OPERATIONAL SOLUTION OF INHOMOGENEOUS WAVE EQUATION

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ABSTRACT

The Laplace transformation, the three-dimensional Fourier transformation, and the convolution theorem are used to solve the inhomogeneous scalar wave equation in unbounded space. The solutions obtained consist of the retarded potential and a function which is expressed as a superposition of the plane wave solutions. The retarded potential solution contains a unit step function which makes the physical explanation of the result more obvious. Some comments on the four-dimensional Fourier transformation method of approach are also included.

1. Introduction.

An important differential equation governing fields and their sources in electromagnetic theory and other branches of mathematical physics is the inhomogeneous scalar wave equation. It has been solved by various approaches in the past. To solve it Green's theorem and limiting process[1,2] were used as well as Green's theorem and Green's function[3], Fourier transforms and Green's function [4,5] were introduced, or the four-dimensional Fourier transformation [6] was applied. In section 2 another approach which makes use of the Laplace transformation, the three-dimensional Fourier transformation, and the convolution theorem will be used to solve the same equation. Some consideration on the four-dimensional Fourier transformation approach will be described in section 3.

2. Solution by Laplace and Fourier transformation

The inhomogeneous wave equation to be solved is a scalar equation of general form:

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\,\phi(\vec{x},t) = -\,g(\vec{x},t), \tag{1}$$

in which ϕ is a scalar potential to be solved and g is a known source function. Both ψ and g are scalar functions of time t, a real scalar variable, and position $\vec{x}$, a vector variable whose rectangular coordinates x_1, x_2, x_3 are real and whose direction is specified by three corresponding mutual perpendicular unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3. \tag{2}$$

In Eq.(1) c is a positive real known constant and ∇^2 denotes the Laplacian operator,

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \tag{3}$$

Assume the functions ψ and g are such that they can be analyzed by the Laplace and Fourier transformation. let

$$L_{t\to s}[\psi(\vec{x},t)] = \int_0^\infty \psi(\vec{x},t)e^{-st}\,dt = \overline{\psi}(\vec{x},s) \tag{4}$$

$$L_{t\to s}[g(\vec{x},t)] = \int_0^\infty g(\vec{x},t)e^{-st}\,dt = \overline{g}(\vec{x},s) \tag{5}$$

be the Laplace transforms of ψ and g, and

$$L^{-1}_{s\to t}[\overline{\psi}(\vec{x},s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \overline{\psi}(\vec{x},s)e^{st}\,ds = \psi(\vec{x},t) \tag{6}$$

$$L^{-1}_{s\to t}[\overline{g}(\vec{x},s)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \overline{g}(\vec{x},s)e^{st}\,ds = g(\vec{x},t) \tag{7}$$

be their inverse Laplace transforms, in which s a complex variable whose real part must be positive, γ a suitable positive real constant, and $i=\sqrt{-1}$. By Laplace transformation, Eq.(1) becomes

$$\left(\nabla^2 - \frac{s^2}{c^2}\right)\overline{\psi}(\vec{x},s) + \frac{1}{c^2}\left[s\psi(\vec{x},0) + \frac{\partial}{\partial t}\psi(\vec{x},0) \right] = -\overline{g}(\vec{x},s), \tag{8}$$

with initial value of ψ and $\frac{\partial\psi}{\partial t}$ at time t=0 being accompanied. To solve Eq.(8) one introduces the three dimensional Fourier transforms:

$$F_{\vec{x}\to\vec{k}}[\overline{\psi}(\vec{x},s)] = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int \overline{\psi}(\vec{x},s)e^{-i\vec{k}\cdot\vec{x}}d^3x = \Psi(\vec{k},s) \tag{9}$$

$$F_{\vec{x}\to\vec{k}}[\overline{g}(\vec{x},s)] = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int \overline{g}(\vec{x},s)e^{-i\vec{k}\cdot\vec{x}}d^3x = G(\vec{k},s) \tag{10}$$

$$F_{\vec{x}\to\vec{k}}[\phi(\vec{x},0)] = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int \phi(\vec{x},0)e^{-i\vec{k}\cdot\vec{x}}d^3x = A(\vec{k}) \tag{11}$$

$$F_{\vec{x}\to\vec{k}}\left[\frac{\partial}{\partial t}\phi(\vec{x},0)\right] = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int \frac{\partial}{\partial t}\phi(\vec{x},0)e^{-i\vec{k}\cdot\vec{x}}d^3x = B(\vec{k}) \tag{12}$$

The corresponding inverse transforms of $\Psi(\vec{k},s)$ and $G(\vec{k},s)$ are

$$F^{-1}_{\vec{k}\to\vec{x}}[\Psi(\vec{k},s)] = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int \Psi(\vec{k},s)\ e^{i\vec{k}\cdot\vec{x}}d^3k = \bar{\psi}(\vec{x},s) \tag{13}$$

and

$$F^{-1}_{\vec{k}\to\vec{x}}[G(\vec{k},s)] = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int G(\vec{k},s)\ e^{i\vec{k}\cdot\vec{x}}d^3k = \bar{g}(\vec{x},s) \tag{14}$$

respectively, where

$$\vec{k} = k_1\vec{e}_1 + k_2\vec{e}_2 + k_3\vec{e}_3 \tag{15}$$

is another vector variable introduced, having real rectangular components k_1, k_2, k_3, and

$$\vec{k}\cdot\vec{x} = k_1x_1 + k_2x_2 + k_3x_3 \tag{16}$$

denotes the inner product of $\vec{k}$ and $\vec{x}$. Throughout this paper d^3x and d^3k will denote the volume element in $\vec{x}$-space and $\vec{k}$-space respectively,

$$d^3x = dx_1dx_2dx_3 \qquad d^3k = dk_1dk_2dk_3 \tag{17}$$

and the region of each integration will extend to all its corresponding space, ie., all $\vec{x}$-space or all $\vec{k}$-space. Applying Fourier transformation to Eq.(8) and solving for $\Psi(\vec{k},s)$, one may find that

$$\Psi(\vec{k},s) = \frac{sA(\vec{k}) + B(\vec{k})}{s^2 + c^2k^2} + \frac{G(\vec{k},s)}{k^2 + \frac{s^2}{c^2}}\ , \tag{18}$$

where

$$k^2 = k_1^2 + k_2^2 + k_3^2 \tag{19}$$

The solution for scalar function $\phi(\vec{x},t)$ is obtained after applying the inverse Laplace transformation and the inverse Fourier transformation. Assuming the order of operation $L^{-1}_{s\to t}$ and $F^{-1}_{\vec{k}\to\vec{x}}$ is reversible and applying them to the first term on the right of Eq.(18), one has

$$\phi_c(\vec{x},t) = F^{-1}_{\vec{k}\to\vec{x}}\ L^{-1}_{s\to t}\left[\frac{sA(\vec{k}) + B(\vec{k})}{s^2 + c^2k^2}\right] \tag{20}$$

The inverse Laplace transform of $\frac{sA(\vec{k})+B(\vec{k})}{s^2+c^2k^2}$ will be

$$L^{-1}_{s\to t}\left[\frac{sA(\vec{k})+B(\vec{k})}{s^2+c^2k^2}\right]=\left[c(\vec{k})e^{ickt}+D(\vec{k})e^{-ickt}\right]u(t), \tag{21}$$

where $u(t)$ is a unit step function:

$$U(t)=\begin{cases}0\ , & t<0\\ 1\ , & t>0\end{cases} \tag{22}$$

$$C(\vec{k})=\frac{A(\vec{k})+\frac{B(\vec{k})}{ick}}{2} \tag{23}$$

and

$$D(\vec{k})=\frac{A(\vec{k})-\frac{B(\vec{k})}{ick}}{2} \tag{24}$$

The operation of the inverse Fourier transformation leads Eq.(21) to the result:

$$\psi_c(\vec{x},t)=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int\left[C(\vec{k})e^{i(\vec{k}\cdot\vec{x}+ckt)}+D(\vec{k})e^{i(\vec{k}\cdot\vec{x}-ckt)}\right]U(t)d^3k \tag{25}$$

Next applying $L^{-1}_{s\to t}$ and $F^{-1}_{\vec{k}\to\vec{x}}$ to the second term of Eq.(18), one has

$$\psi_p(\vec{x},t)=L^{-1}_{s\to t}\ F^{-1}_{\vec{k}\to\vec{x}}\left[\frac{G(\vec{k},s)}{k^2+\frac{s^2}{c^2}}\right]. \tag{26}$$

By the convolution theorem

$$F^{-1}_{\vec{k}\to\vec{x}}\left\{F_{\vec{x}\to\vec{k}}[g(\vec{x})]\cdot F_{\vec{x}\to\vec{k}}[f(\vec{x})]\right\}=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int g(\vec{x}')f(\vec{x}-\vec{x}')d^3x' \tag{27}$$

and the result

$$F^{-1}_{\vec{k}\to\vec{x}}\left[\frac{1}{k^2+\frac{s^2}{c^2}}\right]=\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\frac{e^{-\frac{s}{c}|\vec{x}|}}{|\vec{x}|} \tag{28}$$

the inverse Fourier transform of $\frac{G(\vec{k},s)}{k^2+\frac{s^2}{c^2}}$ may be written as

$$F^{-1}_{\vec{k}\to\vec{x}}\left[\frac{G(\vec{k},s)}{k^2+\frac{s^2}{c^2}}\right]=\frac{1}{4\pi}\int\bar{g}(\vec{x}',s)\,\frac{e^{-\frac{s}{c}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x' \tag{29}$$

Equation (28) is obtained by first integrating over angles and then use Cauchy's residue theorem. By means of applying the inverse Laplace transformation to Eq.(29), changing the order of integration, and comparing with Eq.(7) it is found that

$$\phi_p(\vec{x},t)=\frac{1}{4\pi}\int\frac{g\left(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}\,u\left(t-\frac{|\vec{x}-\vec{x}'|}{c}\right)d^3x' \tag{30}$$

in which u is a unit step function defined by Eq.(22).

3. Solution by Fourier transformation.

In this section, the four-dimensional Fourier transformation method will be considered and a mathematical difficulty will be pointed out. Several attempts for saving the situation formally will also be stated.

To solve Eq.(1) the required Fourier transforms and their inverse transforms are introduced:

$$F_{t\to\omega}[\phi(\vec{x},t)]=\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\phi(\vec{x},t)e^{-i\omega t}dt=\bar{\phi}(\vec{x},\omega), \tag{31}$$

$$F_{t\to\omega}[g(\vec{x},t)]=\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}g(\vec{x},t)e^{-i\omega t}dt=\bar{g}(\vec{x},\omega), \tag{32}$$

$$F^{-1}_{\omega\to t}[\bar{\phi}(\vec{x},\omega)]=\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\bar{\phi}(\vec{x},\omega)e^{i\omega t}d\omega=\phi(\vec{x},t), \tag{33}$$

$$F^{-1}_{\omega\to t}[\bar{g}(\vec{x},\omega)]=\left(\frac{1}{2\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\bar{g}(\vec{x},\omega)e^{i\omega t}d\omega=g(\vec{x},t), \tag{34}$$

$$F_{\vec{x}\to\vec{k}}[\Psi(\vec{x},\omega)]=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int\Psi(\vec{x},\omega)e^{-i\vec{k}\cdot\vec{x}}d^3x=\Psi(\vec{k},\omega), \tag{35}$$

$$F_{\vec{x}\to\vec{k}}[\bar{g}(\vec{x},\omega)]=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int\bar{g}(\vec{x},\omega)e^{-i\vec{k}\cdot\vec{x}}d^3x=G(\vec{k},\omega), \tag{36}$$

$$F^{-1}_{\vec{k}\to\vec{x}}[\Psi(\vec{k},\omega)]=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int\Psi(\vec{k},\omega)e^{i\vec{k}\cdot\vec{x}}d^3k=\bar{\phi}(\vec{x},\omega), \tag{37}$$

$$F^{-1}_{\vec{k}\to\vec{x}}[G(\vec{k},\omega)]=\left(\frac{1}{2\pi}\right)^{\frac{3}{2}}\int G(\vec{k},\omega)e^{i\vec{k}\cdot\vec{x}}d^3k=\bar{g}(\vec{x},\omega), \tag{38}$$

where ω is a real variable and the vector $\vec{k}$ has the same meaning as before. The Fourier transformation defined by Eqs. (31), (32), (35), and (36) bring Eq.(1) to the result:

$$\Psi(\vec{k},\omega) = \frac{G(\vec{k},\omega)}{k^2 - \frac{\omega^2}{c^2}}, \tag{39}$$

if

$$k^2 - \frac{\omega^2}{c^2} \neq 0 \ , \tag{40}$$

where k^2 has the previous expression, section 2 Eq.(19). To accomplish the inverse transformation, the integral

$$F^{-1}_{\vec{k}\to\vec{x}}\left[\frac{1}{k^2 - \frac{\omega^2}{c^2}}\right] = \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \int \frac{e^{i\vec{k}\cdot\vec{x}}}{k^2 - \frac{\omega^2}{c^2}} d^3k \tag{41}$$

is considered first. This is a divergent improper integral in the ordinary sense so that the four-dimensional Fourier transformation approach is proved in vain from the mathematical point of view. Several attempts are given below, trying to solve the difficulty formally. One defines [6] the value of the integral as the limit of the function given in Eq.(28) as s approaching $i\omega$ or $-i\omega$, or one takes the Cauchy principal value as the sum of the integral by integrating first over angles and then indenting the contour around the poles at $\pm\frac{\omega}{c}$. It will be seen that the different definitions lead the integral to the different result and give another mathematical trouble, the multiplicity of Fourier transformation:

$$F^{-1}_{\vec{k}\to\vec{x}}\left[\frac{1}{k^2 - \frac{\omega^2}{c^2}}\right] = \begin{cases} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{e^{-i\frac{\omega}{c}|\vec{x}|}}{|\vec{x}|} \\ \text{or } \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{e^{i\frac{\omega}{c}|\vec{x}|}}{|\vec{x}|} \\ \text{or } \frac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{e^{-i\frac{\omega}{c}|\vec{x}|} + e^{i\frac{\omega}{c}|\vec{x}|}}{|\vec{x}|} \end{cases} \tag{42}$$

Carrying on formally by applying the convolution theorem, Eq.(27), and Eq.(42), one yields

$$F^{-1}_{\vec{k}\to\vec{x}}\left[\frac{G(\vec{k},\omega)}{k^2-\frac{\omega^2}{c^2}}\right]=\begin{cases}\frac{1}{4\pi}\int\overline{g}(\vec{x}',\omega)\,\frac{e^{-i\frac{\omega}{c}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x' \\ \text{or }\frac{1}{4\pi}\int\overline{g}(\vec{x}',\omega)\,\frac{e^{i\frac{\omega}{c}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}\,d^3x' \\ \text{or }\frac{1}{2}\left[\frac{1}{4\pi}\int\overline{g}(\vec{x}',\omega)\frac{e^{-i\frac{\omega}{c}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}d^3x'+\frac{1}{4\pi}\int\overline{g}(\vec{x}',\omega)\frac{e^{i\frac{\omega}{c}|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}d^3x'\right]\end{cases}\quad(43)$$

The solution $\psi(\vec{x},t)$ is obtained by applying the inverse Fourier transformation $F^{-1}_{\omega\to t}$ to Eq.(43) and reverting the order of integration. Comparing the result with Eq.(34) one may verify that

$$\psi(\vec{x},t)=\begin{cases}\frac{1}{4\pi}\int\frac{g\left(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}\,d^3x' \\ \text{or }\frac{1}{4\pi}\int\frac{g\left(\vec{x}',t+\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}\,d^3x' \\ \text{or }\frac{1}{2}\left[\frac{1}{4\pi}\int\frac{g\left(\vec{x}',t-\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}d^3x'+\frac{1}{4\pi}\int\frac{g\left(\vec{x}',t+\frac{|\vec{x}-\vec{x}'|}{c}\right)}{|\vec{x}-\vec{x}'|}d^3x'\right]\end{cases}$$

4. Discussion

In this paper the functions ψ and g are assumed to have such properties that they can be analyzed by Laplace and Fourier transformation, ie, both ψ and g are assumed to tend to zero as $|\vec{x}|\to\infty$ and $|t|\to\infty$. The conditions will be satisfied if all sources are located within a finite space and if they have been established within some finite period in the past. It is also assumed that the changes of the order of integration are allowable. These require the functions ψ and g to possess certain continuous properties. All justifications are omitted since these solutions having been derived are well-known.

Owing to the very nature of the three-dimensional Fourier transformation, the pure operational approach gives solutions in unbounded space only. The four-dimensional Fourier transformation method described in section 3 has been found to encounter a mathematical difficulty, a direct consequent of the divergent improper integral Eq.(41), and which is in turn due to the restriction among the four variables k_1, k_2, k_3, and ω as given in Eq.(40). Several attempts, trying to give the formal solutions, are treated in section 3 and the results, containing the retarded and advanced potential solutions, are found in Eq.(44). It should be noted that the results, Eq.(44), happen to consist with an other approach of solution[1,3,5]. This is due to the proper definitions having been given to the divergent improper integral Eq.(41), another definition will undoubtedly give a meaningless result, a consequence of the multiplicity of Fourier transformation. Therefore the four-dimensional Fourier transformation method is neither satisfactory nor applicable in solving the inhomogeneous wave equation. It suggests, however, from the origin of the mathematical trouble Eqs.(40) and Eq.(41), that the Laplace transformation seems much better than the Fourier transformation. The modified method, the Laplace and Fourier transformation approach described in section 2 which is the main body of this paper, gives rigorously the solutions of the inhomogeneous wave equation and satisfactorily overcomes the previous shortcoming. The particular integral solution obtained in section 2 involves the retarded potential solution Eq.(30) only. This is a nature consequent of the Laplace transformation---a mathematical equivalent of the physical notion of causality. The unit step function appearing in Eq.(30) indicates that the effect observed at the point $\vec{x}$ at time t is due to these sources which originated at an earlier time $t'=t-\frac{|\vec{x}-\vec{x}'|}{c}$ and at point $\vec{x}'$ being located within a sphere of radius ct with its center at the observation point $\vec{x}$, ie., $|\vec{x}-\vec{x}'| < ct$. Although the region of integration in the final results, Eqs.(30) and(44), is expressed explicitly extending to all the three-dimensional space, it is understood that the actual region required will be the entire volume, finite in space, occupied by the sources. A solution of homogeneous wave equation, which has been interpreted as a superposition of the plane wave solutions, is also derived in section 2 Eq.(25)

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