ON A GENERAL OPTIMIZATION
PRINCIPLE FOR DISCRETE SYSTEMS

by

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SUMMARY

A general theorem about a function of a matrix vector is presented and proved. It is shown that this theorem forms the basis of a general optimization principle for certain performance index in a discrete system. With this principle, it is possible to determine both the optimum (maximum or minimum) value of the performance index and the parameters of the system in order to achieve the optimum condition. An example is given which illustrates the optimization procedure as well as demonstrates the versatility of the principle.

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Introduction

A problem of considerable importance in electrical engineering is that of determining a set of system parameters which will maximize or minimize a specified system performance index. Quite frequently one deals with discrete systems in which certain system parameters have significance only at discrete values of a chosen variable. Examples are the maximization of the directive gain or the signal-to-noise ratio of an antenna array,[1] the optimization of power transfer between transmitting and receiving adaptive array antennas,[2] and the design of a predetection filter for the optimum detection of a sampled random signal in additive noise.[3] It appears that, because of the complexity of the mathematics involved, published studies often had to impose undesirable restrictions on a given situation in order to obtain a reasonably useful solution. The present paper states and proves a general optimization principle for discrete systems.

The basic theorem will be presented in three parts as the properties of a function of a matrix vector. By a straightforward manipulation, the vector function converts easily to a form which assimilates the expression of certain performance index of important discrete engineering systems. It will be shown that the general theorem provides a way to determine not only the optimum (maximum or minimum) value of the performance index but also the parameters of the system in order to achieve the optimum condition. Some aspects of the basic theorem have been inferred in the literature. However, the present authors have not been able to find a statement in the general form, nor a complete proof. The problem of gain maximization for a linear array with non-isotropic, non-uniformly spaced elements will be used as an example to illustrate the optimization procedure. It will become clear that the general optimization principle is capable of leading to solutions for new and presently unsolved situations.

The Theorem

Let a vector function $G(\vec{a})$ be defined as follows:

$$G(a) = \frac{\overset{\sim}{\vec{a}}{}^{*}\,\overline{A}\,\vec{a}}{\overset{\sim}{\vec{a}}{}^{*}\,\overline{B}\,\vec{a}},$$

(1)

where

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}$$

(2)

is an $N \times 1$ column vector, $\sim$ represents transposition, $*$ means 'the complex conjugate of',

$$\overline{A} = [\alpha_{mn}]$$

(3)

and

$$\overline{B} = [\beta_{mn}]$$

(4)

are both Hermitian $N \times N$ square matrices. If $\overline{B}$ is positive definite, then

 (i) the roots of the equation

$$\det(\overline{A} - \lambda\overline{B}) = O,$$

(5)

$$\lambda_1 \geq \lambda_2 \geq \cdots\cdots \geq \lambda_N, \text{ are real;}$$

 (ii) λ_1 and λ_N represent the bounds of the value of $G(\vec{a})$:

$$\lambda_1 \geq G(\vec{a}) \geq \lambda_N; \text{ and}$$

(6)

 (iii) the left equality in (6) is attained when $\vec{a}$ satisfies

$$\overline{A}\vec{a} = \lambda_1 \overline{B}\vec{a},$$

(7)

and the right equality in (6) is attained when $\vec{a}$ satisfies

$$\overline{A}\vec{a} = \lambda_N \overline{B}\vec{a}.$$

(8)

This three-part theorem will be proved in Appendix I.

Discussion

By employing (2), (3), and (4) in straight-forward matrix multiplication, we can easily show that

$$\overset{\sim}{\vec{a}}{}^{*}\,\overline{A}\,\vec{a} = \sum_{m=1}^{N} \sum_{n=1}^{N} a_m^{*}\,\alpha_{mn}\,a_n$$

(9)

and

$$\beta_{mn} = \frac{1}{4\pi} \int_0^{2\pi} d\phi \int_{-1}^{1} e^{-j(D_n-D_m)u} g(u,\phi) du. \tag{18}$$

Furthermore, we see that both $\overline{A} = [\alpha_{mn}]$ and $\overline{B} = [\beta_{mn}]$ are Hermitian and that $\overline{B}$ is positive definite, the foregoing optimization principle is therefore applicable in this case.

Although it is not the intention of this paper to present the numerical solution of a particular situation, we shall indicate below the mathematical procedure of the problem. We write

$$\overline{A} = \vec{\xi}\,\tilde{\vec{\xi}}^* \tag{19}$$

with

$$\vec{\xi} = \begin{pmatrix} e^{jD_1u_o} \\ e^{jD_2u_o} \\ \vdots \\ e^{jD_Nu_o} \end{pmatrix} \tag{20}$$

The roots of

$$\det(\overline{A}-\lambda\overline{B})=0 \tag{5}$$

are (see Appendix II)

$$\lambda_1 = \tilde{\vec{\xi}}^* \overline{B}^{-1}\vec{\xi} > 0 \tag{21}$$

and

$$\lambda_2=\lambda_3= \cdots =\lambda_N=0. \tag{22}$$

Hence the maximum gain, C_M, is

$$G_M=\lambda_1=\tilde{\vec{\xi}}^* \overline{B}^{-1}\vec{\xi} \tag{23}$$

The optimum excitations in the N elements are found from (7) by determining the column vector $\vec{a}$.

$$\vec{\xi}\,\tilde{\vec{\xi}}^*\vec{a} = \lambda_1\overline{B}\vec{a}. \tag{24}$$

Now, from (2), (14) and (20),

$$\tilde{\vec{\xi}}^*\vec{a} =E(u_o). \tag{25}$$

The combination of (24) and (25) then gives

$$\vec{a} = \frac{E(u_o)}{\lambda_1} \overline{B}^{-1} \vec{\xi},$$
(26)

which determines all the complex excitation amplitudes in the N elements.

Note that the above formulation enables us to determine both the amplitude and the phase of the excitation in the elements of a linear array which will yield a maximum gain in the direction (u_o, ϕ_o). This formulation is general and therefore very versatile in that the elements do not have to be uniformly spaced, that the main beam of the array may point to an arbitrary direction, and that the elements may be nonisotropic. Calculations for the special case of equal spacing and broadside radiation check exactly with the results obtained by Tai.[5] The amplitude and phase distributions required for optimizing the gain of 8-elements endfire arrays have been computed.[8] The present formulation can also be extended to include arbitrary planar and volumetric arrays.

Conclusion

A general optimization principle for discrete systems has been presented which will prove to be useful in many engineering situations. An example is given which illustrates its versatility in that it is capable of yielding solutions which are hitherto unavailable.

Appendix I - Proof of Theorem.

We shall prove the three-part basic theorem by making use of some known relations in linear algebra.

(i) It is seen that (5) is the condition for the existence of a nontrivial solution for the matrix equation

$$\overline{A}\vec{x} = \lambda \overline{B}\vec{x}, \quad \vec{x} \neq 0.$$
(27)

The complex conjugate of (27) is

$$\overline{A}^* \vec{x}^* = \lambda^* \overline{B}^* \vec{x}^*.$$
(28)

Taking the transposition of both sides of (28) and noting that $\overline{A}$ and $\overline{B}$ are Hermitian, we obtain

$$\widetilde{\vec{x}}^* \overline{A} = \lambda^* \widetilde{\vec{x}}^* \overline{B}.$$
(29)

Now, post-multiplication of (29) by $\vec{x}$ and pre-multiplication of (27) by $\vec{\tilde{x}}^*$ yield respectively

$$\vec{\tilde{x}}^* \tilde{A} \vec{x} = \lambda^* \vec{\tilde{x}} \tilde{B} \vec{x} \qquad (30)$$

and

$$\vec{\tilde{x}}^* \tilde{A} \vec{x} = \vec{\tilde{x}}^* \lambda \tilde{B} \vec{x}. \qquad (31)$$

Subtracting the corresponding sides of (30) and (31), we have

$$0 = (\lambda - \lambda^*) \vec{\tilde{x}}^* \tilde{B} \vec{x}. \qquad (32)$$

The positive definiteness of the matrix $\tilde{B}$ implies that

$$\vec{\tilde{x}}^* \tilde{B} \vec{x} > 0 \qquad (33)$$

for nonzero column vector $\vec{x}$. Hence

$$\lambda = \lambda^*, \qquad (34)$$

from which we infer that all roots of (5), $\lambda_1, \lambda_2, \cdots, \lambda_N$, are real.

(ii) Two known theorems will be used for the proof of part (ii). They are:

    (a) Every positive definite Hermitian form such as $\vec{\tilde{a}}^* \tilde{B} \vec{a}$ can be changed into $\vec{\tilde{v}}^* \vec{v}$ by a complex non-singular linear transformation. (See, for instance, Corollary (i), Theorem 13.2.1 in Reference 4.)

    (b) If λ_1 and λ_N are respectively the greatest andl east eigenvalues of a Hermitian matrix $\bar{D}$, then for all $\vec{v}$,

$$\lambda_1 \vec{\tilde{v}}^* \vec{v} \geq \vec{\tilde{v}}^* \bar{D} \vec{v} \geq \lambda_N \vec{\tilde{v}}^* \vec{v}. \qquad (35)$$

The left and right equalities are attained when $\vec{v}$ is the eigenvector of $\bar{D}$ corresponding to λ_1 and λ_N respectively. (See, for instance, Theorem 12.6.5 in Reference 4.)

By virtue of theorem (a), there exists a non-singular matrix $\bar{C}$ such that

$$\vec{a} = \bar{C} \vec{v} \qquad (36)$$

and

$$\vec{\tilde{a}}^* \tilde{B} \vec{a} = \vec{\tilde{v}}^* \vec{v} \qquad (37)$$

Eqs. (36) and (37) imply that

$$\widetilde{C}^{*}\overline{B}\,\overline{C} = \overline{I},\tag{38}$$

where $\overline{I}$ is the unitary matrix. The following relation also results from (36):

$$\overrightarrow{a}^{*}\overline{A}\,\overrightarrow{a} = \overrightarrow{v}^{*}\,(\widetilde{C}^{*}\overline{A}\,\overline{C})\,\overrightarrow{v}.\tag{39}$$

Now, since $\widetilde{C}^{*}\overline{A}\,\overline{C}$ is Hermitian, we have, from (35),

$$\lambda_1\,\overrightarrow{v}^{*}\overrightarrow{v} \ge \overrightarrow{v}^{*}\,\widetilde{C}^{*}(\overline{A}\,\overline{C})\,\overrightarrow{v} \ge \lambda_N\,\overrightarrow{v}^{*}\overrightarrow{v}\tag{40}$$

for all $\overrightarrow{v}$, where λ_1 and λ_N are respectively the maximum and minimum eigenvalues of $\widetilde{C}^{*}\overline{A}\,\overline{C}$. λ_1 and λ_N are two of the roots of the equation

$$\det(\widetilde{C}^{*}\overline{A}\,\overline{C} - \lambda\overline{I}) - 0.\tag{41}$$

By using (38), (41) can be rewritten as

$$\det(\widetilde{C}^{*}\overline{A}\,\overline{C} - \lambda\,\widetilde{C}^{*}\overline{B}\,\overline{C}) = 0,\tag{42}$$

which is readily seen to be reducible to (5). Combination of (37), (39) and (40) yields (6) and proves part (ii) of the theorem.

(iii) By theorem (b), the left and right equalities in (40) are attained when $\overrightarrow{v}$ is the eigenvector of $\widetilde{C}^{*}\overline{A}\,\overline{C}$ corresponding to the eigenvalues λ_1 and λ_N respectively. Leaving out the subscript on λ, we have

$$(\widetilde{C}^{*}\overline{A}\,\overline{C})\,\overrightarrow{v} = \lambda\overrightarrow{v}\tag{43}$$

In view of (36) and (38), (43) is the same as

$$\widetilde{C}^{*}\overline{A}\,\overrightarrow{a} = \lambda\overline{C}^{-1}\overrightarrow{a} = \lambda\widetilde{C}^{*}\overline{B}\,\overrightarrow{a},\tag{44}$$

which yields (7) or (8) directly, and hence part (iii) of the theorem is proved.

Appendix II – The Roots of Equation (5).

Here again we start with the fact that (5) is the condition for the existence of a non-trivial solution for the matrix equation

$$\overline{A}\,\overrightarrow{x} = \lambda\overline{B}\,\overrightarrow{x}, \qquad \overrightarrow{x} \ne 0.\tag{27}$$

From (19) and (20), it is easily verified that

$$\det(\overline{A}) = \det(\overrightarrow{\xi}\;\overset{\sim}{\overrightarrow{\xi}}{}^*) = 0, \tag{45}$$

hence $\lambda = 0$ is certainly a solution of (5). As a matter of fact, manipulation of the determinant $\det(\overline{A}-\lambda\overline{B})$ shows that a term λ^{N-1} can be factored out and it follows that

$$\lambda_2 = \lambda_3 = \cdots\cdots = \lambda_N = 0$$

which is (22). Substituting (19) in (27), we have

$$\overrightarrow{\xi}\;\overset{\sim}{\overrightarrow{\xi}}{}^*\overrightarrow{x} = \lambda\overline{B}\overrightarrow{x}. \tag{46}$$

It is noted that if $\overset{\sim}{\overrightarrow{\xi}}{}^*\overrightarrow{x} = 0$, $\lambda = 0$. Pre-multiplying both sides of (46) by $\overset{\sim}{\overrightarrow{\xi}}{}^*\overline{B}^{-1}$, we obtain

$$(\overset{\sim}{\overrightarrow{\xi}}{}^*\overline{B}^{-1}\overrightarrow{\xi})\;\overset{\sim}{\overrightarrow{\xi}}{}^*\overrightarrow{x} = \lambda\overset{\sim}{\overrightarrow{\xi}}{}^*\overrightarrow{x}. \tag{47}$$

Equation (47) clearly indicates that, if $\overset{\sim}{\overrightarrow{\xi}}{}^*\overrightarrow{x} \neq 0$,

$$\lambda = \lambda_1 = \overset{\sim}{\overrightarrow{\xi}}{}^*\overline{B}^{-1}\overrightarrow{\xi} > 0,$$

which is (21).

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