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LOWER BOUNDS OF INNER PRODUCTS

TENG CHING-CHENG
College of Engineering, National Chiao Tung University

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Abstract: *Converse of Schwarz inequality which provides lower bounds for real-value inner products is stated and proved. Application to the lower bounds of an integral inner product is given in example.*

1. INTRODUCTION

In this paper, wə shall discuss the lower bounds for real positive definite inner products in vector spaces V over R. The inner product of two vectors v and w in V is denoted by $<v, w>$. The upper bound of this inner product is given by the famous Schwarz inequality which reads $|<v, w>|\leq||v|| \ ||w||$ for all v, w in V. Here, We intend to reverse the Schwarz inequality. Consequently, we have a lower bound of this inner product which gives a useful application to the approximation of the evaluation of integrations.

2. CONVERSE OF SCHWARZ INEQUALITY

The angle between two vectors v and w in vector space V as defined is expressed by

$$\cos\theta=\frac{<v, w>}{||v|| \ ||w||}$$

In this case Schwarz inequality merely amounts to the statement that the cosine of a real angle is ≤ 1. For the sake of reversing the Schwarz inequality, we assume that there exists a real number K such that $|<v,w>|\geq K||v|| \ ||w||$ for all v,w in V. The existence of K is guaranted by the following theorem.

Theorem 1. Let V be as defined, and let $v,w \in V$ be nonzero vectors. Then $|<v,w>|\geq K||v||||w||$

where $$K=\frac{<v,w>^2}{||v||^2 \ ||w||^2}$$

Equality holds if v,w are linearly dependent.

Proof: By Schwarz inequality, it is obvious.

In fact $K=\cos^2\theta\leq 1$, where θ is the angle between v and w. The obvious fact of theorem 1 is that the length of a projection is less than or equal to that of the vector projected. In this case, v is projected to w, and the projection of v along w is projected to v again.

A more useful theorem concerning three vectors is as follows:

Theorem 2. Let V be as defined, ana let $v,w,u \ \varepsilon V$ be any nonzero vectors such that

$$|<v,w>|\geq K_1||v||\ ||w||$$

$$|<v,u>|\geq K_2||v||\ ||u||$$

where $K_1=\cos^2\theta$, $K_2=\cos^2\phi$ as defined in theorem 1. Then

$$|<w,u>|\geq K_3||w||\ ||u||$$

where $K_3=K_1K_2+(1-K_1)(1-K_2)-2\sqrt{K_1K_2(1-K_1)(1-K_2)}$

Proof: From theorem 1, we see that K_3 is the square of cosine of the angle β between vectors w and u. It is clearly sufficient to consider only the case $\beta=\theta+\phi$. Hence, by the identity of cosine function $\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$ we immediately obtain

$$K_3=\cos^2\beta=\cos^2(\theta+\phi)=K_1K_2+(1-K_1)(1-K_2)-2\sqrt{K_1K_2(1-K_1)(1-K_2)}$$

As a comparison with Moore's result[1], let $K_1=1-\varepsilon$, $K_2=1-\varepsilon$, then $K_3=1-4\varepsilon+4\varepsilon^2=(1-2\varepsilon)^2$

where $0\leq\varepsilon\leq1$, here, we have not any annoying for the constraint for $\varepsilon<0.268$ as in (1). If $\varepsilon=\frac{1}{2}$, then $K_3=0$. This is jast the case the two vectors w and u are orthogonal.

3. APPLICATION

The application of converse of Schwarz inequality to the lower bound of integral inner product is illustrated by using the same example as in (1), for the sake of comparison. Example: Let $V=L_2(0,\infty)$ with the usual integral inner product. Consider the two vectors w and u in V given by

$$w(t)=\exp\left[-\left(\frac{1}{2}t^2\right)\right]\cos[(1+\alpha)t]$$

$$u(t)=\exp\left[-\left(\frac{1}{2}t^2\right)\right]\cos[(1-\alpha)t]$$

where α is a parameter, small in absolute value. Suppose that we are interested in the inner product

$$I(\alpha)=<w,\ u>=\int_0^\infty w(t)\,u(t)dt=\int_0^\infty \exp[-t^2]\cos[(1+\alpha)t]\cos[(1-\alpha)t]dt$$

which we pretend to be difficult to evaluate exactly. We therefore desire a lower bound $L(\alpha)$ for $|I(\alpha)|$ in addition to the usual upper bound provided by the Schwarz inequality. Vector v is properly chosen to be $v(t)=\exp\left[-\left(\frac{1}{2}t^2\right)\right]$ so that we can find easily that

$$||v||^2=\sqrt{\frac{\pi}{2}}$$

$$||w||^2=\sqrt{\frac{\pi}{2}}\exp\left[-\frac{1}{2}(1+\alpha)^2\right]\cosh\left[\frac{1}{2}(1+\alpha)^2\right]$$

$$||u||^2 = \sqrt{\frac{\pi}{2}} \exp\left[-\frac{1}{2}(1-\alpha)^2\right] \cosh\left[\frac{1}{2}(1-\alpha)^2\right]$$

$$<v, w> = \sqrt{\frac{\pi}{2}} \exp\left[-\frac{1}{4}(1+\alpha)^2\right]$$

$$<v, u> = \sqrt{\frac{\pi}{2}} \exp\left[-\frac{1}{4}(1-\alpha)^2\right]$$

and that $K_1 = \frac{<v,w>^2}{||v||^2\ ||w||^2} = \frac{1}{\cosh\left[\frac{1}{2}(1+\alpha)^2\right]}$

$$K_2 = \frac{<v,u>^2}{||v||^2\ ||u||^2} = \frac{1}{\cosh\left[\frac{1}{2}(1-\alpha)^2\right]}$$

$$K_3 = K_1K_2 + (1-K_1)(1-K_2) - 2\sqrt{K_1K_2(1-K_1)(1-K_2)}$$

From theorem 2 therefore, a lower bound $L(\alpha)$ for $|I(\alpha)|$ is given by

$$L(\alpha) = K_3(\alpha)\ ||w||\ ||u||$$

$$= K_3(\alpha)\sqrt{\frac{\pi}{2}} \exp\left[-\frac{1}{2}(1+\alpha^2)\right]\sqrt{\cosh\left[\frac{1}{2}(1+\alpha)^2\right] \cosh\frac{1}{2}(1-\alpha)^2}$$

On the other hand, the integral for $I(\alpha)$ is well known, it is

$$I(\alpha) = \sqrt{\frac{\pi}{2}} \exp\left[-\frac{1}{2}(1+\alpha^2)\right] \cosh\left[\frac{1}{2}(1-\alpha^2)\right]$$

Curves for comparison of $I(\alpha)$ and $L(\alpha)$ as functions of α are shown in Fig. 1. A curve $L_1(\alpha)$ found by Moore[1] is also shown in the figure. It is clear that curve $L(\alpha)$ is better than $L_1(\alpha)$ for the lower bound of $I(\alpha)$. We also see that $L_1(\alpha)$ is zero for $|\alpha| > 0.536$. It is the worst case for the lower bound of $I(\alpha)$.

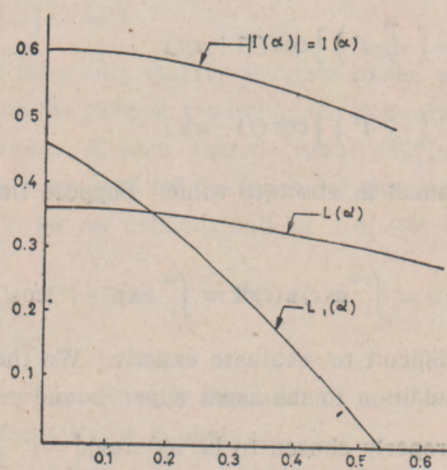

Fig. 1. Plot of $|I(\alpha)|$, $L(\alpha)$ and $L_1(\alpha)$ versus α.

REFERENCES

1. Michael H. Moore "An inner product inequality," SIAM J. MATH. ANAL. Vol. 4, No. 3, August 1973.