New Identities on the Riemann Tensor※

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Abstract

A set of new identities which involve second covariant derivatives and quardratic forms of the Riemann tensor are proved. These new identities can be thought of as integrability conditions derived from the equations that define the Riemann tensor in terms of the affine connections.

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1. INTRODUTION

We all know the system of equations which define the Riemann tensor $R^{l}{}_{ijk}$ in terms of the affine connections $\Gamma^{l}{}_{ij}$. These equations are

$$(\partial_j \Gamma^{l}{}_{ik} - \partial_k \Gamma^{l}{}_{ij}) + (\Gamma^{a}{}_{ij} \Gamma^{l}{}_{aj} - \Gamma^{a}{}_{ij} \Gamma^{l}{}_{ak}) = R^{l}{}_{ijk} \qquad (1)$$

It is obvious from eqs. (1) that $R^{l}{}_{ijk}$ is antisymmetric with respect to the indices j and k, that is

$$R^{l}{}_{ijk} = -R^{l}{}_{ikj} \qquad (2)$$

If we assume that the connections are symmetric, that is $\Gamma^{l}{}_{ij} = \Gamma^{l}{}_{ji}$ (From now on we all assume that the connections are symmetric.), then we have

$$R^{l}{}_{ijk} + R^{l}{}_{jki} + R^{l}{}_{kij} = 0 \qquad (3)$$

Eqs. (3) involve no derivative of the Riemann tensor. The famous Bianchi identities which involve first order covariant derivatives of the Riemann tensor are

$$R^{l}{}_{ijk;\,m} + R^{l}{}_{ikm;\,j} + R^{l}{}_{imj;\,k} = 0 \qquad (4)$$

Eqs. (4) can be proved to be independent of eqs. (2) and (3). That is, eqs. (4) can not be obtained from the derivatives of eqs. (2) and (3). We can give these identities some meanings if we regard $R^{l}{}_{ijk}$ as given functions and eqs. (1) as partial differential equations for $\Gamma^{l}{}_{ij}$.[1,2] From this point of view, eqs. (2), (3) and (4) are integrability conditions for eqs. (1). Now, besides eqs. (2), (3) and (4), are there any other integrability conditions? The answer to this question is yes if the Riemannian space is of high dimentions. For example, in the two dimentional space, eqs. (3) and (4) are trivial. But in three dimentional space, eqs. (3) and (4) are not trivial at all. It is the purpose of this paper to derive new integrability conditions of eqs. (1) for high dimentional space and in turn they are the new identities on the Riemann tensor. We give the new identities in section 2 and some discussions and comments in section 3. We prove the identities in appendix A.

2 . THE NEW IDENTITIES

We can obtain the Bianchi identities, eqs. (4), from the following procedure:

Step 1. Differentiate eqs. (1) and get

$$\partial_m \partial_j \Gamma^{l}{}_{ik} - \partial_m \partial_k \Gamma^{l}{}_{ij} = \partial_m R^{l}{}_{ijk} + \partial_m (\Gamma^{a}{}_{ij} \Gamma^{l}{}_{ak} - \Gamma^{a}{}_{ik} \Gamma^{l}{}_{aj}) \qquad (5)$$

Step 2. Change the order of indices j, k, m, from eqs. (5) using the rule $j \rightarrow k \rightarrow m \rightarrow j$ and get

$$\partial_j\partial_k \Gamma^1{}_{im} - \partial_j\partial_m \Gamma^1{}_{ik} = \partial_j R^1{}_{ikm} + \partial_j (\Gamma^a{}_{ik}\Gamma^1{}_{am} - \Gamma^a{}_{im}\Gamma^1{}_{ak}) \tag{6}$$

Step 3. Repeat step 2 from eqs. (6) and get

$$\partial_k\partial_m \Gamma^1{}_{ij} - \partial_k\partial_j \Gamma^1{}_{im} = \partial_k R^1{}_{imj} + \partial_k (\Gamma^a{}_{im}\Gamma^1{}_{aj} - \Gamma^a{}_{ij}\ \Gamma^1{}_{am}) \tag{7}$$

Step 4. Add eqs. (5), (6) and (7) together and get

$$LHS = RHS$$

The *LHS* is equal to zero since $\partial_m\partial_j \Gamma^1{}_{ik} = \partial_j\partial_m \Gamma^1{}_{ik}$, and the *RHS* contains first derivatives of the Riemann tensor. After some algebrac manipulations, we can convert the *RHS* into covariant form. The covariant form version of the *RHS* is the famous Bianchi identities.

Now we can use this procedure again,

Step 1. Differentiate eqs. (5) and obtain

$$\partial_n\partial_m\partial_j \Gamma^1{}_{ik} - \partial_n\partial_m\partial \Gamma_k{}^1{}_{ij} = \partial_n\partial_m R^1{}_{ijk} + \partial_n\partial_m (\Gamma^a{}_i \Gamma_j{}^1{}_{ak} - \Gamma^a{}_{ik}\Gamma^1{}_{aj}) \tag{8}$$

Step 2. Change the indices j, k, n, and m from eqs. (8) using the rule $j \to k \to n \to m \to j$ and get

$$\partial_m\partial_j\partial_k \Gamma^1{}_{in} - \partial_m\partial_j\partial_n \Gamma^1{}_{ik} = \partial_m\partial_j R^1{}_{ikn} + \partial_m\partial_j (\Gamma^a{}_{ik}\Gamma^1{}_{an} - \Gamma^a{}_{in}\Gamma^1{}_{ak}) \tag{9}$$

Step 3. Repeat step 2 from eqs. (9) and get

$$\partial_j\partial_k\partial_n \Gamma^1{}_{im} - \partial_j\partial_k\partial_m \Gamma^1{}_{in} = \partial_j\partial_k R^1{}_{inm} + \partial_j\partial_k (\Gamma^a{}_{in}\Gamma^1{}_{am} - \Gamma^a{}_{im}\Gamma^1{}_{an}) \tag{10}$$

Step 4. Repeat step 2 from eqs. (10) and get

$$\partial_k\partial_n\partial_m \Gamma^1{}_{ij} - \partial_k\partial_n\partial_j \Gamma^1{}_{im} = \partial_k\partial_n R^1{}_{imj} + \partial_k\partial_n (\Gamma^a{}_{im}\Gamma^1{}_{aj} - \Gamma^a{}_{ij}\Gamma^1{}_{am}) \tag{11}$$

Step 5. Add eqs. (8), (9), (10) and (11) all together and get

$$LHS = RHS$$

The *LHS* is equal to zero since $\partial_n\partial_m\partial_j \Gamma^1{}_{ik} = \partial_j\partial_n\partial_m \Gamma^1{}_{ik}$. The right hand side contains second order derivatives of the Riemann tensor. We can convert the *RHS* into covariant form. The covariant version of the *RHS* is

$$\begin{aligned}
&R^1{}_{ijk;\,mn} + R^1{}_{ijk;\,nm} + R^a{}_{inj}R^1{}_{amk} + R^a{}_{jmk}R^1{}_{ian} \\
&+ R^1{}_{ikm;\,nj} + R^1{}_{ikm;\,jn} + R^a{}_{ijk}R^1{}_{anm} + R^a{}_{knm}R^1{}_{iaj} \\
&+ R^1{}_{imn;\,jk} + R^1{}_{imn;\,kj} + R^a{}_{ikm}R^1{}_{ajn} + R^a{}_{mjn}R^1{}_{iak} \\
&+ R^1{}_{inj;\,km} + R^1{}_{inj;\,mk} + R^a{}_{imn}R^1{}_{akj} + R^a{}_{nkj}R^1{}_{iam} = 0
\end{aligned} \tag{12}$$

Or in short hand notation

$$F(R^1{}_{ijk;\,mn} + R^1{}_{ijk;\,nm} + R^a{}_{inj}R^1{}_{amk} + R^a{}_{jmk}R^1{}_{ian}) = 0 \tag{12}$$

Eqs. (12) are the new identities on the Riemann tensor.

3. DISCUSSIONS AND COMMENTS

(1). The identities, eqs. (12), may be derived from differentiation of the known identities, such as eqs. (2), (3) and (4). These new identities can be though of as integrability conditions derived from eqs. (1).

(2). These identities become trivial when eqs. (1) become linear, that is when the $(\Gamma\Gamma-\Gamma\Gamma)$ terms are absent. So these identities are a kind of measure of the non-linearity of eqs. (1).

(3). For three dimentional space, eqs. (12) can be derived from eqs. (2), (3) and (4). (See appendix B.)

(4). For five dimentional space or higher, there are still new identities on the Riemann tensor.

(5). Perira[2] derived some identities on the Riemann tensor which from the point of view of the auther are all dependent on eqs. (2), (3), and (4).

Appendix A

We define the notation $F(\cdots)$ first. $F(\cdots)$ is an operator which acts on the indices j, k, m and n with the following property

$$F(nmjk)=(nmjk)+(mjkn)+(jknm)+(knmj) \tag{13}$$

For example,

$$F(R^1{}_{ijk;\,mn})=R^1{}_{ijkmn}+R^1{}_{ikn;\,jm}+R^1{}_{inm;\,kj}+R^1{}_{imj;\,nk} \tag{14}$$

With this definition, we can easily derive two useful properties of the operator $F(\cdots)$.

$$\text{(i)}\ \ F(A+B)=F(A)+F(B) \tag{15}$$

$$\text{(ii)}\ \ F(nmjk)=F(mjkn)=F(jknm)=F(knmj) \tag{16}$$

Since
$$R^1{}_{ijk;\,mn}=\partial_n(R^1{}_{ijk;\,m})+\Gamma^1{}_{an}R^a{}_{ijk;\,m}-\Gamma^a{}_{jn}R^1{}_{iak;\,m}-\Gamma^a{}_{in}R^1{}_{jak;m}-\Gamma^a{}_{kn}R^1{}_{ija;\,m}-R^1{}_{ijk;\,a} \tag{17}$$

we get $F(R^1{}_{jk;\,nm}+R^1{}_{ijk;\,nm})$

$=F(\partial_nR^1{}_{ijk;\,m}+\Gamma^1{}_{an}R^a{}_{ijk;\,m}-\Gamma_{ia}{}^aR^1{}_{ajkm}-\Gamma^a{}_{jn}R^1{}_{i}{}^a{}_{k;m}\Gamma$

+terms with n and m interchanged.) (18)

But $F(\Gamma^a{}_{jn}R^1{}_{iak;\,m}+\Gamma^a{}_{kn}R^1{}_{ija;\,m}+\Gamma^a{}_{jm}R^1{}_{iak;\,n}+\Gamma_{km}R^1{}_{ija;\,n}+\Gamma^a{}_{mn}R^1{}_{ijk;\,a})$

$$=F(\Gamma^a{}_{jn}R^1{}_{ika;\,m^a}+\Gamma^a{}_{nj}R^1{}_{iak;\,m})+F(\Gamma^a{}_{mj}(R^1{}_{ina;\,k}+R^1{}_{iak;\,n}+R^1{}_{ikn;\,a}))=0 \tag{19}$$

Thus eqs. (18) ecome $F(\mathrm{R}^1{}_{ijk;\,mn}+R^1{}_{ijk;\,nm})=F(\partial_nR^1\mathrm{b}_{jk;\,m}+\Gamma^1{}_{an}R^a{}_{ijk;\,m}$

$$-\Gamma^a{}_{in}R^1{}_{ajk;\,m}+\partial_mR^1{}_{ijki;\,n}+\Gamma^1{}_{am}R^a{}_{ijk;\,n}-\Gamma^a{}_{im}R^1{}_{ajk;\,n}-\Gamma^a{}_{nm}R^1{}_{ijk;\,a}) \tag{20}$$

Substituting the following equation

$$R^1{}_{ijk;\,m} = \partial_m R^1{}_{ijk} + \Gamma^1{}_{am} R^a{}_{ijk} - \Gamma^a{}_{im} R^1{}_{ajk} - \Gamma^a{}_{jm} R^1{}_{iak} - \Gamma^a{}_{km} R^1{}_{ija} \quad (21)$$

into eqs. (20), we finally get[3]

$$F(R^1{}_{ijk;mn} + R^1{}_{ijk;\,nm} + R^a{}_{inj} R^1{}_{amk} + R^a{}_{mnj} R^1{}_{iak})$$
$$= 2F(\partial_n \partial_m R^1{}_{ijk} + \partial_n \partial_m (\Gamma^a{}_{ij} \Gamma^1{}_{ak} - \Gamma^a{}_{ik} \Gamma^1{}_{aj})) = 0 \quad (22)$$

Appendix B

For three dimentional space, at least two of the indices n, m, j, k, must be equal. Let us say $n=m$, then the *LHS* of eqs. (12) become

$$R^1{}_{ijk;\,mm} + R^1{}_{ijk;\,mm} + R^1{}_{ikm;\,jm} + R^1{}_{ikm;\,mj} + R^1{}_{imm;\,kj} + R^1{}_{imm;jk} + R^1{}_{imj;\,mk}$$
$$+ R^1{}_{imj;\,km} + R^a{}_{imj} R^1{}_{amk} + R^a{}_{jmk} R^1{}_{iam} + R^a{}_{imk} R^1{}_{ajm} + R^a{}_{kjm} R^1{}_{iam} + R^a{}_{ijm} R^1{}_{akm}$$
$$+ R^a{}_{mkm} R^1{}_{iaj} + R^a{}_{ikm} R^1{}_{amj} + R^a{}_{mmj} R^1{}_{iak} = (R^1{}_{ijk;\,m} + R^1{}_{ikm;\,j} + R^1{}_{imj;\,k})_{;\,m} \times 2 = 0.$$

Thus we prove the statement that eqs. (12) are derivable from eqs. (2), (3) and (4) in the three-dimentional case.

REFERENCE

1. C. M. Pereira, J. Math. Phys. 13, 152 (19742).

2. C. M. Pereira, J. Math. Phys. 15, 269 (1974).

3. We can use normal coordinates to simplify the calculation.