ANALYSIS OF SAMPLED-DATA SYSTEMS IN THE W-PLANE

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SUMMARY: A computer program for doing bilinear transformation is presented, and the relations between z-plane and w-plane are investigated. Various methods for analyzing sampled-data control systems in the w-plane are considered.

I. INTRODUCTION

It is well known that the commonly used methods such as Nyquist criterion, Bode diagram and Routh criterion are not suitable for the analysis of sampled–data systems; and it is believed by most of the authors that, after a bilinear transformation, the aforementioned methods can be applied easily, since the stability boundary is converted from a unit circle in the z–plane to an imaginary axis in the w–plane[1]. However, the analyses in this paper indicate that to analyze a sampled–data system in w–plane by no means an easy task.

The relations between z–plane and w–plane are studied in the next two sections, and the methods for analyzing sampled–data systems in the w–plane are considered in section IV.

II. RELATIONS BETWEEN Z-PLANE AND W-PLANE

The relations between z–plane and w–plane are defined by an equation

$$z=(1+w)/(1-w) \tag{1}$$

Consider an Nth order characteristic equation as

$$F(z)= \sum_{i=0}^{N} a_i z^i =0 \tag{2}$$

After transformation, Eq. (2) becomes

$$F(w) = \sum_{I=0}^{N} A_I w^I = 0 \tag{3}$$

where $\qquad A_0 = \sum_{i=0}^{N} a_i$, $\qquad\qquad\qquad$ for $I=0$

$$A_I = (-1)^I \sum_{i=0}^{N} a_i \left[\sum_{j=0}^{I} (-1)^j \binom{i}{j} \binom{N-i}{I-j} \right], \qquad \text{for } N > I > 1$$

where $\qquad i > j \qquad\qquad N-i > I-j$

$$A_N = \sum_{i=0}^{N} (-1)^{i+N} a_i, \qquad\qquad\qquad \text{for } I=N$$

with $\qquad \binom{i}{j} = \dfrac{i!}{j!(i-j)!}$

$$\binom{N-i}{I-j} = \dfrac{(N-i)!}{(I-j)!(N-i-I+j)!}$$

thus
$$F(w) = \sum_{i=0}^{N} a_i + \sum_{I=1}^{N-1} A_I w^I + \sum_{i=0}^{N} (-1)^i a_i w^N = 0 \tag{4}$$

Eq. (4) indicates that the transformation can be accomplished either by manual calculation or with a digital computer. A flow diagram for doing this is given in appendix I.

III. USEFUL CONTOURS IN THE W-PLANE

The same as in the s-plane and z-plane, some of the contours in w-plane are useful for analysis and design.

1) Contours of constant damping ratios (ρ):

The contours of constant damping ratios in s-plane are straight lines defined by

$$\alpha = - \frac{\rho}{\sqrt{1-\rho^2}} \omega \tag{5}$$

where $s = \alpha + j\omega$. In z-plane, the contours are heart-shaped curves defined by

$$z = e^{-\dfrac{\rho\omega}{\sqrt{1-\rho^2}}T} (\cos\omega T + j\ \sin\omega T) \qquad (6)$$

Thus in w-plane, the contours are defined by

$$w = (z-1)/(z+1) = \frac{\left[\cos\omega T - e^{\rho\omega T/\sqrt{1-\rho^2}}\right] + j\ \sin\omega T}{\left[\cos\omega T + e^{\rho\omega T/\sqrt{1-\rho^2}}\right] + j\ \sin\omega T}$$

$$= \frac{\left[\cos^2\omega T - e^{2\rho\omega T/\sqrt{1-\rho^2}} + \sin^2\omega T\right] + j2e^{\rho\omega T\sqrt{1-\rho^2}} \sin\omega T}{\left[\cos T + e^{\rho\omega T/\sqrt{1-\rho^2}}\right]^2 + \sin^2\omega T}$$

$$(7)$$

which are also heart-shaped curves.

2) Contours of constant damping factors (α):

The contours of constant damping factors in the s-plane are straight lines defined by

$$s = -\alpha \qquad (8)$$

while that in the z-plane are circles defined by

$$z = e^{\alpha T}/\omega T = e^{\alpha T} (\cos\omega T + j\ \sin\omega T) \qquad (9)$$

Thus in the w-plane

$$w = (e^{-\alpha T}e^{j\omega T} - 1)/(e^{-\alpha T}e^{j\omega T} + 1)$$

$$= \frac{(\cos\omega T - e^{\alpha T}) + j\ \sin\omega T}{(\cos\omega T + e^{\alpha T}) + j\ \sin\omega T}$$

$$= \frac{(1 - e^{2\alpha T}) + j2e^{\alpha T}\sin\omega T}{(\cos\omega T + e^{\alpha T})^2 + (\sin\omega T)^2} \qquad (10)$$

which represents a family of circles with centers on the real axis of the w-plane (see Appendix II).

A master chart which consists of both kinds of contours is given in Fig. 1, where the general shapes of the contours are independent to the sampling period (T), except in a scale change. If only the primary strip in

the s-plane is considered, the master chart in Fig. 1 represents a direct relation between s-plane and w-plane. In other words, if a characteristic root in w-plane is found, then its position in s-plane, in terms of damping ratio and damping factor, can be read directly; thus the root locus method can be applied, and design work can be accomplished in the w-plane.

IV. ANALYSIS OF SAMPLED-DATA SYSTEMS IN THE W-PLANE

In this section, methods for analysis of sampled-data systems in w-plane are considered. It is assumed that the transfer functions or characteristic equations has been transformed to w-plane already, and the main purpose is to find which method is the best one for analysis.

1) Nyquist criterion and Bode diagram:

In order to use these two methods, the transfer functions must be in a factored form, which is usually very complex, especially for high order systems; thus both of these methods are not suitable for analysis.

2) Routh criterion:

To apply Routh criterion in the w-plane is the same as that for continuous systems in the s-plane. Although the result of this method can be used only as an indication of the absolute stability of a system, to do a bilinear transformation and then to apply the Routh criterion would be much simpler than the methods of testing stability in the z-plane, such as Shu-Cohn's criterion and Jury's method,[1,2]

3) Stability curve method:[3]

This method gives the approximate locations of the characteristic roots in the w-plane by finding the relations among the real roots of two stability equations, which are the real and imaginary parts of the characteristic equation in w-form; thus more information can be obtained than that of the Routh criterion. Since the order of the stability equation is half of that of the characteristic equation, to find their real roots would be easier than to solve the characteristic equation.

4) Solving characteristic equations:

Using a digital computer to solve characteristic equations is a direct method for analysis, since characteristic roots can give all the required information; thus a bilinear transformation is not necessary, because the

characteristic roots in w–plane represent the same characteristics as that in the z–plane.

For an illustration, the aforementioned methods are applied to a system as in the following example.

Example: Consider a system with a 7th order characteristic equation as

$$F(z) = z^7 - 1.1z^6 + 0.46z^5 - 0.46z^4 + 0.152z^3 + 0.215z^2$$
$$- 0.163z + 0.0326 = 0 \tag{11}$$

After transformation, Eq. (11) becomes

$$F(w) = 2.765w^7 + 15.88w^6 + 27.14w^5 + 38.14w^4 + 29.8w^3$$
$$+ 11.93w^2 + 2.2w + 0.133 = 0 \tag{12}$$

By using Routh criterion, the result is

2.765	27.14	29.8	2.2
15.88	38.14	11.93	0.133
20.5	27.8	1.975	
16.6	10.45		
14.89			

which indicates that the system is stable.

Using stability curve method, the stability equation are

$$F_e(y) = 15.88y^3 + 38.14y^2 + 11.93y + 0.133 \tag{13}$$

$$F_o(y) = 2.765y^3 + 27.14y^2 + 29.8y + 2.2 \tag{14}$$

where $y = \nu^2$ and $w = j\nu$. Assume z_i and p_i are the absolute values of the real roots of Eqs. (13) and (14) (in terms of ν) respectively, then

$$z_1 = 0.108, \quad z_2 = 0.596, \quad z_3 = 1.43$$
$$p_0 = 0, \quad p_1 = 0.282, \quad p_2 = 1.08, \quad p_3 = 2.93$$

The root loci are sketched in Fig. 2, which indicate that the system has a reasonable damping.[3]

In case, Eq. (11) is solved directly, the characteristic roots are found at

$$z_{1,2} = 0.4 \pm j0.2, \quad a_3 = 0.5, \quad z_4 = -0.6$$
$$z_{5,6} = -0.2 \pm j0.8, \quad z_7 = 0.8$$

thus all the characteristics can be defined.

Although the above analyses indicate that to solve the characteristic equation in the z–form is better than all the methods used in the w–plane, the main purpose of this paper is to investigate the usage of the w–plane which has not been thoroughly studied in the current literature.

Appengix I
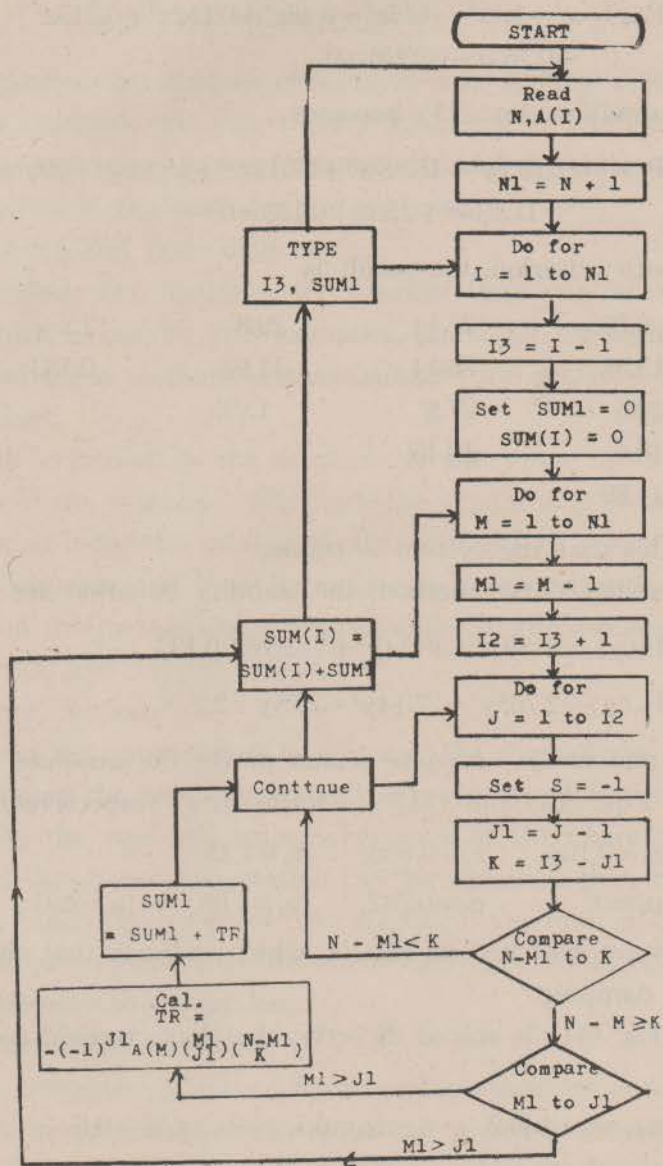
CONCLUSION

In this paper, a computer programming for doing bilinear transformation is presented; a master chart which relates the w-plane to the s-plane is given; and methods for analyzing sampled-data systems in the w-plane are considered. It is hoped that the results in this paper will be useful for those readers who want to know the actual usage of the bilinear transformation and to select a method of using a digital computer in analyzing sampled-data systems.

APPENDIX II

From Eq. (10),

$$w = Re + j\ Im \tag{15}$$

where Re and Im are the real and imaginary parts of Eq. (10) respectively. For $\omega = 0$ and $\omega_s/2$, Eq. (15) becomes, respectively,

$$w\ \Big|_{\omega=0} = (1 - e^{\alpha T})/(1 + e^{\alpha T}) \tag{16}$$

$$w\ \Big|_{\omega=\omega_s/2} = (1 + e^{\alpha T})/(1 - e^{\alpha T}) \tag{17}$$

Let

$$a = \Big| w\ \big|_{\omega=0} + r \Big| = (1 + e^{2\alpha T})/(-1 + e^{2\alpha T}) \tag{18}$$

where

$$r = \frac{1}{2}\Big[w\ \big|_{\omega=\omega_s/2} - w\ \big|_{\omega=0} \Big] = -2e^{\alpha T}/(e^{2\alpha T} - 1) \tag{19}$$

then

$$(Re + a)^2 + Im^2 = \Big[\frac{1 - e^{2\alpha T}}{(\cos\omega T + e^{\alpha T})^2 + (\sin\omega T)^2} + \frac{e^{2\alpha T} + 1}{e^{2\alpha T} - 1} \Big]^2$$

$$+ \Big[\frac{2e^{\alpha T} \sin\omega T}{(\cos\omega T + e^{\alpha T})^2 + (\sin\omega T)^2} \Big]^2 \tag{20}$$

$$= \frac{4e^{2\alpha T}}{(e^{2\alpha T} - 1)^2} = r^2 \tag{21}$$

Thus Eq. (10) represents a family of circles with centers at (a, 0) and with radii (r) defined by Eqs. (18) and (19) respectively.

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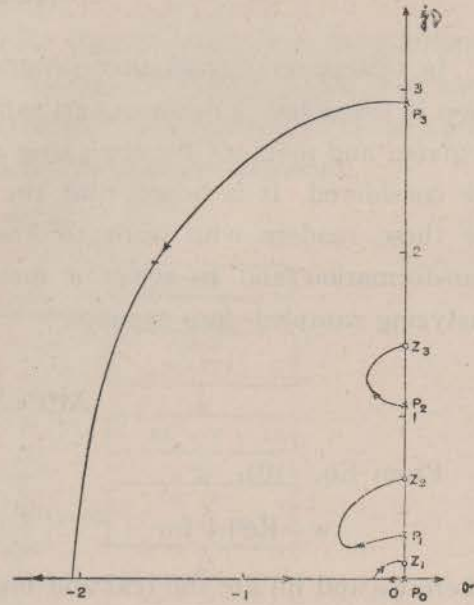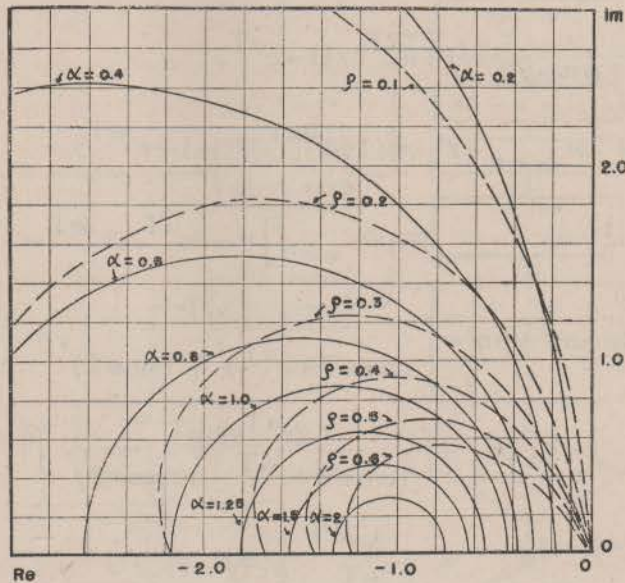
FIG. 2

ROOT LOCUS IN W-PLANE

FIG. 1

CONSTANT DAMPING RATIO AND DAMPING FACTOR
IN W-PLANE SAMPLING PERIOD T=1.0 SEC.