4\sigma}\right]$	$B^{\oplus}_{\{2,1,0,-1\}}, B^{\otimes}_{\{2,1,0,-1,-2\}}, B^{\ominus}_{\{1,0,-1,-2\}}$
$I_{-2}^3=\left[-\frac{f(-1+\sigma)}{3},-\frac{f(-\sigma)}{2}\right]$	$B^{\oplus}_{\{2,1,0,-1\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{1,0,-1,-2\}}$
I_{-3}	
$I_{-3}^1=\left[-\frac{f(-1+\sigma)}{3-4\sigma\sigma},-\frac{f(-1+\sigma)}{3}\right]$	$B^{\oplus}_{\{2,1,0,-1\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{1,0,-1,-2\}}$
$I_{-3}^2=\left[-\frac{f(-\sigma)}{1+4\sigma},-\frac{f(-1+\sigma)}{3-4\sigma}\right]$	$B^{\oplus}_{\{2,1,0\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{0,-1,-2\}}$
$I_{-4}=\left[-\frac{f(-1+\sigma)}{2},-\frac{f(-\sigma)}{1+4\sigma}\right]$	$B^{\oplus}_{\{2,1,0\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{0,-1,-2\}}$

Table 3 (*Continued*)

Parameter space	Possibe basic patterns
I_{-5}	
$I_{-5}^1 = \left[-f(-\sigma), -\frac{f(-1+\sigma)}{2}\right]$	$B^{\oplus}_{\{2,1,0\}}, B^{\otimes}_{\{1,0,-1\}}, B^{\ominus}_{\{0,-1,-2\}}$
$I_{-5}^2 = \left[-\frac{f(-1+\sigma)}{2-4\sigma}, -f(-\sigma)\right]$	$B^{\oplus}_{\{2,1,0\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{0,-1,-2\}}$
$I_{-5}^3 = \left[-f(-1+\sigma), -\frac{f(-1+\sigma)}{2-4\sigma}\right]$	$B^{\oplus}_{\{2,1\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-1,-2\}}$
I_{-6}	
$I_{-6}^1 = \left[-\frac{f(-1+\sigma)}{1-4\sigma}, -f(-1+\sigma)\right]$	$B^{\oplus}_{\{2,1\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-1,-2\}}$
$I_{-6}^2 = \left[-\frac{f(-\sigma)}{4\sigma} - \frac{f(-1+\sigma)}{1-4\sigma}\right]$	$B^{\oplus}_{\{2\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-2\}}$
$I_{-7} = \left[-\infty, -\frac{f(-\sigma)}{4\sigma}\right]$	$B^{\oplus}_{\{2\}}, B^{\otimes}_{\{0\}}, B^{\ominus}_{\{-2\}}$

Table 4

$\underline{\lambda}_1$: the largest eigenvalue of the transition matrix corresponding to attaching affirmatively feasible basic patterns, and $\overline{\lambda}_1$: the largest eigenvalue of the transition matrix corresponding to attaching both affirmatively feasible basic patterns and possible basic patterns

Parameter space	$\underline{\lambda}_1$	$\overline{\lambda}_1$	h
$I_7 = \left[\frac{f(-1+\sigma)}{4\sigma}, \infty\right]$	1	1	0
I_6			
$I_6^2 = \left[f(1+\sigma), \frac{f(-1+\sigma)}{4\sigma}\right]$	1	1	0
$I_6^1 = \left[\frac{f(1+\sigma)}{1+4\sigma}, f(1+\sigma)\right]$	1	1.4656	$0 \le h \le 0.3823$
I_5			
$I_5^3 = \left[\frac{f(-\sigma)}{1-4\sigma}, \frac{f(1+\sigma)}{1+4\sigma}\right]$	1.4656	1.4656	0.3823
$I_5^2 = \left[\frac{f(1+\sigma)}{2}, \frac{f(-\sigma)}{1-4\sigma}\right]$	1.4656	1.8972	$0.3823 \le h \le 0.6404$
$I_5^1 = \left[f(-\sigma), \frac{f(1+\sigma)}{2}\right]$	1.4656	2.3165	$0.3823 \le h \le 0.8401$
$I_4 = \left[\frac{f(1+\sigma)}{2+4\sigma}, f(-\sigma)\right]$	1.8972	2.3165	$0.6404 \le h \le 0.8401$
I_3			
$I_3^2 = \left[\frac{f(1+\sigma)}{3}, \frac{f(1+\sigma)}{2+4\sigma}\right]$	2.3165	2.3165	0.8401
$I_3^1 = \left[\frac{f(1+\sigma)}{3+4\sigma}, \frac{f(1+\sigma)}{3}\right]$	2.3165	2.5921	$0.8401 \le h \le 0.9525$
I_2			
$I_2^3 = \left[\frac{f(-\sigma)}{2-4\sigma}, \frac{f(1+\sigma)}{3+4\sigma}\right]$	2.5921	2.5921	0.9525
$I_2^2 = \left[\frac{f(1+\sigma)}{4}, \frac{f(-\sigma)}{2-4\sigma}\right]$	2.5921	2.8312	$0.9525 \le h \le 1.0407$
$I_2^1 = \left[\frac{f(1+\sigma)}{4+4\sigma}, \frac{f(1+\sigma)}{4}\right]$	2.5921	3	$0.9525 \le h \le 1.0986$

Table 4 (*Continued*)

Parameter space	$\underline{\lambda}_1$	$\overline{\lambda}_1$	h
$I_1 = \left[\frac{f(-\sigma)}{2}, \frac{f(1+\sigma)}{4+4\sigma}\right]$	2.7693	3	$1.0186 \le h \le 1.0986$
$I_0 = \left[-\frac{f(-\sigma)}{2+4\sigma}, \frac{f(-\sigma)}{2}\right]$	3	3	1.0986
$I_{-1} = \left[-\frac{f(-1+\sigma)}{4}, -\frac{f(-\sigma)}{2+4\sigma}\right]$	2.7693	3	$1.0186 \le h \le 1.0986$
I_{-2}			
$I^1_{-2} = \left[-\frac{f(-1+\sigma)}{4-4\sigma}, -\frac{f(-1+\sigma)}{4}\right]$	2.5921	3	$0.9525 \le h \le 1.0986$
$I^2_{-2} = \left[-\frac{f(-\sigma)}{2}, -\frac{f(-1+\sigma)}{4-4\sigma}\right]$	2.5921	2.8312	$0.9525 \le h \le 1.0407$
$I^3_{-2} = \left[-\frac{f(-1+\sigma)}{3}, -\frac{f(-\sigma)}{2}\right]$	2.5921	2.5921	0.9525
I_{-3}			
$I^1_{-3} = \left[-\frac{f(-1+\sigma)}{3-4\sigma\sigma}, -\frac{f(-1+\sigma)}{3}\right]$	2.3165	2.5921	$0.8401 \le h \le 0.9525$
$I^2_{-3} = \left[-\frac{f(-\sigma)}{1+4\sigma}, -\frac{f(-1+\sigma)}{3-4\sigma}\right]$	2.3165	2.3165	0.8401
$I_{-4} = \left[-\frac{f(-1+\sigma)}{2}, -\frac{f(-\sigma)}{1+4\sigma}\right]$	1.9052	2.3165	$0.6446 \le h \le 0.8401$
I_{-5}			
$I^1_{-5} = \left[-f(-\sigma), -\frac{f(-1+\sigma)}{2}\right]$	1.4656	2.3165	$0.3823 \le h \le 0.8401$
$I^2_{-5} = \left[-\frac{f(-1+\sigma)}{2-4\sigma}, -f(-\sigma)\right]$	1.4656	1.9052	$0.3823 \le h \le 0.6446$
$I^3_{-5} = \left[-f(-1+\sigma), -\frac{f(-1+\sigma)}{2-4\sigma}\right]$	1.4656	1.4656	0.3823
I_{-6}			
$I^1_{-6} = \left[-\frac{f(-1+\sigma)}{1-4\sigma}, -f(-1+\sigma)\right]$	1	1.4656	$0 \le h \le 0.3823$
$I^2_{-6} = \left[-\frac{f(-\sigma)}{4\sigma} - \frac{f(-1+\sigma)}{1-4\sigma}\right]$	1	1	0
$I_{-7} = \left[-\infty, -\frac{f(-\sigma)}{4\sigma}\right]$	1	1	0

6. Effect of boundary conditions on pattern formation and spatial entropy

In Section 2, three typical types of boundary conditions: Neumann (N), periodic (P), and Dirichlet (D), have been introduced. In this section, we plan to discuss the effect of these boundary conditions on pattern formation and spatial entropy. We introduce the following notations to distinguish different considerations of spatial entropy. For the definition of spatial entropy used in Section 5, i.e., from counting the number of patterns projected from global patterns (patterns on $\mathbb{Z}^1$, $\mathbb{Z}^2$), we introduce the notation $\Gamma^{\infty}_{\mathbf{k}}$ to represent the number of such patterns on $T_{\mathbf{k}}$. Since our formulation includes the situations of patterns obtained from affirmatively feasible basic patterns as well as from possible basic patterns additionally, we further denote

Table 5
Estimations for the lower bounds of spatial entropy for each parameter region, for the two-dimensional system (1.2) with $b_2 = 0$

Parameter region	$\underline{h}$	Parameter region	$\underline{h}$
$I_{13} = \left[\frac{-f(1-\sigma)}{8\sigma}, \infty\right]$	0	$I_{-1} = \left[\frac{f(1-\sigma)}{8}, \frac{f(\sigma)}{4+8\sigma}\right]$	$\frac{\ln 69}{4}$
$I_{12} = \left[\frac{f(1+\sigma)}{1+8\sigma}, \frac{-f(1-\sigma)}{8\sigma}\right]$	0	$I_{-2}[\frac{f(1-\sigma)}{7}, \frac{f(1-\sigma)}{8}]$	$\frac{\ln 51}{4}$
$I_{11} = \left[\frac{f(\sigma)}{-1}, \frac{f(1+\sigma)}{1+8\sigma}\right]$	0	$I_{-3} = \left[\frac{f(\sigma)}{3+8\sigma}, \frac{f(1-\sigma)}{7}\right]$	$\frac{\ln 51}{4}$
$I_{10} = \left[\frac{f(1+\sigma)}{2+8\sigma}, \frac{f(\sigma)}{-1}\right]$	0	$I_{-4} = \left[\frac{f(1-\sigma)}{6}, \frac{f(\sigma)}{3+8\sigma}\right]$	$\frac{\ln 49}{4}$
$I_9 = \left[\frac{f(1+\sigma)}{3+8\sigma}, \frac{f(1+\sigma)}{2+8\sigma}\right]$	$\frac{\ln 3}{16}$	$I_{-5} = \left[\frac{f(1-\sigma)}{5}, \frac{f(1-\sigma)}{6}\right]$	$\frac{\ln 3}{2}$
$I_8 = \left[\frac{f(\sigma)}{-2}, \frac{f(1+\sigma)}{3+8\sigma}\right]$	$\frac{\ln 3}{16}$	$I_{-6} = \left[\frac{f(\sigma)}{2+8\sigma}, \frac{f(1-\sigma)}{5}\right]$	$\frac{\ln 3}{2}$
$I_7 = \left[\frac{f(1+\sigma)}{4+8\sigma}, \frac{f(\sigma)}{-2}\right]$	$\frac{\ln 5}{4}$	$I_{-7} = \left[\frac{f(1-\sigma)}{4}, \frac{f(\sigma)}{2+8\sigma}\right]$	$\frac{\ln 3}{16}$
$I_6 = \left[\frac{f(1+\sigma)}{5+8\sigma}, \frac{f(1+\sigma)}{4+8\sigma}\right]$	$\frac{\ln 3}{2}$	$I_{-8} = \left[\frac{f(1-\sigma)}{3}, \frac{f(1-\sigma)}{4}\right]$	$\frac{\ln 3}{16}$
$I_5 = \left[\frac{f(1+\sigma)}{6+8\sigma}, \frac{f(1+\sigma)}{5+8\sigma}\right]$	$\frac{\ln 3}{2}$	$I_{-9} = \left[\frac{f(\sigma)}{1+8\sigma}, \frac{f(1-\sigma)}{3}\right]$	$\frac{\ln 3}{16}$
$I_4 = \left[\frac{f(\sigma)}{-3}, \frac{f(1+\sigma)}{6+8\sigma}\right]$	$\frac{\ln 7}{2}$	$I_{-10} = \left[\frac{f(1-\sigma)}{2}, \frac{f(\sigma)}{1+8\sigma}\right]$	0
$I_3 = \left[\frac{f(1+\sigma)}{7+8\sigma}, \frac{f(\sigma)}{-3}\right]$	$\frac{\ln 51}{4}$	$I_{-11} = \left[f(1-\sigma), \frac{f(1-\sigma)}{2}\right]$	0
$I_2 = \left[\frac{f(1+\sigma)}{8+8\sigma}, \frac{f(1+\sigma)}{7+8\sigma}\right]$	$\frac{\ln 51}{4}$	$I_{-12} = \left[\frac{f(\sigma)}{8\sigma}, f(1-\sigma)\right]$	0
$I_1 = \left[\frac{f(\sigma)}{-4}, \frac{f(1+\sigma)}{8+8\sigma}\right]$	$\frac{\ln 69}{4}$	$I_{-13} = \left[-\infty, \frac{f(\sigma)}{8\sigma}\right]$	0
$I_0 = \left[\frac{f(\sigma)}{4+8\sigma}, \frac{f(\sigma)}{-4}\right]$	$\ln 3$		

$$\overline{\Gamma}^{\infty}_{\mathbf{k}} = \Gamma_{\mathbf{k}}(\overline{\mathcal{S}}) := \mathrm{card}(\pi_{\mathbf{k}}(\overline{\mathcal{S}}))$$
$$\underline{\Gamma}^{\infty}_{\mathbf{k}} = \Gamma_{\mathbf{k}}(\underline{\mathcal{S}}) := \mathrm{card}(\pi_{\mathbf{k}}(\underline{\mathcal{S}})),$$

where $\overline{\mathcal{S}}$, $\underline{\mathcal{S}}$ are as defined in Section 5. The upper and lower bounds for the spatial entropy, $\overline{h} := h(\overline{\mathcal{S}})$, and $\underline{h} := h(\underline{\mathcal{S}})$ have been defined in (5.4).

On the other hand, with considerations of boundary conditions, under the same parameters, we set $\underline{\mathcal{S}}^B_{\mathbf{k}}$ (respectively, $\overline{\mathcal{S}}^B_{\mathbf{k}}$) as the class of mosaic patterns on $T_{\mathbf{k}}$ obtained from attaching all affirmatively feasible (respectively, all affirmatively feasible and possible) basic patterns for (sd-DE)$_B$, where $B =$ N, P, D. Moreover, let $\Gamma^B_{\mathbf{k}} := \Gamma(\mathcal{S}^B_{\mathbf{k}})$ (respectively, $\overline{\Gamma}^B_{\mathbf{k}} := \Gamma(\overline{\mathcal{S}}^B_{\mathbf{k}})$, $\underline{\Gamma}^B_{\mathbf{k}} := \Gamma(\underline{\mathcal{S}}^B_{\mathbf{k}})$) be the number of patterns in $\mathcal{S}^B_{\mathbf{k}}$ (respectively, $\overline{\mathcal{S}}^B_{\mathbf{k}}$, $\underline{\mathcal{S}}^B_{\mathbf{k}}$). Accordingly, we have

$$\begin{aligned} \overline{h}_B &= h(\overline{\mathcal{S}}^B_{\mathbf{k}}) := \lim_{\mathbf{k}\to\infty} \frac{1}{k_1\cdots k_d} \ln \overline{\Gamma}^B_{\mathbf{k}}, \\ h_B &= h(\mathcal{S}^B_{\mathbf{k}}) := \lim_{\mathbf{k}\to\infty} \frac{1}{k_1\cdots k_d} \ln \overline{\Gamma}^B_{\mathbf{k}}, \\ \underline{h}_B &= h(\underline{\mathcal{S}}^B_{\mathbf{k}}) := \lim_{\mathbf{k}\to\infty} \frac{1}{k_1\cdots k_d} \ln \underline{\Gamma}^B_{\mathbf{k}}. \end{aligned} \tag{6.1}$$

We propose a criterion for $h = h_B$, where $B = \mathrm{N}$, P or D, in the following proposition. For $\mathbf{k} = (k_1, \ldots, k_d)$, and $s \in \mathbb{R}$, by $\mathbf{k} - s$, we mean $(k_1 - s, \ldots, k_d - s)$.

Proposition 6.1. *If there are fixed positive integers s, r such that (i) $\underline{\Gamma}^B_{\mathbf{k}} \geq \overline{\Gamma}^{\infty}_{\mathbf{k}-s}$, for all $\mathbf{k} > s$, and (ii) $\overline{\Gamma}^B_{\mathbf{k}} \leq p^c \cdot \underline{\Gamma}^{\infty}_{\mathbf{k}-r}$ for some $p > 0$ and $c = c(\mathbf{k})$ with $\lim_{\mathbf{k}\to\infty} c/(k_1 \cdots k_d) = 0$, then $\underline{h} = \underline{h}_B = h = h_B = \overline{h} = \overline{h}_B$, where $B = \mathrm{N}$, P or D.*

Proof. Let us prove the two-dimensional case. From condition (i), we have

$$\underline{h}_B = \lim_{\mathbf{k}\to\infty} \frac{1}{k_1 k_2} \ln \underline{\Gamma}^B_{\mathbf{k}} \geq \lim_{\mathbf{k}\to\infty} \frac{1}{k_1 k_2} \ln \overline{\Gamma}^{\infty}_{\mathbf{k}-s} = \lim_{\mathbf{k}\to\infty} \frac{(k_1 - 2s)(k_2 - 2s)}{k_1 k_2} \frac{\ln \overline{\Gamma}^{\infty}_{\mathbf{k}-s}}{(k_1 - 2s)(k_2 - 2s)} = h(\overline{\mathcal{S}}) = \overline{h}.$$

Condition (ii) yields that

$$\overline{h}_B = \lim_{\mathbf{k}\to\infty} \frac{1}{k_1 k_2} \ln \overline{\Gamma}^B_{\mathbf{k}} \leq \lim_{\mathbf{k}\to\infty} \frac{1}{k_1 k_2} \ln(p^c \cdot \underline{\Gamma}^{\infty}_{\mathbf{k}-r}) = \lim_{\mathbf{k}\to\infty} \frac{(k_1 - 2r)(k_2 - 2r)}{k_1 k_2} \frac{c \ln p + \ln \underline{\Gamma}^{\infty}_{\mathbf{k}-r}}{(k_1-2r)(k_2-2r)} = h(\underline{\mathcal{S}}) = \underline{h},$$

Therefore,

$$\underline{h}_B \geq \overline{h} \geq \underline{h} \geq \overline{h}_B.$$

On the other hand, $\underline{h}_B \leq \overline{h}_B$, from our definition. The assertion of the proposition thus follows. □

By applying Proposition 6.1, the following result can be derived.

Theorem 6.2. *$h = h_{\mathrm{N}} = h_{\mathrm{P}} = h_{\mathrm{D}}$ for the mosaic patterns of (sd-DE) on one-dimensional lattice $d = 1$.*

The problem of whether if $h = h_{\mathrm{N}} = h_{\mathrm{P}} = h_{\mathrm{D}}$ is much more complicated for the two-dimensional case $d = 2$. Condition (ii) of Proposition 6.1 holds for the situation herein. In several cases, we can carry out the examination for condition (i). We have not found a situation for $h \neq h_B$, as there are two examples for $h \neq h_{\mathrm{D}}$ and for $h \neq h_{\mathrm{N}}$ in cellular neural networks [8]. One observation is that the feasible basic patterns, corresponding to each parameter region, exist in groups in which ⊕ and ⊖ seem to play equal roles. The observation certainly depends on the configuration for the graph of nonlinearity *f*.

7. Numerical Illustrations

In this section, we shall demonstrate several two-dimensional mosaic patterns for the spatially discrete diffusion equations. Herein, we employ our basic pattern formation to produce these patterns for (1.2) with cubic nonlinearity (1.3). In the illustrations, we impose the Dirichlet boundary condition by setting $\tilde{u}_{ij} = 0$, for $(i, j) \in \mathbf{b}$ (see Section 2). We first explore basic patterns needed to compose the desired patterns and locate the parameters for these basic patterns. To justify our construction, we compute the numerical solutions to system (1.2). It can be seen from the computations that each component of the solution lies within the σ-ranges centered at $-1, 0, 1$. We color the patterns as in Figs. 4 and 5 to enhance the effect of demonstration.

Example 2. Checkerboard with horizontal interface.

Fig. 6 is a 7×7 checkerboard with horizontal interface. The 3×3 basic patterns needed to generate this checkerboard, through attaching process, are collected in Fig. 7. If we choose $b_1, b_2 > 0$, the parameters which yield these

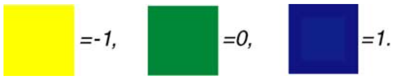

Fig. 4. Colors corresponding to solution values.

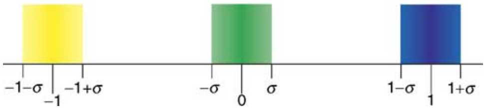

Fig. 5. Colors corresponding to solution values.

basic patterns satisfy the following conditions:

$$\begin{cases} b_1 > 0, \\ b_2 > 0, \\ (7+8\sigma)b_1 + (2+8\sigma)b_2 \le f(1+\sigma), \\ (8+8\sigma)b_1 + 8\sigma b_2 \le f(1+\sigma). \end{cases}$$

Let us choose the parameters $\sigma = 0.01$, $b_1 = 0.002$, $b_2 = 0.002$, which satisfy these conditions, to illustrate this pattern. The computed numerical solution (using Newton's method) is listed in Fig. 8. The associated pattern (in colors) for the numerical solution $\{\overline{u}_{i,j}\}_{1\le i,j\le 7}$ obviously matches the one in Fig. 6.

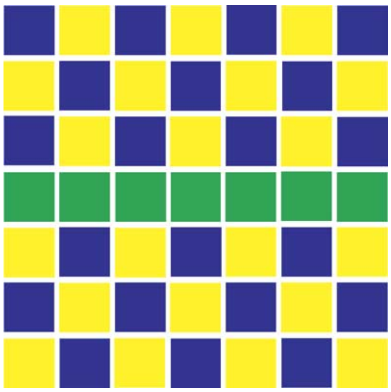

Fig. 6. Checkerboard with horizontal interface.

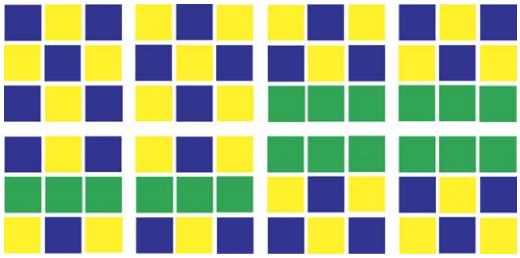

Fig. 7. Basic patterns for the checkerboard with horizontal interface.

1.0091	-1.0091	1.0091	-1.0091	1.0091	-1.0091	1.0091
-1.0091	1.0081	-1.0081	1.0081	-1.0081	1.0081	-1.0091
1.0091	-1.0091	1.0091	-1.0091	1.0091	-1.0091	1.0091
9e-017	1e-013	2e-012	-3e-008	-4e-005	-4e-005	-4e-005
-1.0091	1.0091	-1.0091	1.0091	-1.0091	0.9909	-1.0091
1.0091	-1.0081	1.0081	-1.0081	1.0081	-1.0080	1.0091
-1.0091	1.0091	-1.0091	1.0091	-1.0091	1.0091	-1.0091

Fig. 8. The numerical solution $\{\bar{u}_{i,j}\}_{1\le i,j\le 7}$ to (1.2) associated with the pattern of checkerboard with horizontal interface.

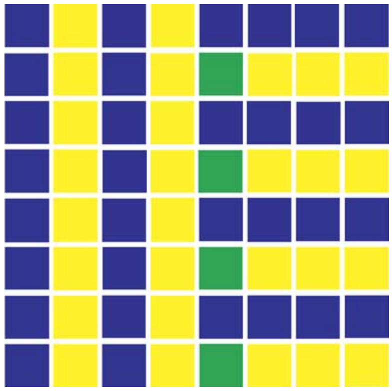

Fig. 9. Vertical and horizontal stripes with vertical interface.

Example 3. Vertical and horizontal stripes with vertical interface.

We need the 3×3 basic patterns in Fig. 10 to generate the vertical and horizontal stripes with vertical interface in Fig. 9 (a 8×8 herein), through the attaching process. For these basic patterns to exist, the following parameter conditions are needed:

$$\begin{cases} b_1 > 0, \\ b_2 > 0, \\ (5+8\sigma)b_1 + (8+8\sigma)b_2 \le f(1+\sigma). \end{cases}$$

As an illustration, we choose the parameters $\sigma = 0.01$, $b_1 = 0.002$, $b_2 = 0.001$. The numerical solution is listed in Fig. 11.

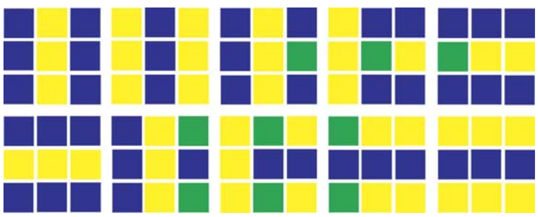

Fig. 10. Basic patterns for the vertical and horizontal stripes with vertical interface.

1.0065	-1.0081	1.0081	-1.0076	1.0071	1.0055	1.0060	1.0065
1.0060	-1.0081	1.0081	-1.0071	-4e-006	-1.0091	-1.0081	-1.0081
1.0060	-1.0081	1.0081	-1.0071	1.0081	1.0071	1.0081	1.0081
1.0060	-1.0081	1.0081	-1.0071	6e-011	-1.0091	-1.0081	-1.0081
1.0060	-0.9920	1.0080	-1.0071	1.0081	1.0071	1.0081	1.0081
1.0060	-1.0081	1.0081	-1.0071	1e-009	-1.0091	-1.0081	-1.0081
1.0060	-1.0081	1.0081	-1.0071	1.0081	1.0071	1.0081	1.0081
1.0065	-1.0081	1.0081	-1.0070	-0.0020	-1.0070	-1.0060	-1.0065

Fig. 11. The numerical solution $\{\overline{u}_{i,j}\}_{1\leq i,j\leq 8}$ to (1.2) associated with the pattern of vertical and horizontal stripes with vertical interface.

Example 4. Checkerboard with diagonal interface.

The pattern of checkerboard with diagonal interface in Fig. 12 could be obtained by attaching the 3×3 basic patterns in Fig. 13 with the chosen parameters $\sigma = 0.01$, $b_1 = 0.002$, $b_2 = 0.002$ which satisfy the conditions:

$$\begin{cases} b_1 > 0, \\ b_2 > 0, \\ 8\sigma b_1 - 8\sigma b_2 \geq f(1-\sigma), \\ (6+8\sigma)b_1 + (2+8\sigma)b_2 \leq f(1+\sigma), \\ (8+8\sigma)b_1 + (1+8\sigma)b_2 \leq f(1+\sigma). \end{cases}$$

Example 5. Quad junction.

The pattern of quad junction in Fig. 14 could be obtained by attaching the 3×3 basic patterns in Fig. 15 with the chosen parameters $\sigma = 0.01$, $b_1 = 0.002$, $b_2 = 0.001$ which satisfy the conditions:

$$\begin{cases} b_1 > 0, \\ b_2 > 0, \\ (4+8\sigma)b_1 + (8+8\sigma)b_2 \leq f(1+\sigma). \end{cases}$$

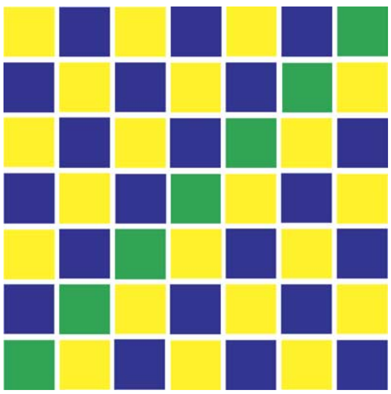

Fig. 12. Checkerboard with diagonal interface.

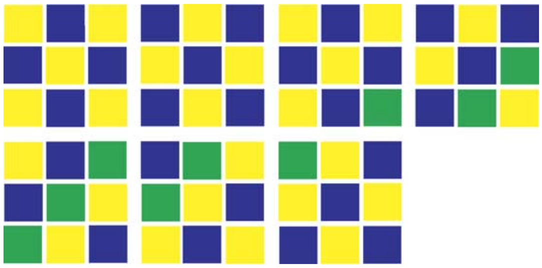

Fig. 13. Basic patterns for the checkerboard with diagonal interface.

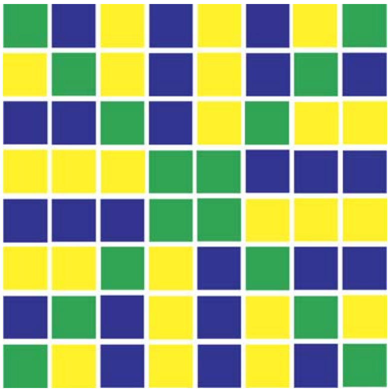

Fig. 14. Quad junction.

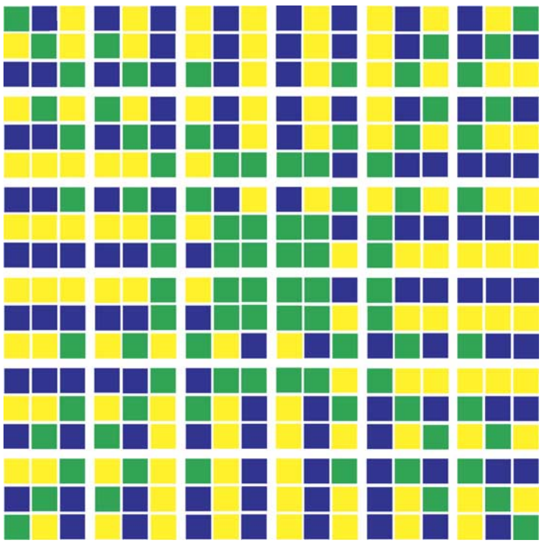

Fig. 15. Basic patterns for the quad junction.

Acknowledgments

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