

OPTIMAL OUTPUT CONTROL OF LINEAR REGULATOR SYSTEMS WITH SOME INACCESSIBLE STATES

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Abstract

In modern control theory designs require feedback of every state variable. However, in practice, many chemical process controls only a few output quantities are available, and thus the practical usefulness of optimal control law is limited in these cases. It is necessary to construct estimates of inaccessible state variables by direct output measurement. A technique based on a very straight forward method leading to an optimal proportional control in terms of measurable output alone is presented.

OPTIMAL CONTROL OF A LINEAR SINGLE-INPUT SYSTEM WITH INACCESSIBLE STATES

1. INTRODUCTION

Many control system designs are based on state vector feedback, where the input to the system is a function only of the state vector. Several new techniques such as dynamic programming^{1, 2, 3}, Pontryagin's maximum principle⁴, and methods based on Lyapunov's theory⁵ have been developed. However, in most control situations, all the state variables are not available for direct measurement. In these cases, either the method must be abandoned or a reasonable substitute for the state vector must be found.

The essential problem in designing an optimal control system with inaccessible state variables is one of an estimation. A great amount of work which describes the optimization of a system with inaccessible states on the basis of operating on a "best" estimate of the unavailable states has been documented by Kalman⁶⁻⁸, Luenberger⁹⁻¹⁰, and others. Kalman has given the treatment of the estimation problem for discrete system. Luenberger has shown that the state vector of a linear system can be reconstructed from "observations" of the system inputs and outputs. Rekasius¹¹ is based on minimizing a secondary performance functional at the "worst" initial state for a specified controller configuration. The definition of the optimal control law proposed by Rekasius is theoretically sound, and Raman and Ramaswam¹² have dealt with the practical ability and the difficulties encountered in designing higher order systems. Pearson and Ding¹³⁻¹⁴ have shown that the optimal control may be realized without measuring every state variable of the plant. Ferguson and Rekasius¹⁵ have obtained results for a single-input plant with a quadratic performance index which is different than the usual form in the sense that it has some terms involving derivatives of the plant. Liou et. al.¹⁶ have presented two methods of obtaining optimal dynamic controllers when some of the state variables are not available for continuous measurement.

In this paper, a very straight forward method based on a very general performance index leading to an optimal proportional control in terms of measurable output alone is developed. And, the development can be easily extended to time-varying problems.

2. OPTIMAL CONTROL OF A LINEAR SINGLE-INPUT SYSTEM WITH INACCESSIBLE STATES

Consider first the completely controllable plant which may be represented by the first order differential equations.

$$\dot{x}(t) = Ax(t) + bu(t) \quad (1)$$

where $x(t)$ is an n -vector, b is also an n -vector, A is an $n \times n$ matrix, and $u(t)$ is a scalar. A linear feedback control law $u(t) = -kx(t)$ is desired such

that the resulting closed-loop system is optimal.

The performance index to be minimized is

$$J = \int_0^{\infty} (x^T Q x + p u^2) dt \quad (2)$$

where Q is an $n \times n$, non-negative definite matrix and p is a positive constant. The solution of this problem is

$$u(t) = - \frac{1}{p} b^T S x(t) \quad (3)$$

where S is an $n \times n$ positive definite, symmetric matrix which satisfies the Riccati equation

$$SA + A^T S + Q = \frac{1}{p} S b b^T S \quad (4)$$

The use of $u(t)$ as defined by Equation (3) is optimal for all initial states with respect to the quadratic performance criterion of Equation (2)

Since the terminal time of the performance index is infinite, the dynamic system is required to be asymptotically stable. Also, the optimal control must be unique for all initial states¹⁷.

In the sequel it is assumed that the plant is completely controllable and has only $(n-r)$ state variables, $x^{(1)}$, available for measurement and that the remaining r state variables, $x^{(2)}$, are not available for measurement.

3. OPTIMAL PROPORTIONAL CONTROL

Considering again the plant, Equation (1), and the performance index, Equation (2), we can obtain the optimal control law

$$u(t) = -kx(t) \quad (5)$$

Substituting Equation (5) into Equation (1), we get

$$\dot{x} = (A - bk)x \quad (6)$$

The solution of Equation (6) is given by

$$x(t) = \phi(t) x_0 \quad (7)$$

where

$$\phi(t) = e^{(A-bk)t}$$

Separating state variables into two parts, measureable and unmeasurable, which are defined as $x^{(1)}(t)$ and $x^{(2)}(t)$ respectively, we can write Equation (7) as

$$x^{(1)}(t) = \phi_{11}(t)x^{(1)}(0) + \phi_{12}(t)x^{(2)}(0) \quad (8)$$

$$x^{(2)}(t) = \phi_{21}(t)x^{(1)}(0) + \phi_{22}(t)x^{(2)}(0)$$

Now assume there is an α such that

$$x^{(2)}(0) = \alpha x^{(1)}(0) \quad (9)$$

where α is an $r \times (n-r)$ matrix.

By elimination and substitution, Equation (8) becomes

$$x^{(1)}(t) = [\phi_{11}(t) + \phi_{12}(t)\alpha]x^{(1)}(0) \quad (10a)$$

$$x^{(2)}(t) = [\phi_{21}(t) + \phi_{22}(t)\alpha]x^{(1)}(0) \quad (10b)$$

Suppose $[\phi_{11}(t) + \phi_{12}(t)\alpha]$ is a non-singular matrix, then Equation (10a) can be written as:

$$x^{(1)}(0) = [\phi_{11}(t) + \phi_{12}(t)\alpha]^{-1} x^{(1)}(t)$$

By substitution, Equation (10b) becomes

$$x^{(2)}(t) = [\phi_{21}(t) + \phi_{22}(t)\alpha][\phi_{11}(t) + \phi_{12}(t)\alpha]^{-1} x^{(1)}(t) \quad (11)$$

Thus we conclude that the unavailable state can be replaced by the available state provided α is given and the inverse indicated in Equation (11) exists. Therefore, the control law $u = -(k_1 x^{(1)} + k_2 x^{(2)})$ can be written as function of measurable part of the state and the initial state ratio α .

$$u(t) = -G(t, \alpha) x^{(1)}(t) \quad (12)$$

where

$$G(t, \alpha) = k_1 + k_2 [\phi_{21}(t) + \phi_{22}(t)\alpha][\phi_{11}(t) + \phi_{12}(t)\alpha]^{-1}$$

Example 1

Consider the control of a second-order system,

$$\frac{d^2 c}{dt^2} + 2\xi \frac{dc}{dt} + c = u(t)$$

and output:

$$y(t) = c(t)$$

The objective is to minimize the performance index

$$J = \frac{1}{2} \int_0^\infty [c^2(t) + p u^2(t)] dt$$

with a given initial condition

$$c(0) = c_0, \quad \frac{dc}{dt}(0) = 0$$

Physically, the problem is really identical to the set-point control of a second-order process. The output variable $c(t)$ is defined as a deviation around the final steady state, so that $u=0$ is ultimately required to hold the process at the desired steady state. Thus, certain set-point problems can be treated as regulator problems if this definition of output variable is made.

Define the state variables as $x_1 = c$, $x_2 = dc/dt$. Then, this problem is described by

$$\dot{x} = Ax + bu$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2\xi \end{pmatrix}$$

$$b = (0 \quad 1)^T$$

We deduce immediately our performance index in terms of state variables

$$J = \frac{1}{2} \int_0^\infty [x_1^2(t) + p u^2(t)] dt$$

A check shows that the system is both controllable and observable.

Now, we determine the feedback gains on the basis of Riccati equation, and we get the optimal control

$$u(t) = -[(\delta - 1)x_1 + (\sqrt{4\xi^2 + 2(\delta - 1)} - 2\xi)x_2]$$

where

$$\delta = \sqrt{1 + p^{-1}}$$

It is equivalent to

$$u(t) = -(k_1 x_1 + k_2 x_2)$$

where

$$k_1 = \delta - 1$$

$$k_2 = \sqrt{4\xi^2 + 2(\delta - 1)} - 2\xi$$

Since x_2 is not available, the problem may now be reduced to determine the optimal control which depends on the output $y(t)$ only

The feedback gain is

$$G(t) = k_1 + k_2 \phi_{21}(t) \phi_{11}^{-1}(t) \quad (13)$$

We now construct the feedback by finding the transition matrix¹⁷.

Then

$$\phi(t) = \begin{pmatrix} e^{-Bt/2} (\cos rt + \frac{B}{2r} \sin rt) & \frac{1}{r} e^{-Bt/2} \sin rt \\ -\frac{\delta}{r} e^{-Bt/2} \sin rt & e^{-Bt/2} (\cos rt - \frac{B}{2r} \sin rt) \end{pmatrix}$$

where

$$B = \sqrt{4\xi^2 + 2(\delta - 1)}$$

$$r = \sqrt{\frac{\delta + 1}{2} - \xi^2}$$

Equation (13) yields

$$G(t) = k_1 - \frac{\delta k_2 \sin rt}{r \cos rt + \frac{B}{2} \sin rt}$$

and then

$$u(t) = \left(\frac{(k_1 + 1)k_2 \sin rt}{r \cos rt + \frac{B}{2} \sin rt} - k_1 \right) c(t)$$

4. DISCUSSIONS AND CONCLUSIONS

Optimal control laws usually require the complete measurement of the plant state. However, in practice, if not all the state variables (e.g. temperatures, pressures, compositions) are available for measurement and feedback, the optimal linear feedback gains depend on the ratio of initial states, α , and time.

Since α is an $r \times (n-r)$ matrix, Equation (9) represents r equations in $r \times (n-r)$ unknowns. If $n-r=1$, then α is determined uniquely, and a unique solution will result. Therefore, the inverse of $[\phi_{11}(t) + \phi_{12}(t)\alpha]$ always exists. If $n-r > 1$, then $r \times (n-r-1)$ of α_{ij} can be assigned arbitrarily, still

maintaining r independent equations, and then the remaining r will be uniquely specified. For the special case, when the initial conditions of inaccessible states are equal to zero, the controller gains are, in a sense, independent of initial states and are uniquely determined.

Several chemical process controls can be applied for this special case. For instance, set-point control of a second-order process is of this type. A typical application is given in Example 1. Also, this method can be easily extended to time-varying problems.

$$G(s) = \frac{b}{s^2 + as + c}$$

A DISCUSSION AND CONCLUSIONS

Optimal control laws usually require the complete measurement of the plant state. However, in practice, if not all the state variables (e.g., temperature, pressure, composition) are available for measurement and feedback, the optimal linear feedback gains depend on the initial states and time. Since it is an $r \times (n-r)$ matrix, Equation (9) represents r equations in $n-r$ unknowns. It is clear that it is determined uniquely and a unique solution will exist. Therefore, the inverse of Equation (9) always exists if $n-r > 1$, that is, $n-r-1$ of us can be assigned arbitrarily, and

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