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A NOTE ON THE SOLUTION OF A DIFFERENCE EQUATION WITH THREE INDICES

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Abstract—*The generating-function technique for the solution of a linear three-index difference equaition with constant coefficients is proposed. In addition, the algorithm for the proposed technique is presented.*

1. INTRODUCTION

The method of using recurrence relations, that are also called difference equations, is a very powerful one in enumerative problems. For illustration, consider an example in coding theory: Find the number of n-digit binary sequences that have exactly l pairs of adjacent 1's and m pairs of adjacent 0's. A direct enumeration of the number becomes complicated for increasing the integer n: however, this can be tackled by setting up difference equation and have it computed step-by-step through a high speed computer. To construct the difference ecuation, let $C_{n,l,m}$ denote that number of such sequences. Also, let $a_{n,l,m}$ denote the number of such sequences that have a 1 as the nth digit, and let $b_{n,l,m}$ denote the number of such sequences that have a 0 as the nth digit. Clearly.

$$C_{n,l,m}=a_{n,l,m}+b_{n,l,m} \tag{1}$$

$$a_{n,l,m}=a_{n-1,l-1,m}+b_{n-1,l,m} \tag{2}$$

$$b_{n,l,m}=a_{n-1,l,m}+b_{n-1,l,m-1} \tag{3}$$

Set $m=m-1$ in Equation (2) and then substract the resultant equation from Equation (3) to obtasn

$$b_{n,l,m}=a_{n,l,m-1}+a_{n-1,l,m}+a_{n-1,l,m}-a_{n-1,l-1,m-1} \tag{4}$$

Elliminating the number b from Equations (2) and (4) yield

$$a_{n,l,m}-a_{n-1,l,m-1}-a_{n-1,l-1,m}-a_{n-2,l,m}+a_{n-2,l-1,m-1}=0 \tag{5}$$

This is a homogeneous linear difference equation with three indices, that is to be solved subject to the boundary conditions:

$$\begin{aligned} &a_{i,0,0}=1 && \text{for } i\geq 1; \\ &a_{i,j,k}=0 && \text{for } i\leq j \text{ or } i\leq k \end{aligned} \tag{6}$$

Following the solution of Equation (5), the number C can be computed through Equations (4) and (1).

The subject of multi-index difference equations is also important in numerical avalysis of multi-variable differential equations, that are related to multi-dimensional physical problems. When digital computer is used to find the approximate solution of a differential equation, the differential equation is approximated by a difference equation. Thus, the multi-index difference equation, the solution of which is a discrete solution, is a quantized version of the multi-variable differential equation, the solution of which is a continuous function.

Only those multi-dimensional problems with homogeneous medium and simple boundary in mathematical physics can be solved exactly. Therefore, pursuing an approximate solution is required. Most problems in mathematical physics can be formulated as a three-variable differential equation together with some boundary conditions. Quantizing the differential equation as well as the boundary dondition, the problem is reduced to that of solving a finite three-index difference equation with the following general form

$$\sum_{i=0}^{p} \sum_{j=0}^{q} \sum_{k=0}^{r} C_{ijk}\ a_{n-i,l-j,m-k} = f(n, l, m) \tag{7}$$

where the $C's$ are constants, and the equation is valid only for $n \geq \alpha_1 \geq p$, $l \geq \alpha_2 \geq q$ and $m \geq \alpha_3 \geq r$. This equation is to be solved subject to the boundary conditions that are obtained approximately from the boundary conditions associated with the differential equation.

In the subsequent section, we shall present a systematic method for solving Bquation (7)

2. THE TECHNIQUE OF GENERATING FUNCTIONS

The technique of using multi dimensional Laplace transformation to solve linear multi-variable differential equations is well known. By analogy with Laplace transformation, there is a technique of ordinary generating function for solving linear multi-index difference equations. The procedure of the technique will be introduced subsequently.

To solve a linear difference equation with three indices, we first define a sequence of generating functions with one function for every two values of two of three indices, that is

$$A_{ij}(W) = a_{i,j,0} + a_{i,j,1}w + a_{i,j,2}w^2 + \ldots\ldots + a_{i,j,r}w^r + \ldots\ldots \text{for } i, j = 0, 1, 2, 3, \ldots\ldots \tag{8}$$

where w is called a formal variable. The generating function $A_{ij}(w)$ can be regarded as a transform of a discrete function (a sequence of numbers), that is a quantized version of one dimensional laplace transform of a continuous function.

Using powers of v, we can also define a generating function of the sequence $\{A_{i0}(w), A_{i1}(w), A_{i2}(w), \ldots\ldots\}$, that is

$$\begin{aligned} A_i(v, w) &= A_{i0}(w) + A_{i1}(w)v + A_{i2}(w)v^2 + \ldots\ldots + A_{iq}(v)v^q + \ldots\ldots \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{i,j,k} v^j w^k \text{ for } i = 0, 1, 2, \ldots\ldots \end{aligned} \tag{9}$$

This is analogous with the two-dimensional laplace transform of a continuous function of two real variables.

Furthermore, we can define a generating function of three fopmal variables u, v, w that is a quantized version of three-dimensional laplace transform of a continuous function of three real variables.

$$Q(u, v, w)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{j,j,k}u^i v^j w^k \tag{10}$$

Now we are ready to introduce a procedure for reducing Equation (7) to an algebraic equation for $Q(u, v, w)$: Let both sides of Equation (7) be multiplied by $u^n v^l w^m$ and then have it summed for n from α_1 to ∞, for l from α_2 to ∞, and for m from α_3 to ∞. This gives

$$\sum_{n=\alpha_1}^{\infty}\sum_{l=\alpha_2}^{\infty}\sum_{m=\alpha_3}^{\infty}\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r} C_{ijk}a_{n-i,\, l-j,\, m-k}u^n v^l w^m=\sum_{n=\ 1}^{\infty}\sum_{l=\alpha_2}^{\infty}\sum_{m=\alpha_3}^{\infty} f(n, l, m)u^n v^l w^m \tag{11}$$

Since

$$\sum_{n=\alpha_1}^{\infty}\sum_{l=\alpha_2}^{\infty}\sum_{m=\alpha_3}^{\infty} C_{ijk}a_{n-1,\, l-j,\, m-k}u^n v^l w^m=C_{ijk}[Q(u, v, w)-\sum_{n=0}^{\alpha_1-1}\sum_{l=0}^{\alpha_2-1}\sum_{m=0}^{\alpha_3-1} a_{n-1,\, l-j,\, m-k}u^n v^l w^m]$$

interchanging the order of summations and simplifying the resultant equation with the above relation. we can reduce Equation (11) to an algebraic equation, by which $Q(u, v, w)$ can be solved. This yields

$$\begin{aligned} Q(n, v, w)=&\sum_{i=0}^{\alpha_1-p-1}\sum_{j=0}^{\alpha_2-q-1}\sum_{k=0}^{\alpha_3-r-1} a_{i,j,k}u^i v^j w^k \\ &+\frac{1}{\sum_{i=0}^{p}\sum_{j=0}^{q}\sum_{k=0}^{r} C_{ijk}u^i v^j w^k}[\sum_{n=\alpha_1}^{\infty}\sum_{l=\alpha_2}^{\infty}\sum_{m=\alpha_3}^{\infty} f(n, l, m)u^n v^l w^n + H.S.] \end{aligned} \tag{12}$$

where
$$\begin{aligned} H.S.=&C_{000}\sum_{i=0}^{p-1}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1} a_{\alpha_1-p+i,\, \alpha_2-q+j,\, \alpha_3-r+k}u^{\alpha_1-p+i}v^{\alpha_2-q+j}w^{\alpha_3-r+k} \\ &+C_{100}\sum_{i=0}^{p-2}\sum_{j=0}^{q-1}\sum_{k=0}^{r-1} a_{\alpha_1-p+1,\, \alpha_2-q+j,\, \alpha_3-r+k}u^{\alpha_1-p+i+1}v^{\alpha_2-q+j}w^{\alpha_3-r+k}+\ldots\ldots \\ &+C_{112}\sum_{i=0}^{p-2}\sum_{j=0}^{q-2}\sum_{k=0}^{r-3} a_{\alpha_1-p+i,\, \alpha_2-q+j,\, \alpha_3-r+k}u^{\alpha_1-p+i+1}v^{\alpha_2-q+j+1}w^{\alpha_3-r+k}+\ldots\ldots \\ &+C_{p-1,\, q-1,\, r-1}a_{\alpha_1-p,\, \alpha_2-q,\, \alpha_3-r}u^{\alpha_1-1}v^{\alpha_2-1}w^{\alpha_3-1} \end{aligned}$$

From Equation (12), we observe that arbitrary choice of the coefficients of the first term on the right side will not effect the solution. In addition, the first term inside the bracket represents the particular solution; while the second term represents the homogeneous solution.

In the last step of the procedure, the required values of the a's can be obtained by comparing the coefficients of the $u^i v^j w^k$'s on the left side of Equation (12) with those coefficients of the corresponding terms on the right side.

We shall illustrate the procedure described above by solving Equation (5). With the notations given in Equation (7), we have

$$C_{000}=1,\ C_{101}=-1,\ C_{110}=-1,\ C_{200}=-1,\ C_{211}=1;$$

$$p=2,\ q=r=1;$$

$$\alpha_1=3,\ \alpha_2=\alpha_3=1;$$

$$a_{1,0,0}=a_{2,1,0}=a_{2,0,0}=1$$

$$a_{1,1,0}=a_{1,0,1}=a_{1,1,1}=a_{2,1,1}=a_{2,2,1}=a_{2,1,2}=a_{2,2,2}=0;$$

and the value a_{000} is not constrained by the difference equation.
Substituting the above data into Equation (12) yields

$$Q(u,v,w)=\frac{u+u^2}{1-(v+w)u-(1-vw)u^2}$$

Carrying out the long division and then sorting the required coefficient, we can obtain the solution.

3. LONG DIVISION

As demonstrated in the previous section, "long division" is a key step in the generating-function technique. For computerizing the technique, we need to develop a subroutine for long division. In FORTRAN, string manipulation is not permitted. Therefore, the procedure of long division is carried out by using arrays. For illustration; consider the following example:

$$\frac{1-2x+x^2-x^3}{1-2x+x^2-2x^3}$$

The program is written in the following.

```
      DIMENSION    A(10), B(10), C(20)
   10 FORMAT       (I2)
   12 FORMAT       (F4.1)
   14 FORMAT       (F10.3)
   20 READ (1, 10) L, M, N
      READ (1, 12) (A(I), I=1, L)
      READ (1, 12) (B(I), I=1, M)
   22 DO 30 I=1, N
      O=A(1)/B(1)
      L=L-1
   24 DO 28 J=1, L
   28 A(J)=A(J+1)-Q*B(J+1)
      L=L+1
      A(L)=0.0
   30 C(I)=Q
   40 WRITE (1, 14) (C(I), I=1, N)
      STOP
      END
```