Exact Analytic Solutons of Classical Free Gauge Field Equation※

Kuo-shung Cheng

鄭 國 順

Department of Applied Mathematics

National Chiiao-Tung University

H^s^inchu, Taiwan 300

Rep. of China

(Received 29 Januray 1975)

Abstract

A class of exact analytic solutions for the classical free gauge field equation was discovered and the implications were discussed.

※Work partially Supported by the National Science council of Taiwan.

1. INTRODUTION

It has been an open question to find out exact analytic solutions for the gauge field equation since Yang and Mills had writen down their classical isospin gauge field equation in 1954.[1] Due to the highly non-linear character of the equation, the progress is slow and only non-analytic examples are known.[2] But analytic solutions of these equation are still physically interested. One would expect some kinds of wave-like solutions to see how the classical field propagates. The purpose of this paper is to present a class of exact analytic wave-like solutions of these classical free gauge field equation. We will give the main result in section 2 and some discussions in section 3.

2. THE SOLUTIONS

The free gauge field equations are[3]

$$f^{\alpha}_{ij} = \partial_j b^{\alpha}_i - \partial_i b^{\alpha}_j - C^{\alpha}_{\beta\gamma} b^{\alpha}_{i\beta} b^{\alpha}_j \tag{1}$$

$$g^{ik} f^{\alpha}_{ij;k} = g^{ik}(\partial_k f^{\alpha}_{ij} + C^{\alpha}_{\beta\gamma} b^{\beta}_k f^{\alpha}_{ij}) = 0 \tag{2}$$

$$g^{ij}\partial_i b^{\alpha}_j = 0 \text{ (gauge condition)} \tag{3}$$

(We consider the gauge field in Minkowski space. The Latin letter, i, j,$\cdots$ runs from 0 to 4 and the Greek letter, α, β,$\cdots$runs from 1 to N, where N is the dimention of the gauge group. For Yang-Mills field, N is equal to 3.) Now we consider solutions of the form

$$b^{\alpha}_i = A\alpha b_i \tag{4}$$

$$f^{\alpha}_{ij} = A\alpha f_{ij} \tag{5}$$

where $A\alpha$ are arbitrary constant. Substituting eqs. (4) and (5) into eqs. (1), (2) and (3) we get

$$f_j = \partial_j b_i - \partial_i b_j \tag{6}$$

$$g^{ik}\partial_k f_{ij}) = 0 \tag{7}$$

$$g^{ij}\partial_j b_i = 0 \tag{8}$$

Eqs. (6), (7) and (8) are the well known classical free electromagnetic field equations. We can write down the general solutions as

$$b_i = \int d^3k B_i(k) e_i^{(kx)} \tag{9}$$

$$f_{ij} = i\int d^3k (B_i k_j - B_j k_i) e^{i(kx)} \tag{10}$$

where $k_i k^i = 0 = k^i B_i$. Thus we get

$$b_i^\alpha = A\alpha \int d^3k B_i(k) e^{i(kx)} \tag{11}$$

$$f_{ij}^\alpha = A\alpha i \int d^3k (B_i k_j - B_j k_i) e^{i(kx)} \tag{12}$$

Eqs. (11) and (12) are the exact analytic solutions of eqs. (1), (2) and (3).

3. DISCUSSIONS

(1) The solutions, eqs. (11) and (12), are superposition of plane wave solutions with propagation velocity equal to the velocity of light.

(2) These solutions are such a simple form that one seems hard to believe it that they were not discovered ealier.

(3) Is there any solution of eqs. (1), (2) and (3) other than eqs. (11) and (12)? The answer to this question is not clear at all. It is not an easy job to prove that the only solutions for eqs. (1), (2) and (3) are eqs. (11) and (12). On the other hand, there is no other example of solution. (From the point of view of the author, the solutions given by Wu and Yang in reference 2 are not free field solutions, they seem to be the solutions with point source at the origin.)

(4) It is of great interest if one can use the exact classical solutions to obtain the quantized view of the field equation.

REFERENCE

1. C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).
2. T. T. Wu and C. N. Yang, in Properties of Matter under Unusual Conditions, edited by H. Mark and S. Fernbach (wiley, New York,1969) p. 349.
3. C. N. Yang, Phys. Rev. Lett. 33, 445 (1974)