CALCULATION OF ELECTRON MOBILITY IN HIGH FIELDS BY PATH INTEGRAL METHOD※

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Abstract

The mobility of elemental semiconductor is calculated for all fields up to breakdown by means of the path integral method. A self-consistent solution of the Boltzmann transport equation is obtained and analytic expressions for the drift velocity, mobility and the relationship between electron temperature and the electric field is obtained. The validity of this calculation is limited to elemental semiconductors, such as Ge and Si at lattice temperatures high enough so that inter-valley scattering may be neglected, and the scattering is dominated by electron-phonon scattering. A comparison between theory and the experimental results for n-type and p-type Ge is made. It is found that agreement between theory and experiment is good for both n- and p-type Ge. A saturation velocity of 6.10×10^5 cm/sec between 2.1×10^3 v/cm and 5.1×10^3 v/cm is predicted for n-type Ge.

I. INTRODUCTION

The hot electron problem has been extensively studied, both experimentally and theoretically. Two recent review articles have presented the state of the theoretical and experimental results up to about 1968.[1, 2] Some of the more recent hot electron properties in germanium have been discussed by McGroddy, Nathans and Smith.[3] The previous theoretical work has been able to explain the observed experimental phenomena in bulk semiconductors, such as *Ge* and *Si*, quite adequately.[1, 2] One method used to calculate the current or mobility of holes and electrons in strong fields is to assume that the scattering term in the Boltzmann equation is due to isotropic scattering mechanisms. The solution of the Boltzmann equation in this case can be obtained by expanding the distribution function in a series of Legendre Polynomials. From this expansion the distribution function may obtain in a self consistent manner. This method of solution has the advantage that the scattering from various mechanisms can be included in a straight forward way. However, the resulting equations are extremely complicated and have only been solved in certain limits.[1, 2]

Another method is to use the conservation of energy to obtain the relationship between the lattice temperature and the elecIron temperature. This is usually done by averaging energy transferred from the electrons to the lattice through various types of collision mechanisms. The averaging and the calculation of the drift velocity in this method is usually accomplished by means of an isotropic distribution function.

An alternate method is to use the path-integral method.[4-7] For material for which a relaxation time approximation is valid (i.e. elastic or isotropic scattering)[1, 2], this method gives the exact solution of the Boltzmann equation[4, 5], and Budd[5] has outlined a iterative procedure for finding the solution for a general scattering mechanisms. Since both acoustic and optical phonon scattering mechanisms can be considered elastic to a good approximation in *Ge* and *Si*[2, 7, 8] this method should be applicable to these semiconductors. The path-integral method has a decided advantage over the expansion method in that the boundary conditions which the distribution funciton must satisfy are easily included.

There are several problems, where boundary conditions are important, in which one would expect the mobility of the conduction electrons to be that obtained in strong electric fields. For example, in the case of the electron transport in the space charge region of a *p-n* junction[8-9] it has been shown that the inclusion of boundary conditions on the distribution means that the hot electron distribution function is only obtained for the

case in which the width of the space charge region is large compared to the mean free path of the electron.[8] Another important problem, which has been solved, is the transport of electrons in the n-channel of an MOS transistor.[10] In this problem the boundary conditions can be easily incoporated and the distribution function can be calculated for the surface scattering.

With this and other problems of a similar nature in mind, we are motivated to explore the possibility of explaining the hot electron phenomena by means of the path integral method; since, if this mehtod can predict the bulk mobility of semiconductors, such as *Ge* and *Si*, in the high field region, it well be a relatively simple matter to attach the proper boundary conditions to the distribution function for specific problems.

II. PATH INTEGRAL SOLUTION OF THE BOLTZMANN EQUATION

The time dependent Boltzmanntransport equation can be written[4-7]

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\underline{r}} \nabla_{\underline{r}} f + \dot{\underline{k}} \cdot \nabla_{\underline{k}} f = \tag{1}$$

$$[w(\underline{k}', \underline{k})\ f(\underline{k}', \underline{r}, t)\ (1-f(\underline{k}, \underline{r}, t)) - w(\underline{k}, \underline{k}') f(\underline{k}, \underline{r}, t) \times (1-f(\underline{k}', \underline{r}, t))] d^3\underline{k}',$$

where $f=f(\underline{r}, \underline{k}, t)$ is the distribution function at $\underline{k}$, $\underline{r}$ and t (where $\underline{k}$ is the Bloch-wave vector, $\underline{r}$ the position and t the time). The transition probability, per unit time, that an electron at $\underline{k}$ will undergo a transition to $\underline{k}'$ is given by $w(\underline{k}, \underline{k}')$. If we move with the electron along its trojectory the only way that an electron can enter or leave a differential volume in phase space is through the scattering term (the right hand side of *Eq*(1)); hence the total time derivative of f, $\frac{df}{dt}$, must be equal to the scattering term in *Eq*(1). The exact solution of *Eq*(1) is[2, 4, 5]

$$f(\underline{\rho}, t) = \int_{t'}^{t} dt'' \left(\int f(\underline{\rho}'', t'') w(\underline{k}''', \underline{k}'') d^3\underline{k}''' \right) \times$$

$$\exp\left(-\int_{t''}^{t} \frac{dt'''}{\tau(\underline{\rho}''')} \right) + f(\underline{\rho}', t') \exp\left(-\int_{t'}^{t} \frac{dt'''}{\tau(\underline{\rho}''')} \right), \tag{2}$$

where $\underline{\rho} = (\underline{k}, \underline{r})$ and

$$\frac{1}{\tau(\underline{\rho})} = \int w(\underline{k}, \underline{k}')\ d^3\underline{k}'. \tag{3}$$

Budd[5] has outlined a iterative method for finding the solution of *Eq*(2). However, if the scattering is isotropic which should be a good approximation for *Ge* and *Si*[7, 8], then

$$f(\rho, t)=\int_{t'}^{t} dt''\left[f^0(\rho'', t'')/\tau(\underline{\rho}'')\right] \exp\left(-\int_{t''}^{t} \frac{dt'''}{\tau(\underline{\rho}''')}\right)$$

$$+f(\rho', t') \exp\left(-\int_{t'}^{t} \frac{dt'''}{\tau(\underline{\rho}''')}\right), \qquad (4)$$

where

$$f^0 = \left(\int w(\underline{k}', \underline{k})\, f(\underline{\rho}')\, d^3\underline{k}'\right) \Big/ \left(\int w(\underline{k}', \underline{k})\, d^3\underline{k}'\right). \qquad (5)$$

If electron-electron interaction can be neglected[2] the f^0 in *Eq*(5) can be approximated by a Maxwellian distribution of the form

$$f^0 \cong C \exp\left(-\varepsilon/k_B T_e\right) \qquad (6)$$

where T_e is the electron temperature and C is a constant. If we require that the total number of electrons per unit volume n be given by the expression[7] $n=\int f^0 \frac{d^3\underline{k}}{4\pi^3}$ then $C = \frac{n}{2}\left(\frac{h^2}{2\pi m^* k_B T_e}\right)^{3/2}$, which of course gives the usual value when $T_e = T$, the lattice temperature. It has been pointed out by Stratton[9] that the mobility is not very sensitive to the shape of f^0 and that to a good approximation f^0 can be assumed to be of the form given in *Eq*(6).

Under certain conditions *Eq*(6) can be further simplified. Harrison[12] has shown that for electrons in *Ge* and for holes in *Ge* and *Si*, the transition probability $w(\underline{k}, \underline{k}')$ is independant of the wave vector $\underline{k}$. Hence, for this case *Eq*(6) reduces to

$$f^0 = \frac{1}{4\pi}\int f(\underline{k}, \Omega)\, d\Omega, \qquad (7)$$

where $d\Omega$ is the element of sold angle. We will use *Eq*(7) to establish the dependence of the electron temperature T_e on electric field E. This may be done by an iterative process: we will find $f(k, \Omega)$ by assuming f^0 is the equilibrium distribution $f_0^0 = C_0 \exp\left(-\frac{\varepsilon}{k_B T}\right)$, where T is the lattice temperature, substitute the resulting distribution function into *Eq*.(7) and make a correction in T and so forth. We will assume the temperature T_e is determined when the norm[13]

$$\| f^0(T_e) - \frac{1}{4\pi}\int f(T')\, d\Omega \| \to 0, \qquad (8)$$

where T_e is the "true' electron temperature and T' is some test temperature. The average indicated in *Eq*(8) will be carried out over all energy, i.e. $\|A\| = \left[\int_0^\infty d\varepsilon n(\varepsilon) A^2\right]^{1/2}$, where $n(E)$ is the density of states between ε and $\varepsilon + d\varepsilon$ If we integrate *Eq*(4) by parts and define $\delta f \equiv f - f^0$ then

$$\delta f(\underline{\rho}, t) = -\int_{t'}^{t} \frac{df^0}{dt''} \exp\left(-\int_{t''}^{t} \frac{dt'''}{\tau(\underline{\rho}''')}\right) dt'' + \delta f(\underline{\rho}', \underline{t}') \exp\left(-\int_{t''}^{t} \frac{dt'''}{\tau(\underline{\rho}''')}\right) \tag{9}$$

If we let $t' \to \infty$ the distribution function will lose its dependance on the initial consitions and

$$\delta f(\underline{\rho}, t) = -\int_{-\infty}^{t} \frac{df^0}{dt'} \exp\left(-\int_{t''}^{t} \frac{dt'''}{\tau(\underline{\rho}''')}\right) dt'' \tag{10}$$

Equation (10) is only applicable to bulk semiconductors, where the boundary conditions are unimportant. In cases where the boundary conditions are important we must include $\delta f(\underline{\rho}', t')$ in the equation.

Let us assume that the electron is subject to a constant uniform electric field $\underline{E}$ and that the crystal is isotropic, then the distribution function will have axial symmetry, i.e. $f(\underline{k}) = f(k, \theta)$ where θ is the polar angle between the wave vector of the electron $\underline{k}$ and the electric field $\underline{E}$. Next, we must calculate $\frac{df^0}{dt''}$ from *Eq*(6). The energy ε in *Eq*(6) is the energy of the electron at the time t'', hence

$$f^0(t) = C \exp\left(-\frac{\varepsilon - \triangle\varepsilon_t}{k_B T_e}\right) \tag{11}$$

where $\triangle\varepsilon_t$ represents the energy gained by the electron in the interval $(t-t'')$, i.e.,

$$\triangle\varepsilon_t = -e\int_{t'}^{t} \underline{E} \cdot \underline{v}(s)\, ds$$

$$= -\left\{\frac{e^2E^2}{2m^*}(t-t')^2 + \left(\frac{2\varepsilon}{m^*}\right)^{1/2} eE\cos\theta\,(t-t')\right\} \tag{12}$$

where we have use the fact that

$$\underline{E} \cdot \underline{v}(t) = E\left(v_0 \cos\theta + \frac{eE}{m^*}(t-t')\right) \tag{13}$$

with v_0 the velocity at t' making an angle θ with respect to E, and m^* the effective mass. (It should be pointed out that up to this point we have made no assumptions about the shape of the engery bands. Since we use the single effective mass m^*, *Eq*(13) introduces the assumption that we have a single spherical energy minimum. However, though the problem would be more difficult, it is possible to carry out the calculation with the full effective mass tensor.)

If we consider only phonon scattering and restrict ourselves to elemental semiconductors such as *Ge* and *Si*, then[7, 12, 14]

$$\frac{1}{\tau(t)} = v(t)/\lambda = \left(\frac{2}{m^*}\right)^{1/2} (\varepsilon - \triangle\varepsilon_t)^{1/2}/\lambda \tag{14}$$

where λ is the mean free path. Conwell[14] has shown that the mobility can be adequately predicted even through electron-electron scattering is neglected. She also shows that the relaxation time for optical phonon scattering is given by $Eq(14)$ if $T_e \gg \theta \gg T$, where θ is the temperature characterizing the optical mode. The optical mode is characterized by the energy 0.037 ev in Ge[15] and 0.063 ev is Si[16] hence $Eq(14)$ should be valid for these materials at room temperature. The validity of *Eqs.* (9) and (18) does not depend on the constancy of λ, only that a relaxation time can be defined. We use $Eq(14)$ only so that a specific form of relaxation time may be defined.

With the aid of *Eqs* (12) and (14) we write *Eq* (10) as

$$\delta f = \int_{-\infty}^{t} dt'' \frac{df^0(\varepsilon - \triangle\varepsilon)}{d(\varepsilon - \triangle\varepsilon)} \frac{d\triangle\varepsilon}{dt''} \exp\left(-\int_{t''}^{t} \frac{dt'''}{\tau(\rho''')}\right) \tag{15}$$

$$= -\frac{1}{k_B T_e}\int_{-\infty}^{t} \left(\frac{e^2E^2}{m^*}(t-t'') + \left(\frac{2\varepsilon}{m^*}\right)^{1/2} eE\cos\theta\right) f^0(\varepsilon - \triangle\varepsilon)$$

$$\times \exp\left(-\int_{t''}^{t}\left(\frac{1}{\lambda}\left(\frac{2}{m^*}\right)^{1/2}(\varepsilon - \triangle\varepsilon)^{1/2}\, dt'''\right)\right) dt''$$

It is often convenient to make the following changes of variables:[7]

$$\beta = (eE/\sqrt{m^*})\ (t-t'''),$$

$$\triangle^{\varepsilon}\eta = (eE/\sqrt{m^*})\ (t-t''),$$

$$\triangle^{\varepsilon}\eta = -(\tfrac{1}{2}\eta^2 + \sqrt{2\varepsilon}\ \eta\cos\theta)$$

Using these new variables *Eq* (51) becomes

$$\delta f = \frac{C\exp\left(-\frac{\varepsilon}{k_B T_e}\right)}{k_B T_e}\int_{\circ}^{\infty} d\eta\ (\eta + \sqrt{2\varepsilon}\cos\theta)\times$$

$$\exp\left(\frac{\triangle^{\varepsilon}\eta}{k_B T_e} - \frac{\sqrt{2}}{eE\lambda}\int_{\circ}^{\eta}\sqrt{\varepsilon - \triangle\varepsilon_\beta}\, d\beta\right) \tag{16}$$

The current density is given by the expression

$$j_\alpha = \frac{-e}{4\pi^3}\int v_\alpha\ \delta f\ d^3k$$

where α is the direction of the current and δf is given by $Eq(16)$. The integration over $\underline{k}$ - space can be converted to and integral over engery by means of the effective mass approximation, i.e. $d^3k = 1/2\ \left(\frac{2m^*}{\hbar^2}\right)^{3/2}\varepsilon^{1/2}\, d\varepsilon\, d\Omega$, where $d\Omega$ is the element of solid angle. In order to calculate the current in the direction of the applied electric field we must express v_α as a function of ε. This may be done dy writing $v_\alpha = \sqrt{\frac{2\varepsilon}{m^*}}\cos\theta$, which is consistent the assumption that f^0 is Maxwellian (see $Eq(6)$). The normaliza-

tion constant C in $Eq(6)$ ard subsequent equations may be eliminated by requiring that $n = \int f^0 \frac{d^3k}{4\pi^3}$, where n is the number of electrons per unit volume of the solid. For low field (i.e. when $T_e = T$, the lattice temperature) $C = \frac{n}{2}(2\pi m^* k_B T_e)^{3/2}$. If we define the drift velocity v_d by the equation $j = n e v_d$, where the current is paralled to the field, we obtain an equation for the drift velocity:

$$v_d = \frac{C\sqrt{2/m^*}}{n4\pi^2 k_B T_e} \left(\frac{2m^*}{\hbar^2}\right)^{3/2} \int_0^\infty d\varepsilon \int_0^\pi \int d\vartheta \int_0^\infty d\eta \ (\varepsilon \sin\theta \cos\theta)$$

$$\times (\eta + \sqrt{2\varepsilon} \cos\theta) \exp\left(\frac{-\varepsilon + \triangle\varepsilon_\eta}{k_B T_e} - \sqrt{\frac{2}{eE\lambda}} \int_0^\eta \sqrt{\varepsilon - \triangle\varepsilon_\beta} \ d\beta\right) \quad (18)$$

where the temperature T_e of the electron may be determined in a self-consistent manner from $Eq(8)$. We obtain the mobility from $Eq(18)$ by means of the definition

$$\mu = v_d / E \quad (19)$$

We will now proceed to evaluate $Eqs(18)$ and (19) in the various limits of electric field.

A. Low Field Region

In the low field region the current density is proportional to the field (i.e. follows Ohm's Law). Specifically, the energy gained from the field $\triangle\varepsilon$ can be neglected. Hence,

$$\delta f = \frac{-e\lambda}{k_B T_e} E \cos\theta \ f^0(\varepsilon) \quad (20)$$

and from Eqs (18) and (19)

$$v_d = \frac{4e\lambda}{3\sqrt{2\pi m^* k_B T_e}} E \quad (21a)$$

$$\mu = \frac{4e\lambda}{3\sqrt{2\pi m^* k_B T}} = \mu_0 \quad (21b)$$

where we have set $T_e = T$, the lattice temperature. The values of v_d and μ are the usual expressions for the low field drift velocity and mobility for a constant mean free path. In the low field region the electron temperature and the lattice temperature T are the same. The mean free path is generally inversely proportional to the temperature T, hence, $\mu \propto T^{-3/2}$, the usual temperature dependence of the mobility.

B. Medium Field Region

The medium field region begins when the mobility starts to become field dependent (e.g. in Ge when the field is of the order of 10^3v/cm at room temperature). However, at the low field end of the medium field region the

electron does not gain much energy from the field. Hence,

$$\varepsilon \gg eE\left(\frac{2\varepsilon}{m^*}\right)^{1/2}(t) \gg \frac{e^2E^2}{2m^*}(t^2) \tag{22}$$

There are two terms in the integral of *Eq*(18) which involve $\triangle\varepsilon_\eta$. Clearly, the first of these terms is most important since second term involves $\sqrt{\varepsilon-\triangle\varepsilon_\eta}$. As is clear from the inequality of *Eq*(22) we are assuming that the energy gain by the electron ($\triangle\varepsilon$) is small compared to ε. Hence the first term adds a correction of the order $\triangle\varepsilon/\sqrt{\varepsilon}$. For this reason we will not expand the second term, but merely use the low field value $\sqrt{\varepsilon}$ for the $\int_0^\eta \sqrt{\varepsilon-\triangle\varepsilon_\beta}\,d\beta$. Hence we may approximate the exponential term in *Eq*(16) by

$$\exp\left(-\frac{\varepsilon-\triangle\varepsilon_\eta}{k_BT_e} - \frac{\sqrt{2}}{eE\lambda}\int_\eta^\eta \sqrt{\varepsilon-\triangle\varepsilon_\beta}\,d\beta\cong\right. \tag{23}$$

$$\exp\left(-\frac{\varepsilon}{k_BT_e}\right)\exp\left(-\left(\sqrt{2\varepsilon}\left(\frac{\cos\theta}{k_BT_e}+\frac{1}{eE\lambda}\right)\eta\right)\left(1-\frac{\eta^2}{2k_BT_e}\right)\right.$$

(One may quickly convince oneself that *Eq*(23) is the correct expansion by expanding the exponential of the terms dropped in $\sqrt{\varepsilon-\triangle\varepsilon}$ and noting that the resulting terms in δf are smaller than the other terms by a factor of ε/k_BT_e). Hence,

$$\delta f = -\frac{C\exp\left(-\frac{\varepsilon}{k_BT_e}\right)}{k_BT_e}\int_0^\infty d\eta\left(\sqrt{2\varepsilon}\cos\theta+\eta\right)$$

$$-\frac{\eta^2}{2k_BT_e} - \frac{\sqrt{2\varepsilon}\cos\theta\,\eta^2}{2k_BT_e}\Big)\times \tag{24}$$

$$\exp\left(-\frac{\sqrt{2\varepsilon}}{eE\lambda}\left(1+\frac{eE\lambda\cos\theta}{k_BT_e}\right)\eta\right)$$

Making use of standard integrals[17] we obtain the expression

$$\delta f = \frac{-C\exp-\left(\frac{\varepsilon}{k_BT_e}\right)}{k_BT_e}\left\{\left\{\sqrt{2\varepsilon}\cos\theta\frac{1}{a} - \frac{\sqrt{2\varepsilon}}{a^3k_BT_e}\cos\theta\right.\right.$$

$$\left.\left.+\frac{1}{a^2} - \frac{3}{a^4k_BT_e}\right\}\right. \tag{25}$$

where $a=\frac{\sqrt{2\varepsilon}}{eE\lambda}\left(1+\frac{eE\lambda\cos\theta}{k_BT_e}\right)$. Substituting *Eq*(25) into *Eq*(18) we obtain (to order E^5 in v_d)

$$v_d=\mu_0E\left(\frac{T}{T_e}\right)^{1/2}\left[1-\frac{9}{10}\left(\frac{e\lambda}{k_BT_e}\right)^2E^2 - \frac{18}{7}\left(\frac{eE\lambda}{k_BT_e}\right)^4\right] \tag{26}$$

where $\mu_0=4e\lambda/3\sqrt{2\pi m^*k_BT}$. The relationship between the electron temperature and the lattice temperature is obtained by substituting the distribution function obtained by averaging *Eq*(25) over the angle and substituting the result into *Eq*(8). This yields (to the order E^4)

$$\{1-2^{5/2}(1+\frac{T'}{T_e})^{-3/2}[1+(1/3)r^2+(1/5)r^4-r^2(1+T'T_e)-r^4(1+\frac{T'}{T_e})$$

$$-3r^4(1+\frac{T'}{T_e})^2]+(\frac{T_e}{T'})^{3/2}(1-(10/3)r^2-(1477/45)r^4)\} \rightarrow 0 \qquad (27)$$

where $r=(eE/k_BT')$. If we let $T_e/T=1+\alpha r^2$, that is if we use the fact that T_e is just beginning to deviate from T then we get

$$T_e/T=1+3.14\ (\frac{eE\lambda}{k_BT})^2$$

Using the value of T_e obtained from *Eq.*(28) in *Eq.*(26) we obtaain

$$v_d=\mu_0 E\,(1-2.47(\frac{eE\lambda}{k_BT})^2+8.20(\frac{eE\lambda}{k_BT})^4) \qquad (29a)$$

$$\mu=\mu_0\ \ (1-2.47(\frac{eE\lambda}{k_BT})^2+8.20(\frac{eE\lambda}{k_BT})^4) \qquad (29b)$$

In his classic paper on the hot electron Shockley[18] obtained an expression of the form $\mu=\mu_0(1-(E/E_c)^2)$ where $E_c=8c/\sqrt{3\pi}\mu_0$ with c the velocity of sound. However, Shockley's calculation, based essentially on acoustic phonons[14], greatly over estimated the heating effect of the field, and gives an E_c at least an order of magnitude too small[18-20].

In order to show that our results agree with experimental results we will evaluate λ for *n*- and *p*-type *Ge*. If we assume that acoustic phonon scattering dominates at room temperature i.e., if we assume μ_0 is given by *Eq.* (21b) and take $\mu_0=3{,}800$ cm^2/v-sec. and 1,800 cm^2/v-sec: $m^*=0.12\ m$ and $0.3\ m$ for *n*- and *p*- type *Ge*, respectively. We get

$\lambda_n=950$ A, *n*-type *Ge*

$\lambda_p=710$ A, *p*-type *Ge*

A plot of *Eq.* (29b) with the values of λ given by *Eq.* (30) for both *n*- and *p*-type *Ge* is shown in Figs. (1) and (2). In Figs. (1) and (2) we have included the calculation of Conwell[14] and the data presented in her article. It is clear that *Eq*(29b) gives at least as good a fit to the experimental data in the medium field region for *n*-type *Ge* and a better fit for *p*-type *Ge* than does Conwell's calculations. Conwell's calculations are based essentially on the theory developed by Schockley except that she includes the optical phonons in her calculations. Basically all of the deviation from Ohm's law is attributed to the heating of the electron and none the nonlinearity to of the distribution function. We have use only acoustic phonon scattering in the medium field region have been able to explain the experimental data for both *n*- and *p*-type *Ge* to a higher degree of accuracy than is Conwell. We will see in the next section that in the case of high fields this same method predicts that the electrons do become heated and that the electron

temperature increases linearly with the field.

C. High Field Region

The high field region begins when the electron gains more energy from the field than its initial energy. That is the inequality used to define the medium field region is region is reversed: $\varepsilon \ll eE(\frac{2\varepsilon}{m^*})^{1/2}(\tau) \ll \frac{e^2E^2}{2m^*}\tau^2$.

In order to apply this approximation we must first perform the integration of $Eq(16)$. If we make the following change of variable in $Eq(16)$ the integration becomes simplier: let $u=\frac{\eta^2}{2}+\eta\sqrt{2\varepsilon}\cos\theta$. With this change of variables $Eq(16)$ becomes

$$\delta f = \frac{-C\exp(-\frac{\varepsilon}{k_BT_e})}{k_BT_e}\int_{\varepsilon}^{\infty} du \exp\left(-\frac{u}{k_BT_e}-\frac{1}{eE\lambda}\int_0^u \sqrt{\frac{\varepsilon+u'}{\varepsilon\cos^2\theta+u'}}du'\right)$$

$$=\frac{-C\exp(-\frac{\varepsilon}{k_BT_e})}{k_BT_e}\int_{\varepsilon}^{\infty} du \exp\left\{-\frac{u}{k_BT_e}-\frac{1}{eE\lambda}\sqrt{(\varepsilon+u)(\varepsilon\cos^2\theta+u)}\right.$$

$$\left.\frac{-\varepsilon\cos\theta}{eE\lambda}+\frac{\varepsilon\sin^2\theta}{eE\lambda}\,\ell n\left(\frac{\sqrt{\varepsilon\cos^2\theta+u}+\sqrt{\varepsilon+u}}{\sqrt{\varepsilon}\,(1+\cos\theta)}\right)\right\} \qquad (31)$$

If we now expand the square roots in $Eq(31)$ in powers of E^{-1} we obtain to first order,

$$\delta f=\frac{-C\exp(-\frac{\varepsilon}{k_BT_e})}{k_BT_e}\int_{\varepsilon}^{\infty} du\ \exp\left[-ua-\frac{\varepsilon}{2eE\lambda}(1-\cos\theta)^2-\right. \qquad (32)$$

$$\left.\frac{\varepsilon\sin^2\theta}{eE\lambda}\,\ell n\left(\frac{2\sqrt{u}+\frac{\varepsilon}{2\sqrt{u}}(1+\cos^2\theta)}{\sqrt{\varepsilon}\,(1+\cos\theta)}\right)\right]$$

where $a=\frac{1}{k_BT_e}+\frac{1}{eE\lambda}$. If we now drop the log term in the integration over u (since the $\ell n\sqrt{u}$ is samall compared to u) we get

$$\delta f=\frac{-C\exp(-\frac{\varepsilon}{k_BT_e}-\frac{\sqrt{\varepsilon}}{2eE\lambda}(1-\cos\theta)^2)}{k_BT_ea} \qquad (33)$$

If we substitute $Eq(33)$ into $Eq(18)$ we get

$$v_d=\frac{C\sqrt{\frac{2}{m^*}}}{m^*4\pi^2k_BT_e\,a}\left(\frac{2m^*}{\hbar^2}\right)^{3/2}\int_{-1}^{1}dx\,x\int_{\varepsilon}^{\infty}d\varepsilon\,\varepsilon\exp\left(-\varepsilon\,b(x)\right) \qquad (34)$$

where $x=\cos\theta$ and $b(x)=a+\frac{(1-x)^2}{2eE\lambda}$. Straight forward integration leads to the result

$$v_d = \frac{1}{a\,k_B T_e}\sqrt{\frac{2k_B T_e}{\pi m^*}}\left[\sqrt{\frac{2eE\lambda}{k_B T_e}}\tan^{-1}\sqrt{\frac{8k_e T_B}{eE\lambda}} - \frac{1}{1+\frac{2k_B T_e}{eE\lambda}}\right] \tag{35}$$

We must now turn to the calculation of the relationship between T_e and the electric field. As before, we must substitute the distribution function resulting from *Eq*(33) into *Eq*(8). However, there is a difficulty in carrying out the θ-integration before the energy integration (as is suggested by the writing of *Eq*(34). The θ-integration leads to an error function of a small argument. We will expand the error function (or the exponential) and keep terms of order $(\frac{\varepsilon}{eE\lambda})^2$. Following this proceedure we obtain the expression

$$K\left[r^{3/2}+\left(1+\frac{1}{akT'}\right)^2-4\sqrt{2}\left(\frac{r}{1+r}\right)^{3/2}\left(1+\frac{1}{akT''}\right)\right.$$
$$\left.+\frac{1}{eE\lambda a^2 k_B T'}-\frac{4\sqrt{2}}{1+r}\left(\frac{r}{1+r}\right)^{3/2}\frac{1}{eE\lambda a}\right]^{1/2}\to 0 \tag{36}$$

as $T'\to T_e$. We have used $r=T'/T_e$ in *Eq*(36). If we set $r=1$ (i.e. set $T'=T_e$) in *Eq*(36) we get the expression,

$$k_B T_e = \frac{2eE\lambda}{\sqrt{\frac{1}{eE\lambda a}+4}-1} \tag{37}$$

which gives $k_B T_e = 1.8414\,eE\lambda$. Hence *Eq*(35) reduces to

$$v_d = 0.438\ (eE\lambda/m^*)^{1/2}\ \text{cm/sec} \tag{38a}$$
$$\mu/\mu_0 = 0.823\ (k_B T/eE\lambda)^{1/2}\ (\lambda/\lambda_{ac})^{1/2} \tag{38b}$$

If we assume that optical phonon scattering is impossible until $k_B T_e > \hbar\omega_{op}$ where $\hbar\omega_{op}$ is the energy which characterizes the optical mode, then λ in *Eq*(38) would be the acoustic phonon scattering mean free path until $k_B T_e < \hbar\omega_{op}$. After the temperature $\hbar\omega_{op}/k_B$ is reached the mean free path will gradually decrease until

$$\frac{1}{\lambda} = \frac{1}{\lambda_{ac}} + \frac{1}{\lambda_{op}} \tag{39}$$

where λ_{ac} and λ_{op} are the acoustic and optical mean free paths respectively. That is, for $T_e < \frac{\hbar\omega_{op}}{k_B}$ optical scattering is impossible hence $\lambda_{op}\to\infty$, however for $T_e > \frac{\hbar\omega_{op}}{k_B}$ optical scattering starts to play an important role and λ_{op} reduces to its high temperature value. If we assume[18] that $\lambda_{op}=\lambda_{ac}\frac{kT}{\hbar\omega_{op}}$ then $\lambda_{op}=\lambda_{ac}(\frac{0.026}{0.037})$ for germanium. Hence the velocity v_d should saturate, that is remain constant, while changes from λ_{ac} to λ (*Eq*(39)). For *n*-type *Ge* the saturation velocity of about 6.1×10^6 cm/sec should begin at $E=2.1\times10^3$ v/cm and remain saturated until $E=5.1\times10^3$ v/cm. The experimental values

for the saturation velocity of *n*-type *Ge* is between 6.5×10^6 cm/sec (Ref. 19) and 5.5×10^6 cm/sec (Ref 20). Both Refs 19 and 20 give the onset of saturation at about 4×10^3 v/cm and Ref 20 gives the end of the saturation region at about 8×10^3 v/cm. The theory presented here then predicts the saturation velocity quite accurately but predicts that the saturation region should occur at values of electric field which are 50% smaller than those observed in *n*-type *Ge*. The saturation velocity for *p*-type *Ge* predicted by *Eq*(38b) is 4.34×10^6 cm/set beginning at about $E=3\times10^3$ v/cm and ending at about 7.2×10^3 v/cm.

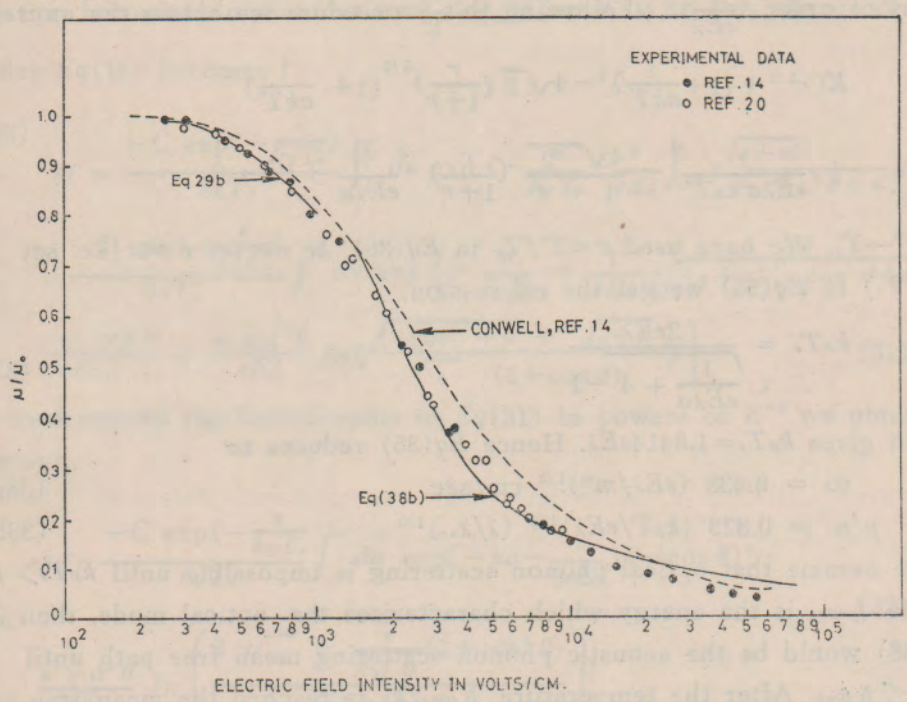

Fig 1 Mobility of n-type Ge verses electric field intensity. Solid lines for path integral theory with broken line after Conwell Ref. 14. Experimental data from Refs 14 and 20.

It can be seen in Figs. (1) and (2) that *Eq*(38b) fits the data of Ref 14 quite well, especially for *n*-type *Ge*. The fit for high field in *p*-type *Ge* is not as good as for the *n*-type yet the fit is at least as good as that obtained br Conwell, especially for $E<5\times10^3$ v/cm. There has been no observed saturation velocity in *p*-type *Ge*, hence if we do not assume that the optical mode is excited (which causes the saturation) very good agreement is obtained for $\lambda=\lambda_{ac}$.

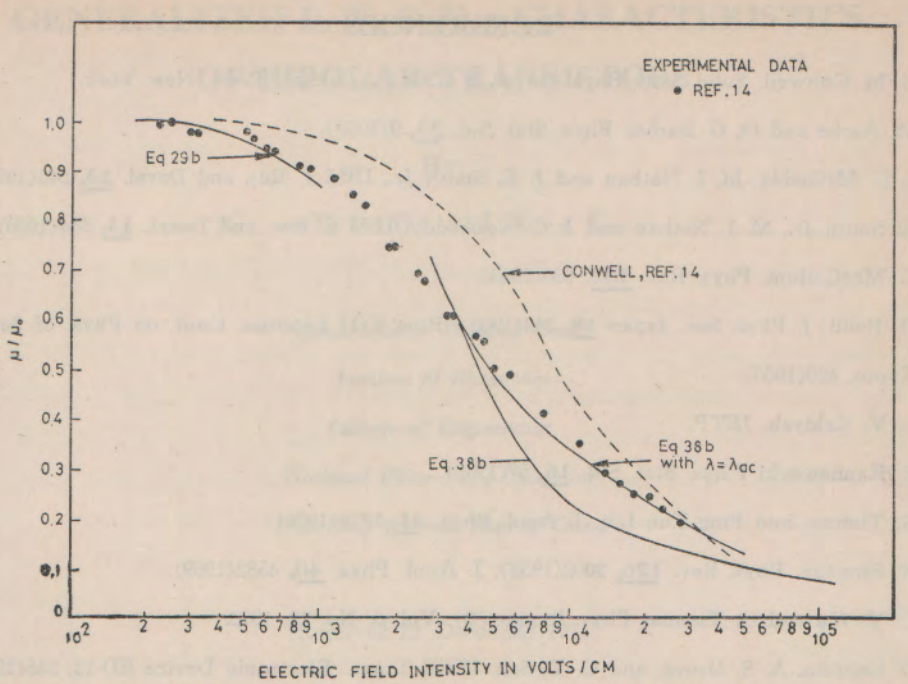

Fig 2 Mobility of p-type Ge verses electric field intensity. Solid lines for path integral theory with broken line after Conwell, Ref. 14. Experimental data from Ref. 14. The mobility inthe high field region is calculated for both $\lambda=(1I\lambda^{ac}=1/\lambda^{op})^{-1}$ and for $\lambda=\lambda^{ac}$.

It should be noted that while the path integral method does not predict as much heating of the electrons at medium fields as the standard conservation of engery method, it does predict about as much heating at high fields. The reason for this seems to be that the standard conservation of energy method uses an isotropic distribution function in order to calculate the energy transfer, whereas, distribution function derived by the path integral method is highly anisotropic. At very high fields the path integral method leads to a more nearly isotropic distribution function and hence the two methods shoud lead to more similar results. Finally the Eq(38a) predicts that the drift velocity should increase as $E^{1/2}$ for $kT_e \gg \hbar\omega_{op}$, while experimental measurement[20] show an increase of only $E^{0.134}$. Part of the reason for this discrepancy might be that intervalley scatterering becomes important in this region. As has been pointed out[3] for lattice temperature below 130°K the high field velocity actually decreases and is most probably due to intervalley scattering.

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