SYNCHRONOUS MACHINE MODELING FOR AUTOMATIC CONTROL THEOREM

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(Received 18 Feb. 1975)

Abstract

In this paper, a mathematical form of synchronous machine model is developed for the requirement of modern automatic control theorem. This model is based on the assumption that the generator feeds a general three-phase tee-form transmission system consisting of a step-up transformer, a shunt load and a transmission line terminating at a bus of known voltage.

INTRODUCTION

Synchronous machine models based on general machine theory are unsuitable for use with modern control theory because they treat a machine as a general form of electrical circuit. A new model is developed by dealing the machine purely as a dynamical system.

The set of state variables chosen to describe the synchronous machine is the flux linkage of the direct and quadrature axis armature and rotor circuits. This set of variables has the advantages over a set of currents and voltages. It is unnecessary to compute new initial values after each change of system parameter, since the flux linkage in the machine remains constant through any instant at which such a change occures.

The models are developed by a general linearized set of differential equations to describe a voltage regulated generator connected to a terminal network with balanced phase parameters.

SYNCHRONOUS MACHINE EQUATIONS

The equations describing the three phase performance of a synchronous machine derived by the use of Park's transformation are summarised as following form:[1]

$$\psi_{fd}=X_{ffd}i_{fd}+X_{fkd}i_{kd}-X_{afd}i_d \tag{1}$$

$$\psi_d=X_{afd}i_{fd}+X_{akd}i_{kd}-X_di_d \tag{2}$$

$$\psi_{kd}=X_{fkd}i_{fd}+X_{kkd}i_{kd}-X_{akd}i_d \tag{3}$$

$$\psi_q=X_{akq}i_{kq}-X_qi_q \tag{4}$$

$$\psi_{kq}=X_{kkq}i_{kq}-X_{akq}i_q \tag{5}$$

$$v_{fd}=\frac{1}{\omega_0}p\psi_{fd}+r_{fd}i_{fd} \tag{6}$$

$$e_d=\frac{1}{\omega_0}p\psi_d-Ri_d-\frac{\omega}{\omega_0}\psi_q \tag{7}$$

$$0=\frac{1}{\omega_0}p\psi_{kd}+r_{kd}i_{kd} \tag{8}$$

$$e_q=\frac{1}{\omega_0}p\psi_q-Ri_q+\frac{\omega}{\omega_0}\psi_d \tag{9}$$

$$0=\frac{1}{w_0}p\psi_{kd}+r_{kq}i_{kq} \tag{10}$$

$$T_g=\psi_d i_q-\psi_q i_d \tag{11}$$

$$Q=e_q i_d-e_d i_q \tag{12}$$

$$e_t^2=e_d^2+e_q^2 \tag{13}$$

These equations, together with a terminal constraint expression derived in next section, give a complete description of the steady-state and transient operation of a synchronous machine for the case of a terminal network with balanced phase parameters.

THE TERMINAL CONSTRAINT

When a generator feeds a bus of known voltage and phase via a long transmission line together with a local load at the sending end, assuming the case of balanced phase parameters, the single line equivalent circuit is shown as in Figure 1. This gives:

$$\frac{e_m-e'_m}{Z_t}=e_m'Y_e+\frac{e_m'-v_m}{Z_e} \qquad m=1,\ 2,\ 3 \tag{14}$$

$$i_m=\frac{e_m-e_m'}{Z_t} \qquad m=1,\ 2,\ 3 \tag{15}$$

where

$$\begin{aligned} Z_t&=R_t+pL_t\\ Z_e&=R_e+pL_e\\ Y_e&=G_e+pC_e \end{aligned} \tag{16}$$

Combining (14) and (15)

$$i_m(B_1+B_2p+B_3p^2+B_4p^3)=e_m(B_5+B_6p+B_7p^2)-v_m,\quad m=1,\ 2,\ 3 \tag{17}$$

where

$$\begin{aligned} B_1&=R_e+R_t+R_eR_tG_e\\ B_2&=L_e+L_t+R_tL_eG_e+R_tR_eC_e+L_tR_eG_e\\ B_3&=L_tL_eG_e+L_tC_eR_e+C_eL_eR_t\\ B_4&=L_eL_tC_e\\ B_5&=1+R_eG_e\\ B_6&=R_eC_e+L_eG_e\\ B_7&=L_eC_e \end{aligned} \tag{18}$$

Park's direct axis transformation of (17) yields

$$\sum_{m=1}^{3}\left[B_1 i_m \cos(\theta-\frac{2\pi m}{3}) + B_2 \frac{di_m}{dt}\cos(\theta-\frac{2\pi m}{3}) + B_3 \frac{d^2 i_m}{dt^2}\cos(\theta-\frac{2\pi m}{3}) + B_4 \frac{d^3{}_m}{dt^3}\cos(\theta-\frac{2\pi m}{3}) \right]$$

$$= \sum_{m=1}^{3}\left[B_5 e_m \cos(\theta-\frac{2\pi m}{3}) + B_6 \frac{de_m}{dt}\cos(\theta-\frac{2\pi m}{3}) + B_7 \frac{d^2 e_m}{dt^2}\cos(\theta-\frac{2\pi m}{3}) \right] - v_d \quad (19)$$

As derived in appendix, substituting for the summation terms in (19) gives

$$B_1 i_d + B_2 (\frac{di_d}{dt} - i_q\omega) + B_3(\frac{d^2 i_d}{dt^2} - i_d\omega^2 - i_q\dot{\omega} - 2\omega\frac{di_q}{d_t})$$
$$+ B_4(\frac{d^3 i_d}{dt^3} - 3\omega\dot{\omega} i_d - \ddot{\omega} i_q + \omega^3 i_q - 3\omega^2\frac{di_d}{dt} - 3\dot{\omega}\frac{di_q}{dt} - 3\omega\frac{d^2 i_q}{dt^2})$$
$$= B_5 e_d + B_6(\frac{de_d}{dt} - e_q\omega) + B_7(\frac{d^2 e_d}{dt^2} - e_d\omega^2 - e_q\dot{\omega} - 2\omega\frac{de_q}{dt}) - v_d \quad (20)$$

where ω is the frequency of the line current. Similarly, the quadrature axis transformation of (17) gives

$$B_1 i_q + B_2(\frac{di_q}{dt} + i_d\omega) + B_3(\frac{d^2 i_q}{dt^2} - i_q\omega^2 + i_d\dot{\omega} + 2\omega\frac{di_d}{dt})$$
$$+ B_4(\frac{d^3 i_q}{dt^3} - 3\omega\dot{\omega} i_q + \ddot{\omega} i_d - \omega^3 i_d - 3\omega^2\frac{di_q}{dt} + 3\dot{\omega}\frac{di_d}{dt} + 3\omega\frac{d^2 i_d}{dt^2})$$
$$= B_5 e_q + B_6(\frac{de_q}{dt} + e_d\omega) + B_7(\frac{d^2 e_q}{dt^2} - e_q\omega^2 + e_d\dot{\omega} + 2\omega\frac{de_q}{dt}) - v_q \quad (21)$$

Equations (20) and (21) describes the behavior of the balanced-phase network of Fig. 1. The steady-state solution of these equation corresponds to the steady sinusoidal performance of the line; and the transient solution will describe any transient initiated by simultaneous identical switchings in each phase of the network.

The transients occurring on the transmission system are of little interest in control studies because of their duration. Therefore, for the purpose of obtaining a boundary condition for use on generator control studies, we can neglect the transient solutions of (20) and (21) and reduce them to their steady-state solution for steady conditions. These can be obtained simply by neglecting all time derivative terms in (20) and (21) to give

$$e_d(B_5-\omega^2 B_7) - e_q\omega B_6 = i_d(B_1-\omega^2 B_3) - i_q(\omega B_2-\omega^3 B_4) + v_d$$
$$e_d\omega B_6 + e_q(B_5-\omega^2 B_7) = i_d(\omega B_2-\omega^3 B_4) + i_q\ (B_1-\omega^2 B_2) + v_q \quad (22)$$

By substituting B_1 to B_7 from (18) into (22), we get a convenient form as

$$\alpha e_d - \beta e_q = \eta i_d - \sigma i_q + v_d \qquad \beta e_d + \alpha e_q = \sigma i_d + \eta i_q + v_q \tag{23}$$

where

$$\alpha = 1 + R_e G_e - \omega^2 L_e C_e = 1 + R_e G_e - X_e B_e$$
$$\beta = \omega (R_e C_e + L_e G_e) = R_e B_e + X_e G_e$$
$$\eta = R_e + R_t + R_e R_t G_e - \omega^2 (L_t L_e G_e + L_t C_e R_e + C_e L_e R_t)$$
$$= R_e + R_t + R_e R_t G_e - (X_t X_e G_e + X_t B_e R_e + B_e X_e R_t)$$
$$\sigma = \omega (L_t + L_e + R_t L_e G_e + R_t R_e C_e + L_t R_e G_e) - \omega^3 L_t L_e C_e$$
$$= X_t + X_e + R_t X_e G_e + R_t R_e B_e + X_t R_e G_e - X_t X_e B_e$$

As ω, the frequency of the line current, is constant at the infinite bus frequency, α, β, η and σ are constants as long as the line is connected to an infinite bus. In the following, ω appears as the instantaneous angular frequency of the generator, but when the line is connected to an infinite bus, α, β, η and σ will take their rated frequency values regardless of the instantaneous generator frequency.

SOLUTIONS OF SYNCHRONOUS MACHINE AND TERMINAL NETWORK EQUATIONS

Combine the terminal network equations (23) with the direct and quadrature axis armature circuit equations (1)～(13) will give the expression of the boundary conditions in terms of the direct and guadrature axis currents and flux linkages only. Substituting (7) and (9) into (23) gives

$$\begin{pmatrix} \eta + \alpha R & -\alpha - \beta R \\ \sigma + \beta R & \eta + \alpha R \end{pmatrix} \begin{pmatrix} i_d \\ i_q \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\omega_0} & \frac{-\beta}{\omega_0} & \frac{-\beta\omega}{\omega_0} & \frac{-\alpha\omega}{\omega_0} & -1 & 0 \\ \frac{\beta}{\omega_0} & \frac{\alpha}{\omega_0} & \frac{\alpha\omega}{\omega_0} & \frac{-\beta\omega}{\omega_0} & 0 & -1 \end{pmatrix} \begin{pmatrix} p\psi_d \\ p\psi_q \\ \psi_d \\ \psi_q \\ v_d \\ v_q \end{pmatrix} \tag{24}$$

From the relation of rotor angle and speed $p\delta = \omega - \omega_0 = p\triangle\delta$ and the definition of per unit speed deviation $n = \frac{\omega - \omega_0}{\omega_0}$, so for disturbances from a steady operating point, there exists:

$$p\triangle\delta = n\omega_0 \tag{25}$$

Again, the quantities v_d and v_q are given[2] in terms of the referance bus voltage v and rotor angle as

$$v_d = v \sin\delta$$
$$v_q = v \cos\delta$$

so for small disturbances from a fixed operating point,

$$v_d = v \sin\delta + (v \cos\delta) \triangle\delta$$
$$v_q = v \cos\delta - (v \sin\delta) \triangle\delta \tag{26}$$

Equation (26) allow (24) to be written for disturbed values of current, flux and voltage, such as $i_{d0} + \triangle i_d$, $i_{q0} + \triangle i_q$.

Substraction of the original equation (24) and neglect of second order terms yield for the current disturbances as the following matrix equation

$$\begin{pmatrix} \eta+\alpha R & -\sigma-\beta R \\ \sigma+\beta R & \eta+\alpha R \end{pmatrix} \begin{pmatrix} \triangle i_d \\ \triangle i_q \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\omega_0} & \frac{-\beta}{\omega_0} & -\beta & -\alpha & -\alpha\psi_{q0}-\beta\psi_{d0} & -v\cos\delta \\ \frac{\beta}{\omega_0} & \frac{\alpha}{\omega_0} & \alpha & -\beta & \alpha\psi_{d0}-\beta\psi_{q0} & v\sin\delta \end{pmatrix} \begin{pmatrix} p\triangle\psi_d \\ p\triangle\psi_q \\ \triangle\psi_d \\ \triangle\psi_q \\ n \\ \triangle\delta \end{pmatrix} \quad (27)$$

When the transmission line is connected to an infinite bus, α, β, η and σ are constants. In the small disturbed case, it is also permissible to assume that they are constants for an impedance load. For convenience, (27) can be expressed as

$$\begin{pmatrix} \triangle i_d \\ \\ \triangle i_q \end{pmatrix} = [C^{-1}][W] \begin{pmatrix} p\triangle\psi_d \\ \vdots \\ \triangle\delta \end{pmatrix} = [U] \begin{pmatrix} p\triangle\psi_d \\ \vdots \\ \triangle\delta \end{pmatrix} \quad (28)$$

where

$$[C] = \begin{pmatrix} \eta+\alpha R & -\sigma-\beta R \\ \sigma+\beta R & \eta+\alpha R \end{pmatrix}$$

$$[W] = \begin{pmatrix} \frac{\alpha}{\omega_0} & \frac{-\beta}{\omega_0} & -\beta & -\alpha & -\alpha\psi_{q0}-\beta\psi_{d0} & -v\cos\delta \\ \frac{\beta}{\omega_0} & \frac{\alpha}{\omega_0} & \alpha & -\beta & \alpha\psi_{d0}-\beta\psi_{q0} & v\sin\delta \end{pmatrix}$$

$$[U] = [C^{-1}][W] = \begin{pmatrix} U_{11} & U_{12} \cdots\cdots & U_{16} \\ U_{21} & \cdots\cdots & U_{26} \end{pmatrix}$$

By definition, excitation voltage $E = X_{afd0}\dfrac{v_{fd}}{r_{fd}}$, the field circuit equation (6) for disturbances can be written as

$$\triangle i_{fd} = \frac{\triangle v_{fd}}{r_{fd}} - \frac{p\triangle\psi_{fd}}{\omega_0 r_{fd}} = \frac{\triangle E}{X_{afd0}} - \frac{p\triangle\psi_{fd}}{\omega_0 r_{fd}} \quad (29a)$$

Similarly for disturbances about an operating point, (8) and (10) become

$$\triangle i_{kd} = \frac{-p\triangle\psi_{kd}}{\omega_0 r_{kd}} \quad (29b)$$

$$\triangle i_{kq} = \frac{-p\psi_{kq}}{\omega_0 r_{kq}} \quad (29c)$$

Substitute equations (29) and the expressions for $\triangle i_d$ and $\triangle i_q$ from (28) into the disturbances forms (1) to (5) to give equations of the following type

$$p\triangle\psi_{fd} = X_{ffd}\left(\frac{\triangle E}{X_{afd0}} - \frac{p\triangle\psi_{fd}}{\omega_0 r_{fd}}\right) + X_{fkd}\left(\frac{-p\triangle\psi_{kd}}{\omega_0 r_{kd}}\right) - X_{afd}(U_{11}p\triangle\psi_d + U_{12}p\triangle\psi_q + U_{13}\triangle\psi_d + U_{14}\triangle\psi_q + U_{15}n + U_{16}\triangle\delta) \quad (30)$$

In order to include the voltage regulator effect in the general linearized model, it is necessary to obtain the disturbed terminal voltage in terms of the machine flux linkages. From (13), for small disturbance

$$\triangle e_t = \frac{e_{d0}}{e_0}\triangle e_d + \frac{e_{q0}}{e_0}\triangle e_q \tag{31}$$

and (7) and (9) become

$$\triangle e_d = \frac{1}{\omega_0} p\triangle\psi_d - R\triangle i_d - \psi_{q0} n - \triangle\psi_q$$

$$\triangle e_q = \frac{1}{\omega_0} p\triangle\psi_q - R\triangle i_q + \psi_{d0} n + \triangle\psi_d \tag{32}$$

From (28), (31) and (32), it gives

$$\triangle e_t = \frac{1}{e_0}\Big(V_1 p\triangle\psi_d + V_2 p\triangle\psi_q + V_3\triangle\psi_d + V_4\triangle\psi_q + V_5 n + V_6\triangle\delta \Big) \tag{33}$$

where

$$V_1 = e_{d0}\left(\frac{1}{\omega_0} - RU_{11}\right) - e_{q0}RU_{21}$$

$$V_2 = -e_{d0}RU_{12} + e_{q0}\left(\frac{1}{\omega_0} - RU_{22}\right)$$

$$V_3 = -e_{d0}RU_{13} + e_{q0}(1 - RU_{23})$$

$$V_4 = -e_{d0}(1 + RU_{14}) - e_{q0}RU_{24}$$

$$V_5 = -e_{d0}(\psi_{q0} + RU_{15}) + e_{q0}(\psi_{d0} - RU_{25})$$

$$V_6 = -e_{d0}RU_{16} - e_{q0}RU_{26}$$

Equation (33) may be used in conjunction with voltage regulator model to describe the action of voltage regulation on the machine flux linkages. By using the single delay equation

$$\triangle E = \frac{K_r}{1 + T_{rg}p}\triangle e_t \tag{34}$$

of the widely used voltage regulator model, a voltage regulation equation in the reguired set of variables is obtained as following

$$p\triangle E = \frac{K_r}{e_0 T_{rg}}(V_1 p\triangle\psi_d + V_2 p\triangle\psi_q + V_3\triangle\psi_d + V_4\triangle\psi_q + V_5 n + V_6\triangle\delta) - \frac{\triangle E}{T_{rg}} \tag{35}$$

Again, from (11), an expression for the incremental air gap torgue is obtained

$$\triangle T_g = \psi_{d0}\triangle i_q + i_{q0}\triangle\psi_d - \psi_{q0}\triangle i_d - i_{d0}\triangle\psi_q \tag{36}$$

Substituting for $\triangle i_d$ and $\triangle i_q$ from (28) gives

$$\frac{dn}{dt} = \frac{1}{T_m}[\triangle T_t - \psi_{d0}(U_{21}p\triangle\psi_d + U_{22}p\triangle\psi_q + U_{23}\triangle\psi_d + U_{24}\triangle\psi_q + U_{25}n + U_{26}\triangle\delta) + \psi_{q0}(U_{11}p\triangle\psi_d + U_{12}p\triangle\psi_q + U_{13}\triangle\psi_d + U_{14}\triangle\psi_q + U_{15}n + U_{16}\triangle\delta) + i_{d0}\triangle\psi_q - i_{q0}\triangle\psi_d] \tag{37}$$

where T_m is the rotor inertia time constant, and the quantity $\triangle T_t$ must be expressed either by an additional set of differential equations[3] describing the governor action, or be set to $\triangle T_t = -D_t n$, for the case of ungoverned turbine with an inherent self regulation.

Now (25), (35), (37) and the five equations of the form (30) for the rates of change of flux linkages are a set of eight first-order differential equations for the eight variables describing the disturbance performance of the ungoverned synchronous machine.

To summarize these equations and put them into matrix form, they are

$$\begin{bmatrix} \frac{-X_{ffd}}{\omega_0 r_{fd}} & -X_{afd}U_{11} & \frac{-X_{fkd}}{\omega_0 r_{kd}} & -X_{afd}U_{12} & 0 & 0 & 0 & 0 \\ 0 & -XqU_{12} & 0 & -X_qU_{22} & \frac{-X_{akq}}{\omega_0 r_{kq}} & 0 & 0 & 0 \\ \frac{-X_{fkd}}{\omega_0 r_{fd}} & -X_{akd}U_{11} & \frac{-X_{kkd}}{\omega_0 r_{kd}} & -X_{akd}U_{12} & 0 & 0 & 0 & 0 \\ \frac{-X_{afd}}{\omega_0 r_{fd}} & -X_dU_{11} & \frac{-X_{akd}}{\omega_0 r_{kd}} & -X_dU_{12} & 0 & 0 & 0 & 0 \\ 0 & -X_{akq}U_{21} & 0 & -X_{akq}U_{22} & \frac{-X_{kkq}}{\omega_0 r_{kq}} & 0 & 0 & 0 \\ 0 & \frac{-K_rV_1}{T_{rg}e_0} & 0 & \frac{-K_rV_2}{T_{rg}e_0} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \psi_{d0}U_{21}-\psi_{q0}U_{11} & 0 & \psi_{d0}U_{22}-\psi_{q0}U_{12} & 0 & 0 & 0 & T_m \end{bmatrix} \begin{bmatrix} p\triangle\psi_{fd} \\ p\triangle\psi_d \\ p\triangle\psi_{kd} \\ p\triangle\psi_q \\ p\triangle\psi_{kq} \\ p\triangle E \\ p\triangle\delta \\ pn \end{bmatrix}$$

$$= \begin{bmatrix} 1 & X_{afd}U_{13} & 0 & X_{afd}U_{14} & 0 & \frac{-X_{ffd}}{X_{afd0}} & X_{afd}U_{16} & X_{afd}U_{15} \\ 0 & X_qU_{23} & 0 & 1+X_qU_{24} & 0 & 0 & X_qU_{26} & X_qU_{25} \\ 0 & X_{akd}U_{13} & 1 & X_{akd}U_{14} & 0 & \frac{-X_{fkd}}{X_{afd0}} & X_{akd}U_{16} & X_{akd}U_{15} \\ 0 & 1+X_dU_{13} & 0 & X_dU_{14} & 0 & \frac{-X_{afd}}{X_{afd0}} & X_dU_{16} & X_dU_{15} \\ 0 & X_{akq}U_{23} & 0 & X_{akq}U_{24} & 1 & 0 & X_{akq}U_{26} & X_{akd}U_{25} \\ 0 & \frac{K_rV_3}{T_{rg}e_0} & 0 & \frac{K_rV_4}{T_{rg}e_0} & 0 & \frac{-1}{T_{rg}} & \frac{K_rV_6}{T_{rg}e_0} & \frac{K_rV_5}{T_{rg}e_0} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_0 \\ 0 & \begin{matrix}-\psi_{d0}U_{23}+\\ \psi_{q0}U_{13}-i_{q0}\end{matrix} & 0 & \begin{matrix}-\psi_{d0}U_{24}+\\ \psi_{q0}U_{14}+i_{d0}\end{matrix} & 0 & 0 & \begin{matrix}-\psi_{d0}U_{26}\\ +\psi_{q0}U_{16}\end{matrix} & \begin{matrix}-\psi_{d0}U_{25}+\\ \psi_{q0}U_{15}-D_t\end{matrix} \end{bmatrix} \begin{bmatrix} \triangle\psi_{fd} \\ \triangle\psi_d \\ \triangle\psi_{kd} \\ \triangle\psi_q \\ \triangle\psi_{kq} \\ \triangle E \\ \triangle\delta \\ n \end{bmatrix} \tag{38}$$

Equation (38) is of the form

$$[H][\dot{X}] = [F][X]$$

or

$$[\dot{X}] = [H^{-1}][F][X] = [A][X]$$

Thus, the dynamic stability of a synchronous generating plant can be studied by the second method of Liapunov.

In fact, due to the saturation of the armature teeth, both the air gap flux linkage and the leakage reactances are affected so that the total flux linkages of all the direct axis circuits are reduced to a common fraction k_d of their unsaturated values and the flux linkages of all the quadrature axis circuits are reduced to a common fraction k_q of their unsaturated values. But it is permissible, as is done in the following paragraph, to neglect saturation effects on the quadrature axis of a salient pole machine.

All that is needed is to assign the values $k_d X_{afd0}$, $k_d X_{akd0}$, $k_d X_{d0}$, $k_d X_{ffd0}$, $k_d X_{kkd0}$, $k_d X_{fkd0}$ to the parameters X_{afd}, X_{akd}, X_d, X_{ffd}, X_{kkd}, X_{fkd} wherever they appear in the matrices [H] and [F] or [A], with the exception of the denomenator of the terms $(-X_{ffd}/X_{afd0})$, $(-X_{akd}/X_{afd0})$, $(-X_{afd}/X_{afd0})$, which remains as X_{afd0} because E is defined as the air gap line open circuit voltage.

The value of k_d, the saturation factor, can be determined from an open circuit magnetization curve as a function of either ψ_d or the direct axis mutual flux linkage since in this case their is no armature leakage flux, From (2) and (9)

$$e_0 = e_q = \psi_d = k_d X_{afd0} i_{fd} \tag{39}$$

THE STEADY-STATE OPERATING POINT

The steady-state operating point must be established before the matrices [H] and [F] or [A] can be formed. A general formula for the steady-state data for the connection of Fig. 1 can be obtained as follow.

In steady-state:

$$i_{kd} = i_{kq} = 0$$

allowing for saturation,

$$\psi_d = k_d \frac{v_{fd}}{r_{fd}} X_{afd0} - k_d X_{d0} i_d = k_d E - k_d X_{d0} i_d \tag{40}$$

$$\psi_q = -X_q i_q \tag{41}$$

Substituting into (24) with $p\psi_d = 0$, $p\psi_q = 0$, $v_d = v \sin\delta$, $v_q = v \cos\delta$, gives

$$\begin{pmatrix} \eta + \alpha R - \beta k_d X_{d0} & -\sigma - \beta R - \alpha X_q \\ \sigma + \beta R + \alpha k_d X_{d0} & \eta + \alpha R - \beta X_q \end{pmatrix} \begin{pmatrix} i_d \\ i_q \end{pmatrix} = \begin{pmatrix} -\beta k_d E - v \sin\delta \\ \alpha k_d E - v \cos\delta \end{pmatrix} \tag{42}$$

Equations (40) and (42) can be used to solve for i_d, i_q and $k_d\psi_d$ for specified values of E, δ, v; and all other steady-state operating data can then be calculated from (1) to (13).

CONCLUSION

A convenient set of equations have been derived to describe the boundary condition imposed on a sychronous machine by a general tee-form three-phase balanced terminal network. The resulting synchronous machine model has the form required by modern automatic control theory and it is suited to the requirements of digital computer simulation. The modeling of a single synchronous machine as given here can be extended to give canonical form descriptions of interconnections of two or more machines.

APPENDIX

Derivative Expressions Involving Park's Transformations

Park's transformation may be written as

$$K_d=\frac{2}{3}\sum_{m=1}^{3}\left(K_m\cos(\theta-\frac{2\pi m}{3})\right)$$

$$K_q=-\frac{2}{3}\sum_{m=1}^{3}\left(K_m\sin\ (\theta-\frac{2\pi m}{3})\right)$$

where K_d, K_q are direct and quadrature axis quantities; K_1, K_2, K_3, phase quantities. In this appendix $[A\cos\theta\, B]$ and $[A\sin\theta\, B]$ are written as shorthand for

$$\sum_{m=1}^{3}\left(A_m\cos(\theta-\frac{2\pi m}{3})\right)B$$

and $$\sum_{m=1}^{3}\left(A_m\ \sin(\theta-\frac{2\pi m}{3})\right)B$$

Also $[A\cos\theta]$ and $[A\sin\theta]$ are written as shorthand for

$$\sum_{m=1}^{3} A_m\cos(\theta-\frac{2\pi m}{3})\ \text{and}\ \sum_{m=1}^{3} A_m\ \sin(\theta-\frac{2\pi m}{3}\ .\ )$$

First derivatives:

Differentiating the direct axis transformation $K_d=\frac{2}{3}[K_1\cos\theta+K_2\cos(\theta-120)+K_3\cos(\theta+120)]$ gives

$$\frac{dKd}{dt}=-\frac{2}{3}\left(K\sin\theta\frac{d\theta}{dt}\right)+\frac{2}{3}\left(\frac{dK}{dt}\cos\theta\right)$$

which is in full

$$\frac{dKd}{dt}=K_q\omega+\frac{2}{3}\left(\frac{dK_1}{dt}\cos\theta+\frac{dK_2}{dt}\cos\ (\theta-120°)+\frac{dK_3}{dt}\cos\ (\theta+120°)\right) \quad (43)$$

since $\frac{d\theta}{dt}=\omega$ and, from the q axis transformation $-\frac{2}{3}[K\sin\theta\ \omega]=K_q\ \omega$.

The quadrature axis transformation

$$K_q = -\frac{2}{3}[K \sin \theta]$$

is differentiated to give

$$\frac{dK_q}{dt} = -\frac{2}{3}\left(K \cos \theta \frac{d\theta}{dt}\right) - \frac{2}{3}\left(\frac{dK}{dt} \sin \theta\right)$$

$$\frac{dK_q}{dt} = -K_d\omega - \frac{2}{3}\left(\frac{dK}{dt} \sin\theta + \frac{dK_2}{dt}\sin(\theta-120°) + \frac{dK_3}{dt}\sin(\theta+120°)\right) \quad (44)$$

Second derivatives:

The second derivative expressions are obtained simply by differentiating the first derivatives.

Differentiating (43)

$$\frac{d^2Kd}{dt^2} = -\frac{2}{3}\left(K \cos\theta \left(\frac{d\theta}{dt}\right)\right)^2 - \frac{2}{3}\left(K \sin\theta \frac{d^2\theta}{dt^2}\right) - \frac{2}{3}\left(\frac{dK}{dt}\sin\theta\left(\frac{d\theta}{dt}\right)\right)$$

$$+\frac{2}{3}\left(\frac{d^2K}{dt^2}\cos\theta\right) - \frac{2}{3}\left(\frac{dK}{dt}\sin\theta\frac{d\theta}{dt}\right)$$

$$\frac{d^2Kd}{dt^2} = -K_d\omega^2 + K_q\dot{\omega} - \frac{4}{3}\omega\left(\frac{dK}{dt}\sin\theta\right) + \frac{2}{3}\left(\frac{d^2K}{dt^2}\cos\theta\right)$$

Substituting for $\left(\frac{dK}{dt}\sin\theta\right)$ from (44)

$$\frac{d^2Kd}{dt^2} = -K_d\omega^2 + K_q\dot{\omega} + 2\omega\left(\frac{dK_q}{dt} + K_d\omega\right) + \frac{2}{3}\left(\frac{d^2K}{dt^2}\cos\theta\right)$$

finally,

$$\frac{d^2Kd}{dt^2} = K_d\omega^2 + K_q\dot{\omega} + 2\omega\frac{dK_q}{dt} + \frac{2}{3}\left(\frac{d^2K_1}{dt^2}\cos\theta + \frac{d^2K_2}{dt^2}\cos(\theta-120°)\right.$$

$$\left.+\frac{d^2K_3}{dt^2}\cos(\theta+120°)\right) \quad (45)$$

Differentiating (44)

$$\frac{d^2K_q}{dt^2} = -\frac{2}{3}\left(\frac{d^2K}{dt^2}\sin\theta\right) - \frac{2}{3}\left[\frac{dK}{dt}\cos\theta\frac{d\theta}{dt}\right] - \frac{2}{3}\left[\frac{dK}{dt}\cos\theta\frac{d\theta}{dt}\right] + \frac{2}{3}$$

$$\left[K\sin\theta\left(\frac{d\theta}{dt}\right)\right] - \frac{2}{3}\left[K\cos\theta\frac{d^2\theta}{dt^2}\right]$$

$$\frac{d^2K_q}{dt^2} = -K_q\omega^2 - K_d\dot{\omega} - \frac{4}{3}\left[\frac{dK}{dt}\cos\theta\right] - \frac{2}{3}\left[\frac{d^2K}{dt^2}\sin\theta\right]$$

whence, using (43)

$$\frac{d^2K_q}{dt^2} = K_q\omega^2 - K_d\dot{\omega} - 2\omega\frac{dK_d}{dt} - \frac{2}{3}\left[\frac{d^2K_1}{dt^2}\sin\theta + \frac{d^2K_2}{dt^2}\sin(\theta-120°)\right.$$

$$\left.+\frac{d^2K_3}{dt^2}\sin(\theta+120°)\right] \quad (46)$$

Third derivatives:

Differentiating the first expression for $\frac{d^2K_d}{dt^2}$ above gives

$$\frac{d^3K_d}{dt^3} = -2\left[\frac{dK}{dt}\cos\theta(\frac{d\theta}{dt})^2\right] + \frac{2}{3}\left[K\sin\theta(\frac{dt}{d\theta})^3\right] - 2\left[K\cos\theta\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}\right]$$

$$-\frac{2}{3}\left[K\sin\theta\frac{d^3\theta}{dt^3}\right] - 2\left[\frac{dK}{dt}\sin\theta\frac{d^2\theta}{dt^2}\right] - 2\ \frac{d^2K}{dt^2}\sin\theta\ \frac{d\theta}{dt} + \frac{2}{3}\left[\frac{d^2K}{dt^2}\cos\theta\right]$$

$$\frac{d^3K_d}{dt^3} = -3\omega\,\dot{\omega}\,K_d + \ddot{\omega}\,K_q - \omega^3K_q - 2\omega^2\left[\frac{dK}{dt}\cos\theta\right] - 2\,\dot{\omega}\,\left[\frac{dK}{dts}\sin\theta\right]$$

$$-2\omega\left[\frac{d^2K}{dt^2}\sin\theta\right] + \frac{2}{3}\left[\frac{d^3K}{dt^3}\cos\theta\right]$$

Using (43), (44) and (46)

$$\frac{d^3K_d}{dt^3} = -3\omega\,\dot{\omega}\,K_d + \ddot{\omega}\,K_q - \omega^3K_q - 3^2\left[\frac{dK_d}{dt} - K_q\omega\right] + 3\,\dot{\omega}\,\left[\frac{dK_q}{dt} + K_d\omega\right]$$

$$+3\omega\left[\frac{d^2K_q}{dt^2} - K_q\omega^2 + K_d\,\dot{\omega} + 2\omega\frac{dK_d}{dt}\right] + \frac{2}{3}\left[\frac{d^3K}{dt^3}\cos\theta\right]$$

Finally

$$\frac{d^3K_d}{dt^3} = 3\omega\,\dot{\omega}\,K_d + \ddot{\omega}\,K_q - \omega^3K_q + 3\omega^2\frac{dK_d}{dt} + 3\,\dot{\omega}\,\frac{dK_q}{dt} + 3\omega\frac{d^2K_q}{dt^2} + \frac{2}{3}\left[\frac{d^3K_1}{dt^3}\cos\theta\right.$$

$$\left. + \frac{d^3K_2}{dt^3}\cos(\theta - 120^\circ) + \frac{d^3K_3}{dt^3}\cos(\theta + 120^\circ)\right] \tag{47}$$

Similarly differentiating the expression for $\frac{d^2K_q}{dt^2}$ gives

$$\frac{d^3K_q}{dt^3} = 2\left[K\sin\theta\frac{d\theta}{dt}\frac{d^2\theta}{dt^2}\right] + \frac{2}{3}\left[K\cos\theta(\frac{d\theta}{dt})^3\right] - \frac{2}{3}\left[K\cos\theta\frac{d^3\theta}{dt^3}\right]$$

$$+2\left[\frac{dK}{dt}\sin\theta(\frac{d\theta}{dt})\right] - 2\left[\frac{dK}{dt}\cos\theta\frac{d^2\theta}{dt^2}\right] - 2\left[\frac{d^2K}{dt^2}\cos\theta\frac{d\theta}{dt}\right] - \frac{2}{3}\left[\frac{d^3K}{dt^3}\sin\theta\right]$$

Using (43), (44) and (45)

$$\frac{d^3K_q}{dt^3} = -3\omega\,\dot{\omega}\,K_q + K_d\omega^3 - \ddot{\omega}\,K_d + 3\omega^2\left[\frac{dK_q}{dt} - K_d\omega\right] - 3\,\dot{\omega}\,\left[\frac{dK_d}{dt} - K_q\,\omega\right]$$

$$-3\omega\left[\frac{d^2K_d}{dt^2} - K_d\omega^2 + K_q\,\dot{\omega} - 2\omega\frac{dK_q}{dt}\right] - \frac{2}{3}\left[\frac{d^3K}{dt^3}\sin\theta\right]$$

Finally

$$\frac{d^3K_q}{dt^3} = 3\omega\,\dot{\omega}\,K_q + K_d\omega^3 - \ddot{\omega}\,K_d + 3\omega^2\frac{dK_q}{dt} - 3\,\dot{\omega}\,\frac{dK_d}{dt} - 3\omega\frac{d^2K_d}{dt^2}$$

$$-\frac{2}{3}\left[\frac{d^3K_1}{dt}\sin\theta + \frac{d^3K_2}{dt^2}\sin(\theta - 120^\circ) + \frac{d^3K_3}{dt^3}\sin(\theta + 120^\circ)\right] \tag{48}$$

The derivative expressions (43) to (48) are used to evaluate the expressions of the form

$$\frac{d^nK}{dt^n}\cos\theta \quad \text{and} \quad \frac{d^nK}{dt^n}\sin\theta$$

in transforming passive circuit equations onto the direct and quadrature axis reference frame.

NOMENCLATURE

i_1, i_2, i_3	currents in phase 1, 2, 3 of the machine
e_1, e_2, e_3	phase voltages
v	reference bus voltage
e_0	machine terminal voltage
e'	sending bus voltage
e_d, e_q, i_d, i_q	direct and quadrature axis voltages and currents
v_{fd}, i_{fd}	field circuit voltage and current
$E = X_{afd0} V_{fd}/R_{fd}$	air gap line open circuit voltage; excitating voltage
ψ_d, ψ_q	direct and quadrature axis armature flux linkages
ψ_{kd}, ψ_{kd}	direct and quadrature axis amortisseur flux linkages
ψ_{fd}	field flux linkages
$i_{d0}, i_{q0}, e_{d0}, e_{q0}, \psi_{d0}, \psi_{q0}$	operating point steady-state values
$X_{afd}, X_{akd}, X_{akq}, X_{fkd}$	rotor-stator and rotor-rotor coupling reactances on direct and quatrature axes
$X_{ffd}, X_{kkd}, X_{kkq}$	rotor circuit self reactances on direct and quadrature axes
r_{fd}, r_{kd}, r_{kq}	rotor circuit resistances on d and q axes
R	armature resistance per phase
X_{afd0}, X_{akd0}, etc.	unsaturated values of direct axis reactances
k_d, k_q	saturation factors for direct and quadrature axes
ω_0, ω	rated and instantaneous angular frequency in electrical radians per second
$n = (\omega_0 - \omega)/\omega_0$	per unit speed variation
δ	rotor angle in radians
Z_e, Z_t, Y_e	per phase parameters of terminal network
$X_e = \omega L_e, X_t = \omega L_t, B_e = \omega C_e$	reactance components of Z_e, Z_t, Y_e for steady sinusoidal operation
R_e, R_t, G_e	resistance components of Z_e, Z_t, Y_e
T_t, T_g	prime mover and generator air gap torque
Q	reactive generation at machine terminals
T_m	rotor inertia time constant in seconds
K_r	voltage regulator gain
T_{rg}	voltage regulator time constant in seconds
$\triangle$	incremental operator
p	time differential operator

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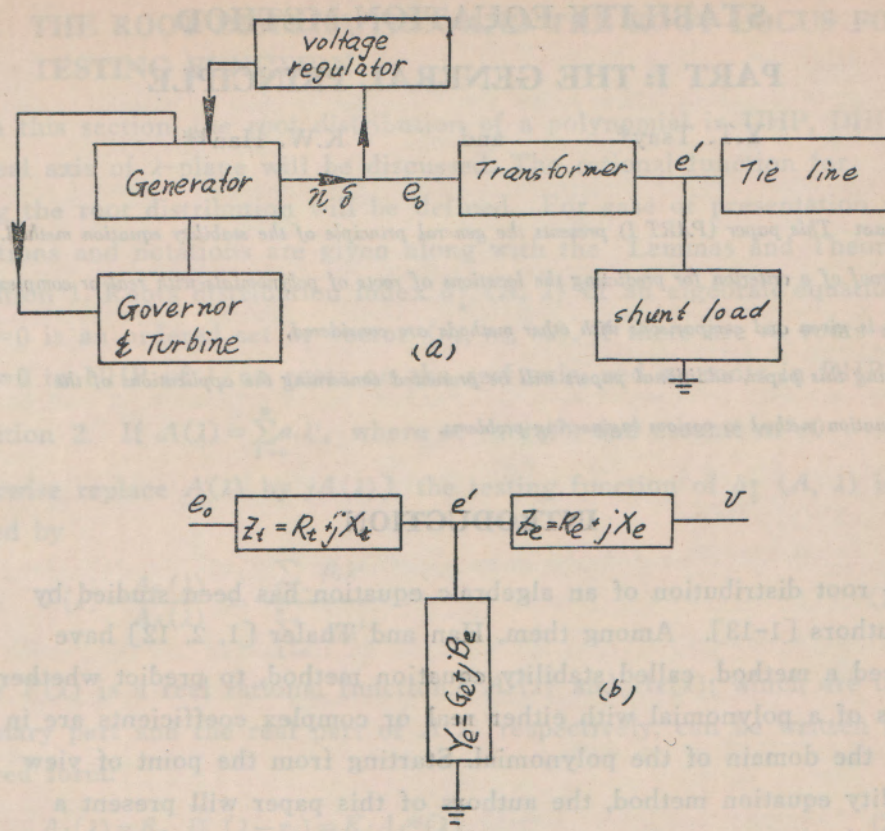

Fig. 1 (a) Block diagram for one machine connected to infinite bus.

(b) Per. phase equivalent circuit for trasmission systemas shown in (a)