Susceptance of Circular Apertures in Wave Guides

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A brief description of the general principles and methods of finding the aperture susceptance of circular apertures in the transverse plane of wave guide is presented. The standing wave method and equivalent magnetic wall method apply only to smaller apertures. For large apertures the variational method is used to extend it to the case of circular wave guide. The results obtained from these methods are found to be compatible.

The problem of the iris or apertures in the wave guide is fundamental to many microwave circuits and coupling devices and also in understanding the properties of a periodically loaded wave guide, such as in the operation of the travelling wave amplifier and linear accelerator.

The iris problem in wave guide is sometime a special problem in the microwave circuit which has waveguides with discontinuities. Any type of discontinuity such as inhomogeneous dielectric, conducting diaphragms, or a change of guide dimension or shape, may be represented as a two terminal pair network with the conventional circuit parameters.

The presence of a discontinuity in a wave guide results in discontinuities in the propagating mode fields. Such discontinuities in the fields can be schematically represented by means of a lumped constant equivalent circuit in the form of two terminal pair T or Π equivalent network with the conventional circuit parameters Z_{11}, Z_{12} and Z_{22} or Y_{11}, Y_{12} and Y_{22}. In the case when the input and output guides are alike then $Z_{11}=Z_{22}$, $Y_{11}=Y_{22}$ the parameters then reduce to two in number. Moreover when the input field or voltage is the same as the output field or voltage such as the case of an iris across a guide the equivalent circuit parameter reduces to only Z_{12} or Y_{12}, i.e. a shunt impedance or susceptance.

The theoretical determination of the equivalent circuit parameters may be found in the references[1,2,3] and will not be developed here.

To describe the field in a uniform wave guide only the dominant mode or propagating mode is required, while in a guide with discontinuity tse complete description of the fieldes in the guide near the discontinuity region with an infinity of nonpropagating higher modes. is required, in addition to the dominant modes.

The nonpropagating or evanescent nature of the higher-mode restricts the field description to the immediate vicinity of the discontinuity. Intreating the iris type of discontinuity it may therefore be regarded as a lumped constant across the uniform wave guide which may be regarded as a transmission line and in fact it can be proved to be a shunt susceptance.

The value of the shunt susceptance can be obtained by different methods of approach all based on the reflection of the dominant mode by the aperture and generation of higher modes to satisfy the boundary conditions at the discontinuity. We may call these methods:

1. The standing wave method
2. The equivalent magnetic wall method
3. The variational method

The first two methods are restricted to the case when the aperture size is small compared to the wavelength.

<u>The Standing Wave Method</u>

In the following development, although the principles involved are quite general, only the applications to the circular apertures are considered.

When a uniform transmission line is terminated by a shunt admittance Y the transverse component of the electric field is given by

$$E_t = A\, e^{j\omega t}\, (e^{-j\beta z} + \rho\, e^{j\beta z})$$

$$= A\, e^{j(\omega t-\beta z)}\, (1+ e^{j2\beta z}) \quad (1)$$

where

A = amplitude of transverse field

$\rho = |\rho|\, e^{-j\theta} = \frac{Y_o - Y}{Y_o + Y}$ is the reflection coefficient

$j\beta = \Gamma$ is the propagation constant of the dominant mode. Maximum value of the electric field occurs at the point where $1 + \rho\, e^{j2\beta z}$ is a maximum or $1 + |\rho|\, e^{j(2\beta z+\theta)}$ is maximum, it occurs when $2\beta z+\theta=0$, $2n\pi$, where n is an integer. The first maximum nearest to the terminal point is at

$$\beta z_0 = -\frac{\theta}{2} \tag{2}$$

where z_0 denotes the distance from the terminal point to the point of maximum electric field.

When the terminal admittance is a pure susceptance jB, by which a loseless diaphragm may be represented, the magnitude of the reflection coefficient $|\rho|$ is unity and the phase angle of the reflection coefficient is given by

$$\theta = \tan^{-1} \frac{2b}{b^2-1} \tag{3}$$

Where b is the normalized shunt susceptance i.e. $b = B/Y_0$ with $Y_0 = \sqrt{\frac{\epsilon_0}{\mu_0}}$ representing, the characteristic admittance of TEM mode.

From Eq. (3) it is readily recognized that

$$\tan\frac{\theta}{2} = b \text{ or from Eq. (2)}$$

$$b = -\tan \beta z_0 \tag{4}$$

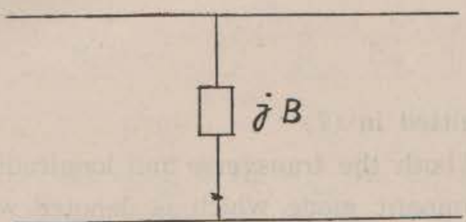

Fig. 1 (a)

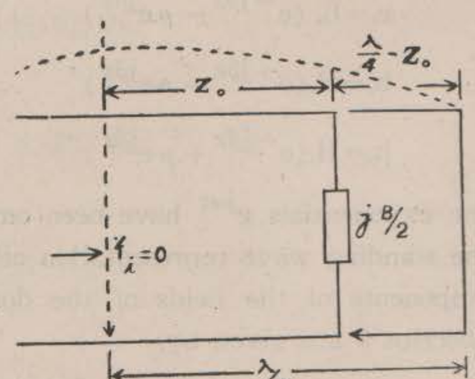

Fig. 1 (b)

Or in other words it is equal to the input admittance of an additional short-circuited line of length $\frac{\gamma}{4} - z_0$ as shown in Fig. 1 (a), i.e. the input admittance at the point of voltage maximum represents the condition of resonance.

The total susceptance of an iris placed across somewhere in an otherwise uniform guide is twice as given by Eq. (4) or

$$b = -2 \tan \beta z_0 \tag{5}$$

We see that the problem of determination of the aperture susceptance reduces to the determination of the quantity z_0.[4]

Let the travelling electric and magnetic fields be represented by their transverse components E_t, H_t and longitudinal components E_z, H_z respectively, then the electric and magnetic fields are:

$$E = (E_t + kE_z)\, e^{j\omega t - \Gamma z}$$

$$H = (H_t + kH_z)\, e^{j\omega t - \Gamma z} \qquad (6)$$

where E_t, H_t are vector functions of the transverse coordinates and k is a unit vector along the longitudinal or the z axis and E_z and H_z are scalar functions of the transverse coordinates.

When reflection occurs, the transverse and longitudinal components are represented by the following relations

$$e_t = E_t\, (e^{-j\beta z} + \rho\, e^{+j\beta z})$$

$$e_z = E_z\, (e^{-j\beta z} - \rho\, e^{j\beta z})$$

$$h_t = H_t (e^{-j\beta z} - \rho\, e^{j\beta z}) \qquad (7)$$

$$h_z = H_z (e^{-j\beta z} + \rho\, e^{j\beta z})$$

The exponentials $e^{j\omega t}$ have been omitted in (7)

The standing wave representation of both the transverse and longitudinal components of the fields of the dominant mode which is denoted with subscript 0 are given by,

$$E_0 = E_{t0}\cos\beta(z - z_0) - k(jE_{z0})\sin\beta(z - z_0)e^{j\omega t}$$

$$H_0 = H_{t0}(-j)\sin\beta(z - z_0) + kH_{z0}\cos\beta(z - z_0)e^{j\omega t} \qquad (8)$$

Now suppose that the actual transverse and longitudinal components of E and H at the aperture are known i.e. at the plane $z=0$

$$E = E_t + KE_z$$

$$H = H_t + KH_z \qquad (9)$$

where notations without subscript 0 indicates the actual field at $z=0$ and the exponentials $e^{j\omega t}$ have been dropped in all the following expressions for simplicity of notation.

These functions E_t, E_z, H_t and H_z can be expressed as series over all mode function E_{tn}, E_{zn}, H_{tn} and H_{zn}, where n being integers is the mode index (with subscript 0 referring to the dominant mode). By the orthogonal and their normalization properties of the field components which amount to

$$\begin{aligned}
&\int (E_{tn}\cdot E_{tm})\, da=\delta=\begin{cases}1 \text{ if } m=n\\ 0 \text{ if } m\neq n\end{cases}\\
&\int (H_{tn}\cdot H_{tm})\, da=Z_n^2\delta\\
&\int (E_{zn}\cdot E_{zm})\, da=\left(\frac{\beta}{\beta_{cn}}\right)^2\delta\\
&\int (H_{zn}\cdot H_{zm})\, da=\left(\frac{\beta}{\beta_{cn}}\right)^2 Z_n^2\,\delta
\end{aligned} \tag{10}$$

where $j\beta_{cn}$ is the mode cut-off propagation constant, z_n is the mode impedance, then we get the expansions as follow:

$$\begin{aligned}
E_t &= \sum_{n=o}^{\infty} a_n E_{tn} = \sum_{n=o}^{\infty} E_{tn} \int E_t\cdot E_{tn}\, da,\\
E_z &= -\sum_{n=o}^{\infty} \frac{\beta^2}{\beta^2_{cn}}\, E_{zn} \int E_z\cdot E_{zn}\, da,\\
H_t &= \sum_{n=o}^{\infty} b_n H_{tn} = \sum_{n=o}^{\infty} H_{tn}\, Z_n^2 \int H_t\cdot H_{tn}\, da,\\
H_z &= -\sum_{n=o}^{\infty} \frac{\beta^2}{\beta_{cn}^2} H_{zn}\, Z_n^2 \int H_z\cdot H_{zn}\, da,
\end{aligned} \tag{11}$$

In the expansion of the z components of E and H, it is understood that E_{zo} must be expanded in terms of the E_{zn} of TM mode only, and H_{zo} in terms of the H_{zn} of TE modes only.

In Eqs. (11) we have the general expansions if the field components in terms of mode functions. But Eq. (8) tells us that the coefficients of the field components in the dominating mode can be expressed in terms of z_o by setting $z=0$. Equating the coefficients between the two expressions we have

$$E_t = \cos\beta z_o = \int E_t \cdot E_{to}\, da = -\frac{\beta^2}{\beta_{co}^2} Z_o^2 \int H_z \cdot H_{zo}\, da$$

$$E_z = \sin\beta z_o = j\frac{\beta^2}{\beta_{co}^2}\int E_z E_{zo}\, da = -j\, Z_o^2 \int H_t \cdot H_{to}\, da$$

Now from Eq. (5) we can write for the aperture susceptance b as follows

$$jb = j\,2\,z_o^2\, \frac{\int H_t \cdot H_{to}\, da}{\int E_t \cdot E_{to}\, da} \quad (12a)$$

$$= -j\,2\,\frac{\beta_{co}^2}{\beta^2}\,\frac{\int H_t \cdot H_{to}\, da}{\int H_z \cdot H_{zo}\, da} \quad (12b)$$

$$= -j\,2\,\frac{\beta^2}{\beta_{co}^2}\,\frac{\int E_z \cdot E_{zo}\, da}{\int E_t \cdot E_{to}\, da} \quad (12c)$$

where the field components without subscript 0 denote the actual field over the plane of the iris, those with subscript 0 denote the dominant mode components. From Eq. (12) we see that the aperture susceptance can be computed once we know the transverse or longitudinal components of electric and magnetic fields in the plane of the aperture.

Equation (12a) is suitable in either the TE or TM mode, Eq. (12b) is suitable only for the TE mode, Eq. (12c) only for the TM mode. The boundary conditions to be satisfied at the plane of the iris are:

1. Only the transverse component of E or E_t and the longitudinal component of H or H_z can exist at the aperture, i.e. the longitudinal component E_z and the transverse component H_t vanish at the aperture opening.
2. Only the longitudinal or normal component of $E(E_z)$ and the transverse or tangential component of H (H_t) can exist at the metallic surface of the diaphragm, i.e. E_t and H_z vanish.

Thus the integrals in the numerators of Eqs (12) must be extended over the metal and the integrals in the denominator over the aperture opening.

Small Circular Aperture in TE Mode

Now as an example let us consider the case of a transverse metal diaphragm with a small circular aperture of radiusho in the transverse plane of a guide operating in the lowest TE mode. It is known that the transverse field components in a wave guide satisfy the condition for the static field i.e. the Laplace equation, now for a small aperture it can be approximated that the longitudinal components also satisfy the Laplace equation and the transverse component H_t differ by a negligible amount from the value of dominant component H_{to} when there were no opening. Eq. (12b) is used in calculating the susceptance to the advantage that only magnetic field components are needed. The longitudinal component H_z actually existing at the aperture may be calculated by the duality correspondance of the electrostatic field problem of a circular disk placed in constant electrostatic field parallel to the surface of the disk, and in the rectangular coordinates it is

$$H_z = \frac{4}{\pi} H_t \frac{x}{\sqrt{a^2-x^2-y^2}}$$

where H_t is the transverse magnetic field assumed constant over the aperture.

The dominant component H_{zo} may be calculated in terms of H_t by expanding H_{zo} in power series about the center of the aperture and taking the first two terms in the expansion for a small aperture, thus

$$H_{zo} = H_{zo}\Big|_o + x\frac{\partial H_z}{\partial x}\Big|_o + y\frac{\partial H_{z0}}{\partial y}\Big|_o + \cdots\cdots$$

$$= H_{zo}\Big|_o + r \bullet \mathrm{grad}_t H_z$$

Now $\mathrm{grad}_t H_z = j\frac{k_c^2}{\beta} H_t$, where $k_c^2 = \Gamma^2 + k_o^2$, $k_o = \omega\sqrt{\mu_o \epsilon_o}$ for TE mode, and $H_{zo}|_o = 0$, $\frac{\partial H_z}{\partial y} = 0$

for H_t and $\mathrm{grad}_t H_z$ has only the x component, we obtain

$$H_{zo} = x\frac{jk_c^2}{\beta}|H_t|$$

where $|H_t|$ is the magnitude of H_t at the center of the aperture.

The integral $\int H_z H_{zo}\, da$ in the denominator of Eq. (12b) becomes

$$\int H_z H_{zo}\, da = j\frac{4}{\pi}|H_t|^2\frac{k_c^2}{\beta}\int\frac{x^2\, da}{\sqrt{a^2-x^2-y^2}}$$

$$= j\frac{8}{3}a^3|H_t|^2\frac{k_c^2}{\beta}$$

substituting into Eq.(12b) we have

$$b_o = -\frac{3}{4r_o^3}\frac{\int |H_t|^2\, da}{\beta|H_t|^2} \qquad (13)$$

By a similar method we can obtain the susceptance aperture for the TM mode

$$b_o = \frac{3}{2 r_o^3} \frac{\beta}{k_c^2} \frac{\int E_z^2 \, da}{E_z^2} \tag{14}$$

The above expressions are valid for both rectangular and circular wave guide as long as the proper values of field components are used.

As application of the Eqs. (13) and (14), we use it to calculate the aperture susceptance of a circular aperture in TE_{o1} mode of a rectangular guide and TE_{11} mode of a circular guide.

<u>TE_{o1} Mode in Rectangular Guide with Small Circular Aperture</u>

With normalized field, the transverse component of the magnetic field is

$$H_t = H_x = K \sin \frac{\pi x}{a},$$

where

$$K = j\beta_{10} \left(\frac{2}{jabk_o z_o \beta_{10}} \right)^{\frac{1}{2}}$$

the factor with the square root is the normalization factor.

Then

$$\int |H_t|^2 \, da = K^2 \int_o^b \int_o^a \sin^2 \frac{\pi x}{a} \, dx \, dy = \frac{K^2 ab}{2}$$

At the center of the aperture $|H_t|_o^2 = K^2$

By Eq. (13), the normalized susceptance is

$$b_o = -\frac{3}{4 r_o^3} \frac{ab}{2 \beta_{10}} = -\frac{3}{8} \frac{ab}{r_o^3 \beta_{10}} \tag{15}$$

<u>TE_{11} Mode in Circular Guide with Small Aperture:</u> For circular guide

$$H_r = K\Gamma_1 k_c J_1'(k_c r) \cos\theta$$

$$H_\theta = \frac{K\Gamma_1}{r} J_1(k_c r) \sin\theta$$

$$|H_t|^2 = H_r^2 + H_\theta^2 = K^2 \Gamma_1^2 \left[K_c^2 J_1'^2 (k_c r) \cos^2\theta + \frac{J_1^2(k_c r)}{r^2} \sin^2\theta \right]$$

$$da = r \, d\theta \, dr$$

where Γ_1, k_c have been written for Γ_{11} and k_{c11} for simplicity. Expanding the Bessel functions into series and integrate, retaining only the lowest order terms for small aperture, it can be shown that

$$\int |H_t|^2 \, da = \frac{\pi}{4} K^2 \Gamma_1^2 k_c^2 a^2$$

where a =radius of the guide, and

$$\beta_1 |H_t|_o^2 = \beta K^2 \Gamma_1^2 \frac{k_c^2}{4}$$

Substituting into Eq. (13) we obtain

$$b_o = -\frac{3}{4 r_o^3} \frac{\pi}{4} k_c^2 r_o^2 \Big/ \frac{\beta k_c^2}{4} = -\frac{3\pi}{4\beta_1} \frac{a^2}{r_o^3}$$

$$= -\frac{3}{8} \frac{\lambda_g}{r_o^3} a^2 \qquad (16)$$

From Eqs. (15) and (16) it is seen that the susceptances are inductive.

TM_{01} Mode in Circular Guide with Small Aperture

Eq. (14) is used for TM mode and only the longitudinal component of electric field is needed for calculation. For TM mode it may be written

$$E_z = K k_c^2 J_o(k_c r)$$

where K is again a normalization factor.

$$J_o(k_c r)|_{r=o} = 1$$

At the center of the aperture we have

$$E_z|_o = K k_c^2,$$

$$\int_0^{2\pi}\int_0^a E_z^2 r \, d\theta \, dr = 2\pi K^2 k_c^4 \int_0^a r J_o^2(k_c r) \, dr$$

$$= 2\pi K^2 k_c^4 \cdot \frac{a}{2} [J_o'^2(k_c a) + J_o^2(k_c a)]$$

The second term in the bracket is zero since the boundary condition requirse that $J_o(k_c a) = 0$, then we have

$$\int E_z^2 \, da = K^2 \pi a^2 k_c^4 J_1^2(k_c a)$$

and by Eq. (14), the susceptance is

$$b_o = \frac{3}{2 r_o^3} \frac{\beta}{k_c^2} \pi a^2 J_1^2(k_c a) = \frac{3\pi^2 a^2}{r_o^2 k_c^2 \lambda g} J_1^2(k_c a)$$

$$= \frac{3\pi^2 J_1^2(k_c a)}{p_{o1}^2} \frac{a^4}{r_o^3 \lambda g} \qquad (17)$$

where $p_{o1} = k_c a$ is the root of J_o ang λg is the guide wavelength.

The Equivalent Magnetic Wall Method

This method is based on the following theorem: Let the fields to the left of the discontinuity be E_o, H_o, when the aperture is closed by a perfect magnetic wall. With the aperture open, the total field to the left of the aperture may be given by the sum of the fields E_o, H_o and the field radiated by a magnetic current $J_m = -nxE_o$ on S_a (see Fig. 2)

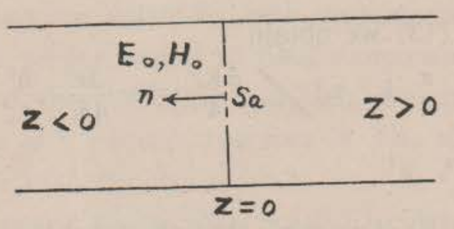

Fig.2

The magnetic wall discontinuity produces a scattered field E_s and H_s which may be expanded in terms of the normal mode for $z<0$ as follows:

$$E_s = \Sigma\, a_n\, E_n$$

$$H_s = \Sigma\, a_n\, H_n$$

For small apertures the coefficients a_n may be represented as coupling to an electric and magnetic dipole

$$a_n = j\omega\,(\mu_o\, H_n \cdot M_o - E_n \cdot P_o) \tag{18}$$

where M_o and P_o are the equivalent magnetic and electric dipole respectively. The expressions of M_o and P_o are difficult to find in general, but for a small aperture a static field solution for the dipole moments of elliptic and circular-shaped aperture can be obtained. These are:

$$P_o = \epsilon_o\, \alpha_e\, nn \cdot E$$

$$M_o = \overline{\alpha}_m \cdot H \tag{19}$$

where E, H are the incident fields at the aperture, α_e is the electric polarizability of the aperture, $\overline{\alpha}_m$ is the dyadic magnetic polarizability of the aperture and are given respectively by

$$\alpha_e = -\frac{\pi\, d_1^3\,(1-e^2)}{3\,E(e)}$$

$$\overline{\alpha}_m = a_u\, a_u \frac{\pi\, d_1^3\, e^2}{3[K(e)-E(e)]} + a_v\, a_v \frac{\pi\, d_1^3\, e^2\,(1-e^2)}{3[E(e)-(1-e^2)K(e)]} \tag{20}$$

where a_u and a_v are unit vectors along u and v coordinates.

d_1=major axis of the ellipse, d_2=minor axis of the ellipse.

$e = (1 - \frac{d_2^2}{d_1^2})^{\frac{1}{2}}$ is the eccentricity of the ellipse.

K(e) and E(e) are the complete elliptic integrals of the first and second kind with modulus e.

For a circular aperture e is zero

then $\alpha_e = -\frac{2}{3} r_o^3$ with r_o =radius of the aperture

$$\bar{\alpha}_m = \frac{3}{4} r_o^3 (a_u a_u + a_v a_v) \tag{22}$$

Let us now apply Eq. (18) to calculate the susceptance of a circular aperture in rectangular and circular guides

<u>TE_{01} Mode in Rectangular Guide—Small Aperture</u>

As in the preceeding section the transverse components of electric and magnetic fields in a guide closed by a conducting wall with a small aperture are given by

$$E_t = E_{10} (e^{-\Gamma_{10} z} - e^{+\Gamma_{10} z})$$

$$H_t = H_{10} (e^{-\Gamma_{10} z} + e^{+\Gamma_{10} z})$$

At the plane of the iris z=0, $H_t |_{z=0} = 2H_{10}$

There is no normal electric field at the aperture so P_o=0 and the magnetic field tangential to the aperture surface gives rise to a magnetic dipole moment given by Eq. (19)

$$M_o = \bar{\alpha}_m \cdot 2H_{10} = \frac{4}{3} r_o^3 \cdot 2H_{10}\, a_x, \qquad a_x = \text{unit vector along x-axis}$$

In the rectangular guide propagating TE_{01} mode the normalized transverse magnetic field is

$$H_{10} = \Gamma_{10} \left(\frac{2}{j\, ab\, k_0\, z_0\, \Gamma_{10}} \right)^{\frac{1}{2}} \sin \frac{\pi x}{a}\, a_x, \qquad a_x = \text{unir vactor alog x axis}$$

where the factor with the square root is the normalization factor. At the center of the aperture

$$H_{10} \Big|_{x=0} = \Gamma_{10} \left(\frac{2}{j\, ab\, k_0\, z_0\, \Gamma_{10}} \right)^{\frac{1}{2}}$$

Since only the dominant TE_{01} mode propagates in the guide, Eq. (18) can be written as

$$a_1 = j\omega\,(\mu_o H_t \cdot M - E_n \cdot P) \tag{23a}$$

and since there is no component of electric field normal to the aperture $E_n = E_z = 0$ and $H_t = H_{ol}$

then we have

$$a_1 = j\,\omega\,\mu_o H_{10} \cdot M_o = \frac{16}{3} \cdot \frac{d^3\,\beta_{10}\,j\,\omega\,\mu_o}{ab\,k_o\,z_o}$$

$$= j\,\frac{16}{3} \cdot \frac{d^3\,\beta_{10}}{ab}, \text{ since } k_o\,z_o = \omega\,\mu_o \tag{23b}$$

where a_l represents that part of the reflected wave which is to be deducted from the reflected wave of a conducting wall, so that the resultant reflected wave is $(a_l-1)\,E_{10}$ and $(1-a_1)\,H_{10}$. An equivalent reflection coefficient may be defined as

$$R = a_1 - 1 \text{ or } a_1 = 1 + R$$

Now a susceptance jb_o connected across a wave guide gives rise to an input admittance $Y_{in} = 1 + jb_o$ at the aperture and a reflection coefficient

$$R = \frac{1 - Y_{in}}{1 + Y_{in}} = \frac{-jb_o}{2 + jb_o}$$

when jb_o is large for small aperture

$$1 + R = \frac{2}{2 + jb_o} \approx \frac{2}{jb_o}$$

or $$jb_o = \frac{2}{1+R} = \frac{2}{a_l} \tag{24}$$

Substituting the value of a_l from Eq. (23b) we have

$$b_o = \frac{2}{ja_1} = -\frac{3ab}{8\,r_o^3\,\beta_{10}} \tag{25}$$

which agrees with Eq. (15)

<u>TE_{11} Mode in Circular Guide with Small Aperture</u>

The normalized magnetic field components of TE_{11} mode in a circular guide are given by

$$H_z = K\,k_c^2\,J_1\,(k_c r)\,\cos\theta$$

$$H_\theta = -K \frac{n\Gamma}{r} J_1(k_c r) \sin\theta$$

$$H_r = K k_c \Gamma J_1'(k_c r) \cos\theta \qquad (26)$$

where K is the normalization factor, given by

$$K^2 = \frac{4}{\pi} \frac{1}{k_o z_o \beta} \left[\frac{1}{(k_c^2 a^2 - 1) J_1^2(k_c a)} \right] = \frac{4K'}{\pi \omega \mu_o \beta}$$

in which $k_c a = p_{11}$ is the root of $J_1(x)$ and for TE_{11} mode $k_c a = 1.84$, K' is the quantity in the bracket.

Again in this mode there is no component of electric field normal to the aperture, so there will not be an induced electric dipole P.

The magnetic field has a tangential component along the radial direction at the center of the aperture given by

$$H_r \Big|_{r=o} = K k_c \Gamma/2$$

The magnetic moment is then

$$M_o = \alpha_m \cdot 2 H_r |_{r=o} = \alpha_m K k_c \Gamma$$

and

$$a_1 = j\omega\mu_o H_{ro} \cdot M$$

$$= j\omega\mu_o \alpha_m K^2 k_c^2 \Gamma^2 / 2$$

Using Eq. (24) we have

$$b_o = \frac{2}{ja_1} = -\frac{1}{K'(k_c a)^2} \cdot \frac{\lambda_g a^2}{\alpha_m} = -\text{constant} \frac{\lambda_g a^2}{\alpha_m} \qquad (27)$$

which also conforms with Eq. (16)

TM_{o1} Mode in Circular Guide with Small Aperture

The field components of the TM_{o1} mode in a circular guide are given by

$$E_z = K k_c^2 J_o(k_c r)$$

$$E_r = K \Gamma k_c J_o'(k_c r)$$

$$H_\theta = K j \omega \epsilon_o k_c J_o'(k_c r)$$

$$E_\theta = H_r = 0 \qquad (28)$$

where K is again a normalization factor which is, for TM_{01} mode, given by

$$K^2 = \frac{4}{\pi \omega \epsilon_0 \beta \cdot P_{01}^2 J_1^2 (p_{01})} = \frac{4K'}{\pi \omega \epsilon_0 \beta}$$

where $p_{01}=2.405$, $k'=1/ P_{01}^2 J_1^2 (p_{01})$

At the center H_θ is zero since $J_0' (k_c r) \mid_{r=0} = 0$, there is no tangential magnetic field component at the center of the aperture, so the magnetic dipole moment M_0 is zero. The component of the electric field normal to the aperture is E_z which when evaluated at the center of the aperture is $E_{z0}=K k_c^2$. By Eq. (23), (19) we have

$$a_1=-j \omega E_{z0}\cdot P_0$$

$$P_0=\epsilon_0 \alpha_e nn\cdot E_{z0}=K \epsilon_0 \alpha_e k_c^2\cdot n$$

or $$a_1=-j \omega \epsilon_0 \alpha_e K^2 k_c^4$$

$$=-j \frac{4K'}{\pi \beta} \alpha_e k_c^4$$

and $$b_0= \frac{2}{j a_1} = \frac{\pi^2 J_1^2 (P_{01}) a^4}{p_{01}^2 \alpha_e \lambda_g} \qquad (29)$$

which conforms also with Eq. (17).

The Variational Method for the Aperture Susceptance

When the aperture size is not small, an approximate method based on the variational principles may be used.

When an incident wave in the dominant mode is incident from the left, due to the discontinuity at the iris, a reflected wave and an infinite number of higher-order modes are excited.

Let ϕ_1, ϕ_n = mode function of the electric field of the dominant and higher-order mode wave respectively.

a_1, a_n = amplitudes of the electric fields of the above waves.

R_1 = reflection coefficient of dominant wave.

Then to the left of the aperture the transverse fields are given by

$$E_t=a_1 (e^{-\Gamma_1 z}+R_1 e^{\Gamma_1 z}) \phi_1 +\sum_{2}^{\infty} a_n \phi_n e^{\Gamma_n z}$$

$$H_t=-\overline{Y}_1\cdot a_1 (e^{-\Gamma_1 z}-R_1 e^{\Gamma_1 z}) \phi_1 +\sum_{2}^{\infty} a_n \overline{Y}_n \cdot \phi_n e^{\Gamma_n z} \qquad (30)$$

where $\overline{Y}_1$, $\overline{Y}_n$ are dyadic wave admittances. To the right of the aperture the transverse fields are:

$$z>o, \quad E_t = \sum_{m=2}^{\infty} b_m \ \phi_m \ e^{-\Gamma_n z} + T_1 \ a_1 \ e^{-\Gamma_1 z}$$

$$H_t = -\left(\sum_{m=2}^{\infty} b_m \overline{Y}_m \bullet \phi_m e^{-\Gamma_n z} + T_1 \ a_1 \bullet \overline{Y}_1 e^{-\Gamma_1 z} \right) \tag{31}$$

where T_1 = Transmission coefficient.

At the aperture $z = 0$, the transverse fields must be continuous, or from Eqs. (30), (31) we have,

$$E = a_1 \ (1+R_1) \ \phi_1 + \sum_{n=2}^{\infty} a_n \phi_n = \sum_{n=2}^{\infty} b_m \ \phi_n + T_1 \ a_1$$

$$a_1 \ (1-R_1) \ \overline{Y}_1 \bullet \phi_1 - \sum_{n=2}^{\infty} a_n \overline{Y}_n \bullet \phi_n = \sum_{n=2}^{\infty} b_m \ \overline{Y}_m \bullet \phi_n + T_1 \ a_1 \bullet \overline{Y}_1 \tag{32}$$

in which E is the electric field at the aperture plane. By the orthogonal properties of the mode functions ϕ_m and ϕ_n, i.e.

$$\int_0^a \phi_m \ \phi_n \ dr = \delta = 1, \text{ for } m = n$$

and zero for $m \neq n$. We may evaluate the coefficients of these mode functions, thus:

$$T_1 a_1 = a_1 \ (1+R_1) = \int_0^a E \ \phi_1 \ dr$$

$$a_n = b_m = \int_0^a E \ \phi_n \ dr$$

Substituting the above coefficient and $T_1 = 1 + R_1$ in the second of Eq. (32) we obtain,

$$\frac{-2R_1}{1+R_1} \ \overline{Y}_1 \bullet \phi_1 \int_0^a E \ \phi_1 \ dr = 2 \int_0^a E \ (r') \ G \ (r, r') \ dr'$$

where $\quad G \ (r, r') = \sum_{2}^{\infty} \overline{Y}_n \bullet \phi_n \ (r) \ \phi_n \ (r')$

Now a susceptance jb across a transmission line gives rise to a reflection coefficient R_1

$$R_1 = -\frac{jb}{2+jb} \quad \text{or} \quad jb = -2R_1/1+R_1$$

So

$$jb = 2\frac{\int_0^a E(r')\, G(r, r')\, dr'}{\overline{Y}_1 \cdot \phi_1 \int_0^a E\, \phi_1\, dr} \tag{33}$$

Since this formula involves the integral of E which is yet unknown, an approximate variational method is used to the fact that a first order approximation of E gives a second order approximation of the expression.[6,7]

A variational integral expression may be obtained by multiplying Eq. (33) by E and integrating over the aperture region.

$$jb = \frac{2\int_0^a \int G(r, r')\, E(r)\, E(r')\, dr\, dr'}{\overline{Y}_1 \cdot \left[\int_0^a E(r)\, \phi_1\, dr\right]^2} \tag{34}$$

Now let us calculate the approximate value of the aperture susceptance for the TM_{01} mode of a circular guide.

<u>TM_{01} Mode of Circular Guide:</u>

For TM_{01} mode by (28) we have,

$E_r = K\, \Gamma\, k_c\, J_0'(k_c\, r)$

The mode function of nth order wave is

$\phi_n(r) = J_0'(k_n\, r),\ \phi_n(r') = J_0'(k_n\, r')$

All the coefficients $K\Gamma k_c$ have been absorbed in the amplitude coefficients a_n and since Eq. (34) is homogeneous in E, the factor $k\Gamma k_c$ cancels out in both the numerator and the denominator. Eq. (34) may be written as

$$jb = \frac{2\int\int_0^{r_0} \sum_2^{\infty} Y_n\, \phi_n(r)\, \phi_n(r')\, dr\, dr'}{Y_L\left[\int_0^{r_0} \phi_1(r)\, dr\right]^2}$$

$$\int_0^{r_o} \phi_n (r)\, dr = \int_0^{r_o} \phi_n (r')\, dr' = \int_0^{r_o} J_o' (k_n r)\, dr$$

$$= \frac{1}{k_n} \Big[J_o (k_n r_o) - 1 \Big] \qquad (35)$$

$$\int_0^{r_o} \phi_1 (r)\, dr = \frac{1}{k_1} \Big[J_o (k_1 r_o) - 1 \Big]$$

$$\frac{Y_n}{Y_1} = \frac{\Gamma_1}{\Gamma_n} \quad \text{for TM mode}$$

For the nonpropagating modes, Γ_n is approximately equal to $k_n = p_{on}/a$, where a is the radius of the guide. Eq. (35) becomes,

$$jb = 2 k_1^2 \Gamma_1 \sum_2^{\infty} \frac{1}{k_n^3} \left[\frac{J_o (k_n r_o) - 1}{J_o (k_1 r_o) - 1} \right]^2$$

$$= j \frac{4\pi a P_{o1}^2}{\lambda_g [(J_o (k_1 r_o) - 1]^2} \sum_2^{\infty} \frac{1}{p_{on}^3} \Big[J_o (k_n r_o) - 1 \Big]^2 \qquad (36)$$

For the TE_{o1} mode the transverse electric field has the θ component only which is,

$$E_\theta (r) = j \omega \mu_o k_c J_o' (k_n r)$$

with the mode function

$$\phi_n (r) = J_o' (k_n r)$$

Similar expression for jb can be obtained, noting that $Y_n/Y_1, = \Gamma_n/\Gamma_1 \doteqdot k_n/\Gamma_1$ approximately. thus we have,

$$jb = -j \frac{p_{o1}^2}{[J_o (k_1 r_a) - 1]^2} \frac{\lambda_g}{\pi a} \sum_2^{\infty} \frac{1}{p_{on}} \Big[J_o (k_n r_a) - 1 \Big]^2$$

For the TE_{11} mode the transverse electric field has both r and θ components. Approximate results may be obtained by treating these components separately and adding the resulting susceptances.

<u>TE_{11} Mode in Circular Guide with Circular Aperture</u>

The mode function for E_θ is,

$$\phi_{n\theta} (r) = J_1' (k_n r)$$

$$\int_0^{r_o} \phi_{n\theta} (r) = \frac{1}{k_n} J_1 (k_n r_o)$$

The mode function for E_r is,

$$\phi_{nr}(r) = \frac{1}{r} J_1(k_n r)$$

$$\int_0^{r_0} \phi_{nr}(r)\, dr = \int_0^{r_0} \frac{J_1(k_n r)}{r}\, dr \approx \frac{r_o}{2 k_n} \text{ for small } r_o$$

the resultant susceptance is then

$$jb = -j\,(p_{11})^2 \frac{\lambda}{\pi a} \sum_{2}^{\infty} \frac{1}{p_{1n}} \frac{J_1^2(k_n r_o) + J_1^2(k_1 r_o)}{J_1^2(k_1 r_o)} \tag{37}$$

In Eqs. (36) and (37), results for small aperture approximation may be obtained by taking first order terms in the series.

Further approximation can be made by assuming a more accurate value of E in Eq. (34) by taking a finite number of terms in the series expansion of the aperture field.

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