ANALYSIS OF NONLINEAR SAMPLED-DATA SYSTEMS USING Z-TRANSFORM METHOD

by

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SUMMARY: A method of testing stability of sampled-data systems with a relay-type nonlinearity is presented. In the z-domain, a linear sampled-data system which is equivalent to a nonlinear system under steady state condition can be found, and the existence as well as the stability characteristics of the limit cycle can be defined.

I. INTRODUCTION

A method of using z-transform theory to analyze nonlinear control systems under steady state condition has been proposed by Mcvey and Nurre.[1] It has been proved that a continuous system with a relay-type nonlinearity can be converted to a linear sampled-data system with a zero order holding circuit. This paper extends the aforementioned method to the analysis of nonlinear sampled-data systems. The basic approach is to regard a sampled-data system (under steady state condition) with a relay-type nonlinearity as a linear sampled-data system with multiple samplers and multiple sampling rates, and the condition for the nonlinear sampled-data system fo have a limit cycle is that a steady state oscillation in the linear sampled-data system exists.

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II. CONTINUOUS SYSTEM WITH AN IDEAL RELAY

This section gives a brief review of the Mcvey and Nurre's method.

Consider the system in Fig. 1, if the output C(t) is a steady state oscillation, then the output of the relay is a square wave with the same frequency as C(t). Let T be the interval between two successive switching instants and the angular frequency of C(t) is w, then T and w are related as $wT = \pi$.

From Fig. 1, the Laplace transform of the output of relay, under steady state oscillation is

$$W(s) = \frac{M(1-e^{-sT})}{s} \sum_{n=0}^{\infty} (-1)^n e^{-nsT} \tag{1}$$

which represents a zero order holding circuit and a sampler with sampling frequency w and sampling period T.

Eq. (1) can be rewritten as

$$W(s) = \frac{1-e^{-sT}}{s} \left[K_T E^*(s) \right] \tag{2}$$

where K_T represents an infinite gain for making the magnitude of the sampled signal equal to M;[1] thus the system can be replaced by an equivalent one as in Fig. 2, and for which the characteristic equation is

$$1+K_T G_{h_0} G(z)=0 \tag{3}$$

Since K_T is equal to infinity, the zeros of the characteristic equation coincide with that of

$$G_{h_0} G(z)=0 \tag{4}$$

which can be written as

$$G_{h_0} G(z)= \frac{[z+a_1(T)][z+a_2(T)]\cdots\cdots}{[z+b_1(T)][z+b_2(T)]\cdots\cdots} \tag{5}$$

If there is a value of $T=T_e$ to make

$$a_i(T_e)=1 \tag{6}$$

then a limit cycle with a half-period T_e exists, because Eq. (6) gives a characteristic root at $z=-1$, which is the condition for the system in Fig.2 to have a steady state oscillation.[1]

In Eq. (3), if an increase in T from T_e moves the zero (originally at $z=-1$) into the unit circle in the z-plane, then the limit cycle is stable; otherwise it is unstable. Thus the stability characteristics of a limit cycle depend upon the derivative of the zero-location (originally at -1) with respect to T. The limit cycle is stable if the derivative is positive, and unstable if negative.[1]

III. NONLINEAR SAMPLED-DATA SYSTEMS

In this section, a nonlinear sampled-data system with an ideal relay is considered. It is assumed that the system is under steady state oscillation, and the sampling frequency w_s is n-times that of the limit cycle.

Consider the system in Fig.3, after the relay is replaced by a sampler and a zero-order holding circuit, the equivalent system is given in Fig.4, where the sampling instants are assumed synchronized with the switching instants of the relay, except that the sampler is n-times faster than the relay. The output of the system is[2]

$$C(s)=F^*(s)K_T G_{h_0}(s)G_2(s) \tag{7}$$

where

$$F(s)=[R(s)-C(s)]^*G_hG_1(s)+\sum_{p=1}^{n-1}\left\{[R(s)-C(s)]e^{\frac{psT}{n}}\right\}^*e^{-\frac{psT}{n}}G_hG_1(s) \tag{8}$$

$$F^*(s)=\frac{R^*(s)G_hG_1^*(s)+\sum_{p=1}^{n-1}\left\{[R(s)e^{\frac{psT}{n}}]^*[e^{-\frac{psT}{n}}G_h(s)G_1(s)]^*\right\}}{1+K_T\left\{G_{h_0}G_2^*(s)G_hG_1^*(s)+\sum_{p=0}^{n-1}[G_{h_0}G_2(s)e^{\frac{psT}{n}}]^*[G_hG_1(s)e^{-\frac{psT}{n}}]^*\right\}} \tag{9}$$

thus

$$F(z)=\frac{R(z)G_hG_1(z)+z\sum_{p=1}^{n-1}R(z,\frac{p}{n})G_hG_1(z,\frac{n-p}{n})}{1+K_T\left[G_hG_1(z)G_{h_0}G_2(z)+z\sum_{p=1}^{n-1}G_{h_0}G_2(z,\frac{p}{n})G_hG_1(z,\frac{n-p}{n})\right]} \tag{10}$$

hence the characteristic equation is

$$1+K_T H(z)=0 \tag{11}$$

where

$$H(z)=G_h G_1(z) G_{h0} G_2(z)+z \sum_{p=1}^{n-1} G_{h0} G_2(z, \frac{p}{n}) G_h G_1(z, \frac{n-p}{n})$$

$$=\frac{K[z+z_1(n,T)][z+z_2(n,T)]\cdots\cdots}{[z+b_1(n,T)][z+b_2(n,T)]\cdots\cdots}$$

$$=P(z)/Q(z) \tag{12}$$

If there is some $n=N$ and $T=T_e$ such that

$$a_1(N,T_e)=1 \tag{13}$$

then a limit cycle with half-period T_e exists.

In order to test stability of the limit cycle, the sign of

$$\frac{d}{dt}\left[a_1(T)\right]_{T=T_e} \tag{14}$$

is required.[1] Since the work of factoring is often very complex, an alternating method is proposed as follows:

From Eqs. (12) and (13), since

$$P(z=-1,\ T=T_e)=0 \tag{15}$$

a Taylor's series expansion gives

$$P(z+\triangle z,\ T+\triangle T)$$

$$=\left.\frac{\partial P}{\partial z}\right|_{z=-1,\ T=T_e}\triangle z+\left.\frac{\partial P}{\partial T}\right|_{z=-1,\ T=T_e}\triangle T \tag{16}$$

It can be seen the condition for having a stable limit cycle is that the signs of the terms in Eq. (16) are different. For example, if the sign of the first term is positive while that of the second term is negative, then an increase of T would cause the zero at $z=-1$ to move into the unit circle, and vice versa; thus the equilibrium point at T_e is stable; otherwise it is unstable.

Based upon the above analysis, a procedure for finding the limit cycle and testing its stability is given below:

1) Find the characteristic equation of the equivalent system as Eq.(11).

2) Let $H(z) = P(z)/Q(z)$, and determine whether there is a relation among the parameters such that a zero of $H(z)$ lies at $z=-1$.

3) Find the signs of the terms in Eq. (16) and define the stability characteristics of the limit cycle.

Example: Consider the system in Figs.3 and 4, if $G_1(s)=1$, $G_2(s) = K/s(s+a)$ and $G_h=G_{h0}=(1-e^{-sT})/s$, then following the suggested procedure

1) $1+K_T H(z)=0$

where

$$H(z)=G_{h0}G_1(z)G_{h0}G_2(z)+z^{n-1}\sum_{p=1} G_{h0}G_1(z,\ p/n)G_{h0}G_1(z,\frac{n-p}{n})$$

$$=\frac{K}{a^2 z(z-1)(z-e^{-\alpha})}[(A+B)z^2+(C-2A)z+(A-D)]$$

with

$$A=\frac{e^{-\alpha/n}-e^{-\alpha}}{1-e^{-\alpha/n}},\quad B=(n-1)(\frac{\alpha}{2}-1)+e^{-\alpha}+\alpha-1,$$

$$C=(n-1)\left[(1+e^{-\alpha})+\frac{\alpha}{2}(1-e^{-\alpha})\right]+1-\alpha e^{-\alpha}-e^{-\alpha},$$

$$D=(n-1)e^{-\alpha}(1+\frac{\alpha}{2})$$

and $\alpha=aT$

2) $P(z)=(A+B)z^2+(C-2A)z+(A-D)$

$f(n,\ \alpha)=P(z=-1)=4A+B-C-D=0$

Using a digital computer, a family of curves in the $f(n,\ \alpha)$ vs α plane with n as a parameter can be plotted as in Fig.5 and from which the relation between n and α for having a limit cycle (i.e. $f(n,\ \alpha)=0$) can be represented by a curve as in Fig.6

3) Let

$$P_{z0}=\frac{\partial P}{\partial z}\Big|_{z=-1,\ T=T_e} \quad \text{and} \quad P_{\alpha 0}=\frac{\partial P}{\partial \alpha}\Big|_{z=-1,\ T=T_e}$$

for various values of n and α, which satisfy the conditions in 2), the following results are obtained.

n	α	P_{z0}	$P_{\alpha 0}$
1	0.01	−.000	+.000
2	2.906	−2.548	+.428
4	7.212	−14.04	+.765
6	11.28	−33.48	+.858

Thus the system has a stable limit cycle (for the considered values of n and α). Assume a=1.8 and the sampling period is $T_s=1$, then two curves, one for nT_s vs n and one for T vs n, can be drawn as in Fig. 7, where the intersection (Q) indicates that the limit cycle has a half-period equal to 4.

Note that the results in 2) and 3) are given in terms of parameters instead of numerical values, because for some values of a and T_s the value of n may not be an integer. In such a case, the proposed method can only give an approximate answer about the existence of a limit cycle.

CONCLUSION

Using the z-transform method, the limit cycle of a sampled-data system with a relay-type nonlinearity can be found, and its stability characteristics can be defined. Although only the case of ideal relay has been considered, the proposed method can be applied to sampled-data systems with other kinds of relay-type nonlinearities.[1]

REFERENCES

1. E. S. Mcvey and G. S. Nurre, "The Application of Z-transform Theory to the Analysis of Switched-type Nonlinear Systems", IEEE Trans. Applications and Industry, Nov. 1964.
2. J. T. Tou, "Digital and Sampled-data Control Systems", (book), McGraw-Hill Book Company, New York, 1958.
3. M. A. Pai, "Oscillations in Nonlinear Sampled-data Systems," IEEE Trans. Applications and Industry, Jan, 1963.
4. H. C. Torng, "Self-sustained Oscillations in Relay Sampled-data Systems," ibid.

LIST OF SYMBOLS

a_i	zero of transfer function
b_i	pole of transfer function
C	output quantity
G_1	transfer function
G_h	transfer function of holding circuit
K	gain constant

M output of relay
n, p integers
Q limit cycle
R input quantity
s Laplace operator
T time between successive relay operations
T_s sampling period
α parameter

LIST OF CAPTIONS

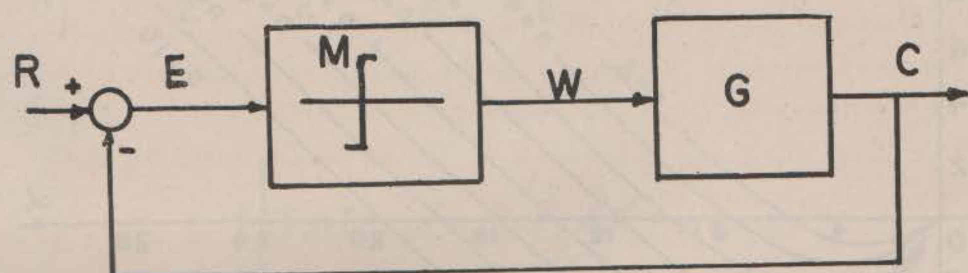

Fig. 1

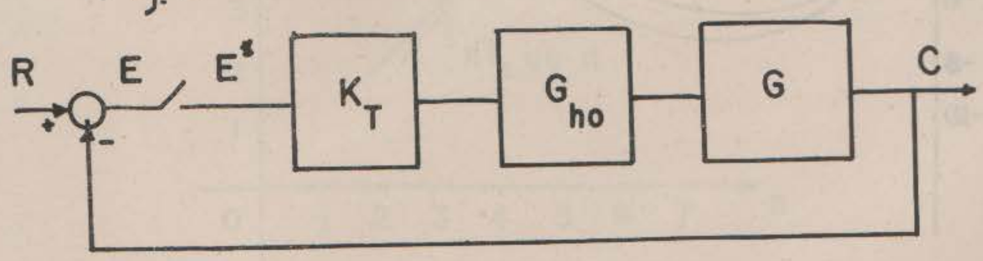

Fig. 2

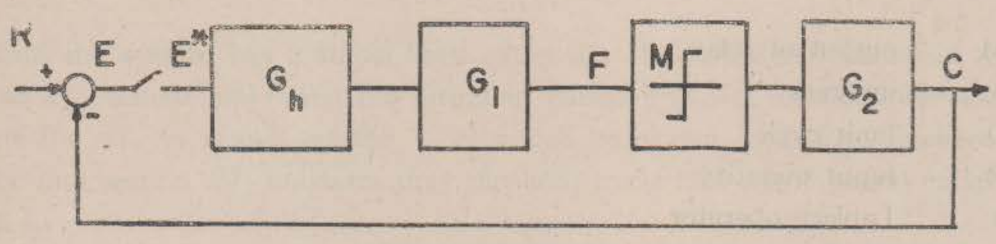

Fig. 3

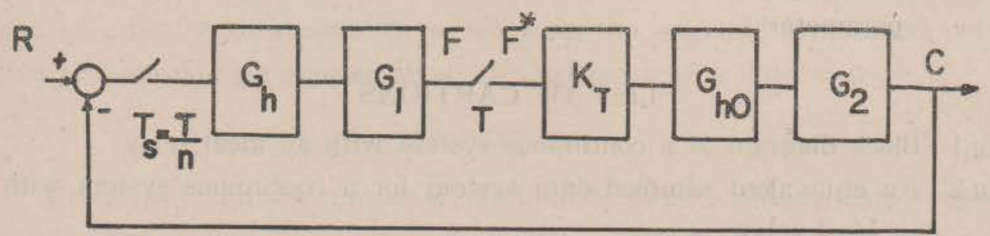

Fig. 4

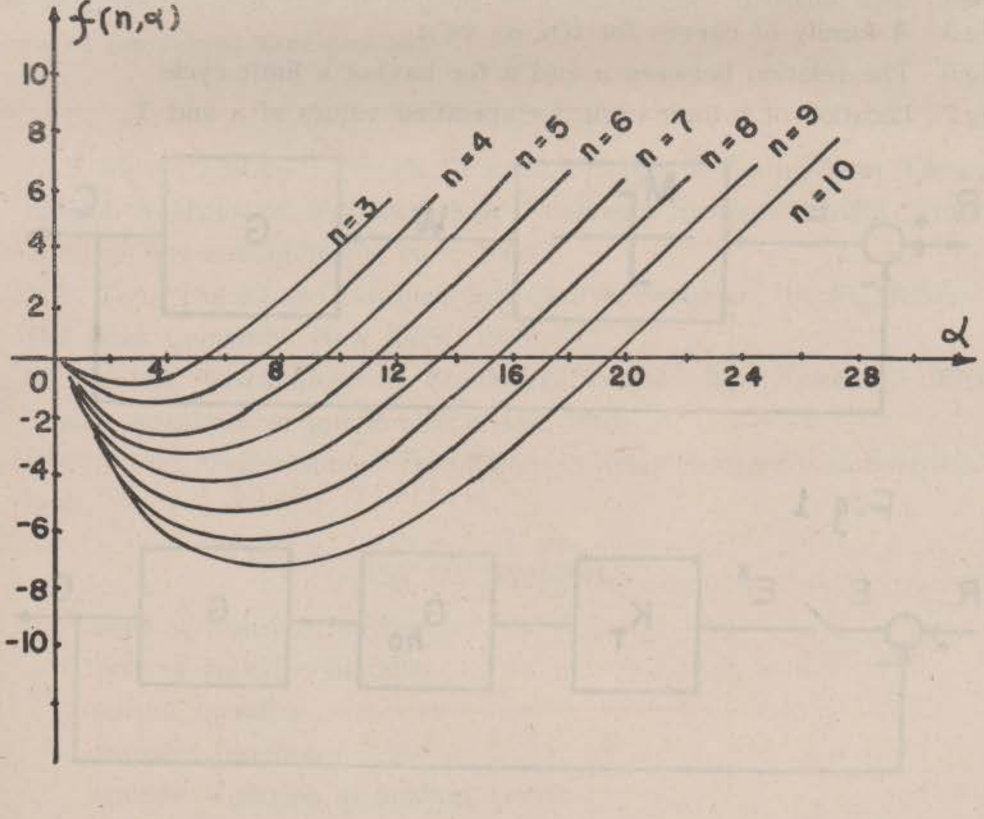

Fig. 5

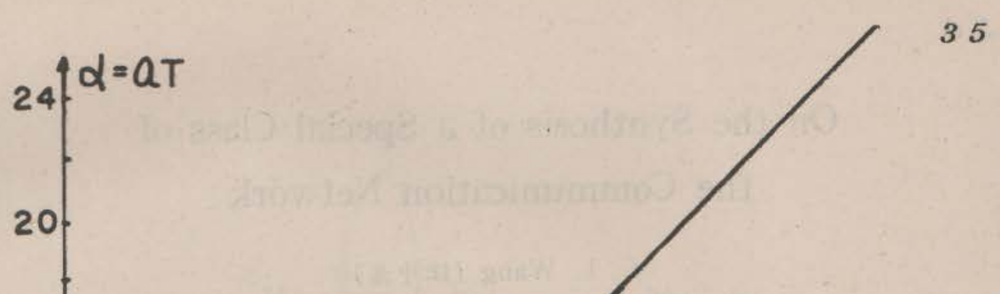
α = aT
24
20
16
12
8
4
0
2
4
6
8
10
12
n

Fig. 6

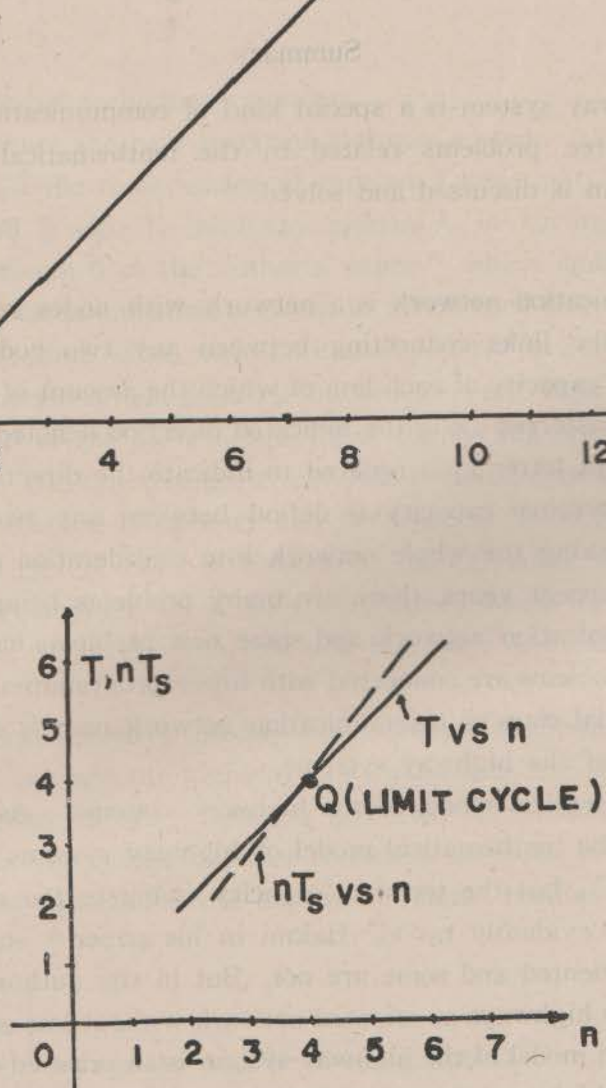
T, nTs
6
5
4
3
2
1
0
1
2
3
4
5
6
7
n
T vs n
Q (LIMIT CYCLE)
nTs vs n

Fig. 7