STABILITY ANALYSIS OF HIGH ORDER SYSTEMS WITH MULTIPLE NONLINEARITIES

K. W. Han* Ph. D., (韓 光 渭)

Professor of Chiao-Tung University

SUMMARY: A method of testing stability in high order systems with multiple nonlinearities is presented. The presented method can be used to find the limit cycles and the relative stability characteristics of a system. A control system for satellite attitude stabilization is considered, and a system with three nonlinearities is analyzed.

I. INTRODUCTION

In current literature, methods of testing nonlinear system stability can be classified into two categories: (1) mathematical analysis to find a general theory which can be applied to all kinds of nonlinear systems, and (2) graphical methods to solve each problem by a set of phase portrait or a family of curves[1-5]. The first category involves complex equations, and the proved results can only be applied to relatively simple cases; the second category is limited due to time consumption, inaccuracy, and is only applicable to those systems having two nonlinear parameters. In view of these limitations, a general method which can be used to analyze the stability characteristics of high order systems with multiple nonlinearities and to present all the results by a minimum number of curves in a plane would be besirable. The main purpose of this paper is to present some results along this approach.

II. STABILITY CRITERION

In this section, a stability criterion for nonlinear systems with real and frequency independent describing functions is presented.

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*K. W. Han is with the Chung Shan Institute, and adjunct associate professor of National Chiao-tung University, The Republic of China.

**G. J. Thaler is Professor of Electrical Engineering, U. S. Naval Postgraduate School, Monterey, California.

Stability criterion 1: For a nonlinear system with any number of nonlinearities, if all the inputs to the nonlinearities are related by constants, all the describing functions are real and independent of frequency and the linear part of the system has low pass characteristics, then the conditions for stability are that all the roots of the odd powered part (poles) and even powered part (zeros) of its linearized characteristic equation are on the imaginary axis of the s-plane for all values of the magnitude (A) of a reference input, and that the absolute values of the poles (p_i') and zeros (z_i') are related as

$$p_0 < z_1 < p_1 < z_2 < p_2 \cdots\cdots\cdots \tag{1}$$

where p_0 is the pole at the origin of the s-plane.

Proof: Since all the inputs to the nonlinearities are related by constants, any one of them can be regarded as a reference input; and for each value of A, the describing functions can be replaced by constants; thus the method of using Eq. (1) to test linear system stability can be applied.[6] If a linearized systemis proved stable for all values of A, then the nonlinear system is stable, since the linear part of the system has low pass characteristics and the method of using describing function is applicable.

Remark: A nonlinear system defined in Criterion 1 will have a limit cycle if for $A=A_l$ there is a pole (p_l) equal to a zero (z_l) and the magnitude of the limit cycle is defined by the value of A_l while its frequency is equal to the value of z_l or p_l.[6]

Example 1. Consider the system in Fig. 1, if $G_1=10/(s+1)$, $G_2=s+10$, $G_3=10/(s+1)^2$ and $b=1$, $M_1=1000$, $M_2=10$, then the linearized characteristic equation is

$$s^3+a_2s^2+a_1s+a_0=0 \tag{2}$$

where

$$a_2=\frac{120K_2(A)+3}{10K_2(A)+1} \qquad a_1=\frac{210K_2(A)+3}{10K_2(A)+1}$$

$$a_0=\frac{100[K_1(A)+K_2(A)]+1}{10K_2(A)+1}$$

and $$K_1(A)=\frac{M_1}{\pi}\left[2\mathrm{Sin}^{-1}b/A-\frac{2b}{A^2}(A^2-b^2)^{1/2}\right]+\frac{4M_1}{\pi A^2}(A^2-b^2)^{1/2}$$

$K_2(A)=4M_2/\pi A$

From criterion 1

$$z_1^2=a_0/a_2=\frac{100[K_1(A)+K_2(A)]+1}{120K_2(A)+3} \tag{3}$$

$$P_1^2=a_1=\frac{210K_2(A)+3}{10K_2(A)+1} \tag{4}$$

for various valuse of A, the values of p_1 and z_1 can be calculated, and the results are plotted in Fig. 2, where Q_1 and Q_2 represent two limit cycles; thus the system is conditionally stable.

In Eqs. (3) and (4), for very small values of A, the values of z_1 and p_1 approach to 0.912 and 4.58 respectively, which indicate that the system is well damped;[6] and for very large values of A, z_1 and p_1 are approximately equal to 0.566 and 1.73 respectively, thus the system is well damped also.

The same example has been analyzed in reference 4, where using a family of root loci to define an "input dependent root locus" has been used instead of two simple curves used in this paper.

III. STABILITY CRITERION FOR NONLINEAR SYSTEMS WITH FREQUENCY DEPENDENT NONLINEARITIES

A system with "frequency dependent nonlinearities" means that at least one of its nonlinearities has a frequency dependent describing function. A stability criterion for such kind of control systems is presented in the following paragraph.

Stability critrion 2: For a nonlinear system with any number of nonlinearities, if all the describing functions are real and their inputs are related by constants or by linear transfer functions with low pass characteristics, then the system is stable if the conditions in stability criterion 1 hold true for all values of ω and A, where A is the magnitude of a sinusoidel input to one of the nonlinearities and ω is the frequency.

Proof:

Since all the nonlinearities are related by linear transter functions, all the inputs to the nonlinearities can be refered to one input. For example, in Fig. 3, where "a" is a reference input, the equivalent describing functions are

$$K_{1e}(A, \omega)=K_1(A', \omega) \tag{5}$$

$$K_{2e}(A, \omega)=K^2(A) \tag{6}$$

$$K_{3e}(A, \omega)=K_3(A'', \omega) \tag{7}$$

where

$$A^1= |G_1|\, A,\ A''=A[K_2(A, \omega)+K_1(\,|G_1|\, A, \omega)]\, |G_2| \tag{8}$$

For each set of values of A and ω there is a set of values of the equivalent describing functions which gives a linearized system. Since stability criterion 1 which is assumed true for all values of A and ω gives the conditions of stability of a linearized system, thus the nonlinear system defined in Criterion 2 is stable.

Remark: A nonlinear system defined in Criterion 2 will have a limit cycle if there is a combination of A and ω which can make a pole (p_i) equal to a zero (z_i). The magnitude of the limit cycle is defined by the value of A, and its frequency (ω) is equal to the values of p_i and z_i.

The statement in this remark is the same as that in the remark of Criterion 1, except that an additional restriction on ω is added, since the magnitude of A alone can not guarante the existance of a limit cycle.

Example 2. Consider the system in Fig. 3, if $G_1=1/s$, $G_2=\alpha/s$, $G_3=\beta$, $G_4=1/s$ and $K_3(A, \omega)=1$, then the characteristic equation is

$$s^3+\alpha\beta K_{2e}s^2+(\alpha\beta K_{1e}+\alpha K_{2e})\ s+\alpha k_{1e}=0 \tag{9}$$

Since $K_{1e}=4\omega M/\pi\ A$, and $K_{2e}=4M/\pi\ A$, thus

$$p_1^2=\alpha[(\beta w+1)\ 4M/\pi\ A] \tag{10}$$

$$z_1^2=w/\beta \tag{11}$$

and condition for having a limit cycle is $p_i=z_i$, i e.,

$$\frac{\omega}{\beta}=\frac{2\alpha M}{\pi A}(\beta\omega+1) \tag{12}$$

Assume $M=10$, $\alpha=0\ 1$ and $\beta=10$, then Eq. (12) becomes

$$\omega=\frac{40}{\pi A}(10\omega+1) \tag{13}$$

Since the frequency of a limit cycle must equal to the value of p_1 or z_1, Eq. (11) is a sufficient condition to decide the value of ω; and then the value of A can be defined by Eq. (13). For this example, a limit cycle is found at $\omega=0.1$ and $A=254$.

It is interesting to note that, from Eq. (11), the frequency of the limit cycle, for this system, is decided by the value of β alone, and this condition may have an important application, since the considered sysem is a typical form for satellite attitude stabilization.[4] It can be realized that the proposed method is very convenient for parameter adjustment, thus it is useful tool for design.

For testing the relative stability of this system, Eq. (13) is plotted as a stability boundary in Fig. 4. From Criterion 2, for all values of ω and A on the lefe side of the boundary the system is stable, thus the limit cycle at Q_1 is unstable. It can be seen that for each set of values of ω and A the relative stability characteristics can be defined by finding the corresponding values of p_1 and z_1[6].

Using commonly proposed methods, to analyze a 3rd order system with two nonlinearities is usually a complex problem; but using the method presented in this section, the results can be found easily. For the analysis of high order systems, a general procedure is presented along with the following example.

Example 3. In example 2, if G_2 is chenged to $\alpha/s\,(s+1)$, then the coefficients of the linearized characteristic equation are

$$a_3=1, \qquad a_2=\alpha\beta K_2(A)$$

$$a_1=\alpha\beta K_{1e}(A,\omega)+\alpha k_2(A)$$

$$a_0=\alpha k_{1e}(A,\omega)$$

For $\alpha=0.1$, $M=10$, and let $\eta=4/\pi A$, then

$$p_1^2=\eta(10\omega+1) \tag{14}$$

$$z_1^2,\ z_1^2=5\eta\mp[(5\eta)^2-\eta\omega]^{\frac{1}{2}} \tag{15}$$

and the general procedure for analysis is to let ω equal to p_i or z_i and plot the loci of the positive real solutions of Eqs. (14) and (15) in a ω vs A plane, where the intersection points among these loci define the limit cycles.

In a later part of this paper, Eqs. (14) and (15) will be called stability equations.

Following the general procedure, Eq. (14) becomes

$$\omega^2=\eta(10\omega+1) \tag{16}$$

i. e.,
$$\omega=P_1=5\eta\pm[(5\eta)^2+\eta]^{\frac{1}{2}} \tag{17}$$

and Eq. (15) gives

$$\omega^2=5\eta\pm[(5\eta)^2-\eta\omega]^{\frac{1}{2}} \tag{18}$$

The results are represented by three curves, (A), (B)and (C), in Fig. 5, where two limit cycles are found at Q_1 and Q_2; and from the relations (defined in Eq. (1)) among the curves in Fig. 5, it can be realized that Q_1 and Q_2 are stable and unstable limit cycles respectively.

The proposed method is suitable for use with a digital computer to test stability of high order systems, since simple equations exist for calculation.

The proposed method has no limitation on the number of nonlinearities, and it can give correct answer on the stability characteristics of a system as long as the linear part of the system is of low-pass characteristic and the nonlinearities are related by linear (low pass) transfer functions. For having an illustration, the system in Fig. 6 has been analyzed, and a stable limit cycle has been found at A=170, ω=1. 7.

All the results in this paper have been checked with an analog computer.

CONCLUSIONS

In this paper, a method of testing stability in high order systems with multiple nonlinearities has been presented. The presented method is useful for finding the limit cycles and the relative stability characteristics of a system, and all the results of analysis can be represented by a few curves in one plane. Verious examples have been given, and a comparison with the method in a current literature has been made.

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LIST OF SYMBOLS

s	Laplace operator
p_i	root of the odd part of characteristic equation (absolute value)
z_i	root of the even part of characteristic equation (absolute value)
$a=A\sin wt$	in put signal to nonlinearities
a_i	coefficient of characteristic equation
G	linear transfer function
G_d	nonlinear transfer function
$K_i(A)$	describing function
$K_{i\omega}(A, \omega)$	frequency dependent describing function
M_i	output of relay or the maximum output of a saturation ainplifier
i	integer
b	input to a saturation amplifier
Q_i	limit cycle
A	magnitude of a sinusoidal input to nonlinearities
η. η'	parameters
ω	frequency

LIST OF CAPTIONS

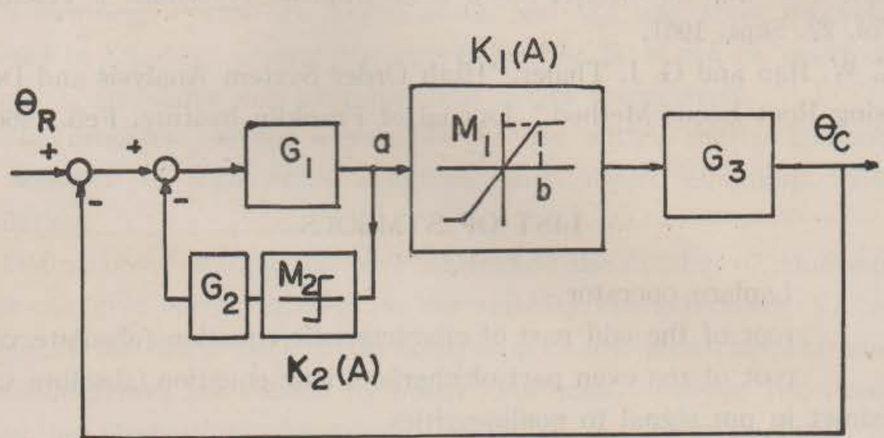


Fig. 1 $a = A\sin\omega t$

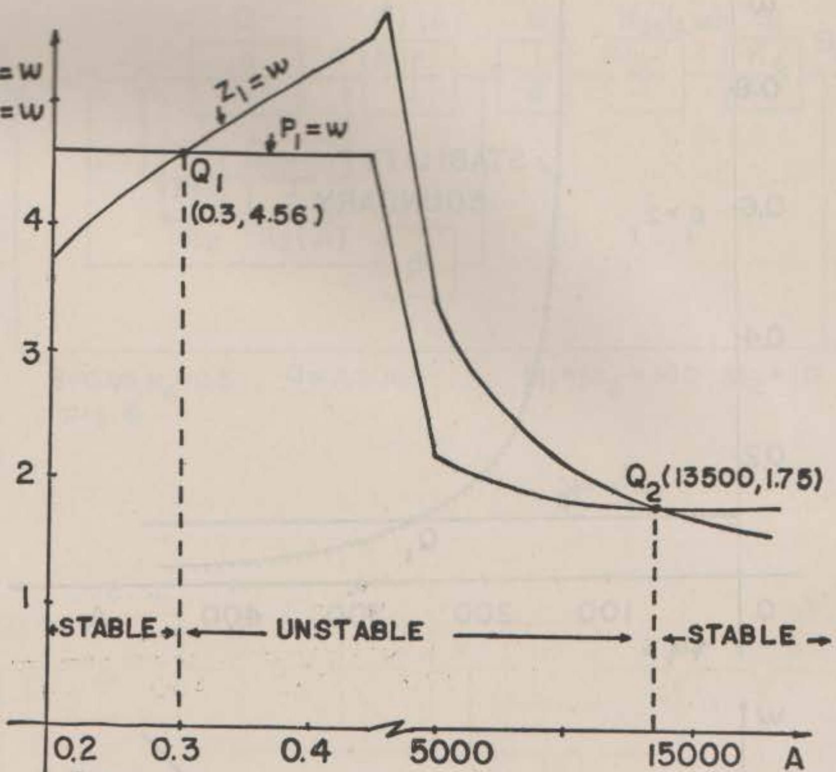


Fig. 2

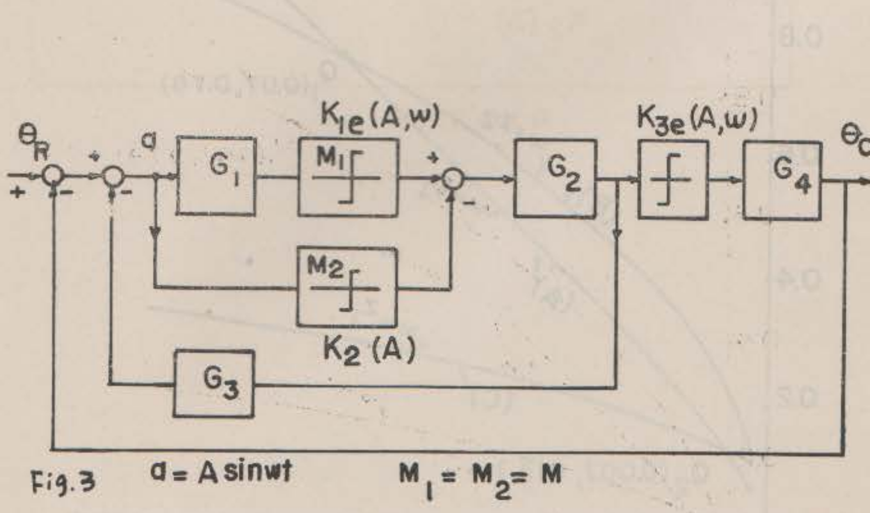


Fig. 3 $a = A \sin wt$ $M_1 = M_2 = M$

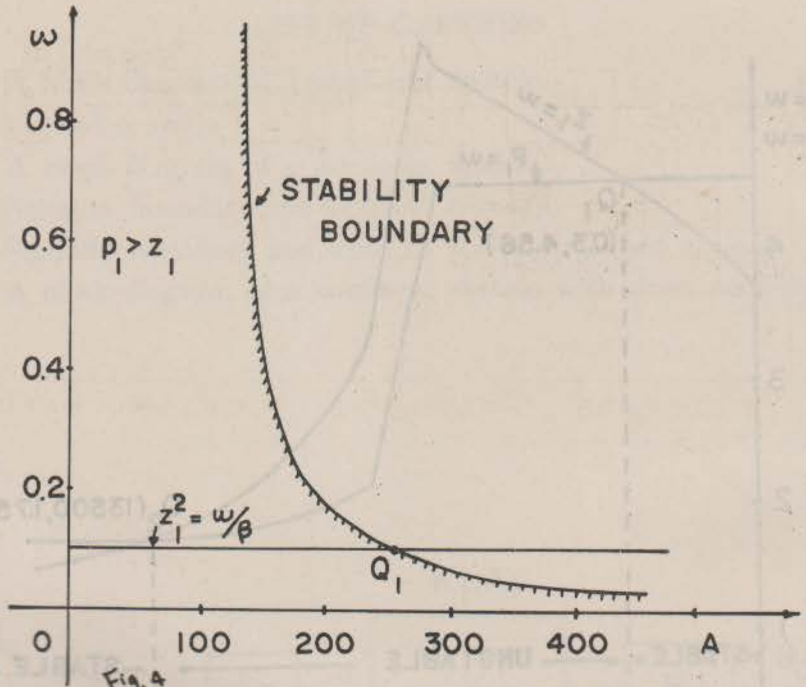

ω
0.8
0.6
0.4
0.2
0
100
200
300
400
A
STABILITY
BOUNDARY
$p_1 > z_1$
$z_1^2 = \omega/\beta$
Q_1

Fig. 4

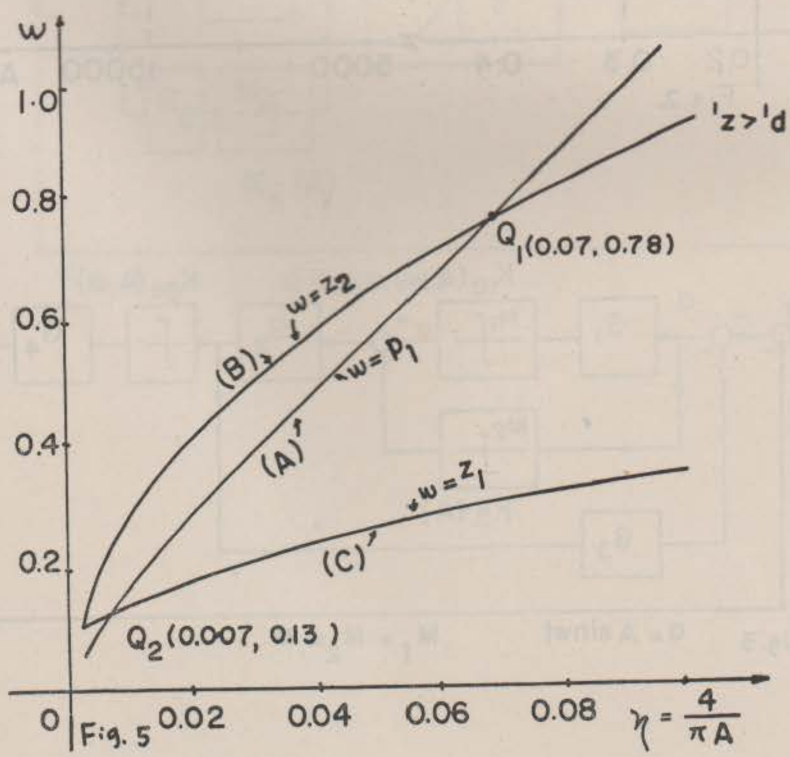
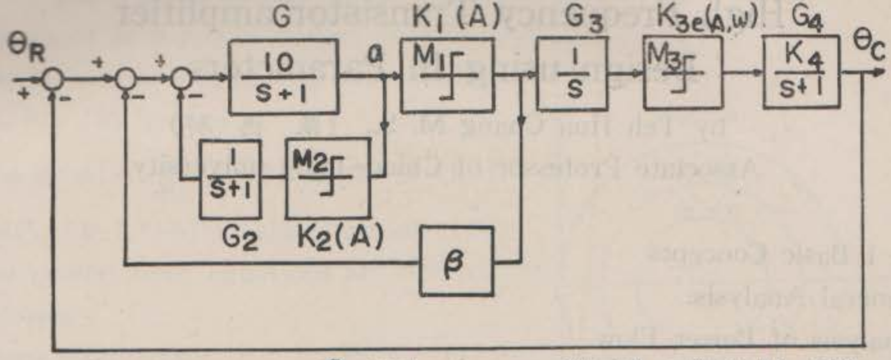

w
1.0
0.8
0.6
0.4
0.2
0
0.02
0.04
0.06
0.08
$\eta = \frac{4}{\pi A}$
$p_1 > z_1$
$Q_1(0.07, 0.78)$
$w = z_2$
$w = p_1$
(B)
(A)
$w = z_1$
(C)
$Q_2(0.007, 0.13)$

Fig. 5



$\beta=0.05, K_4=0.5$, $a=A\sin n\omega t$, $M_1=M_3=100, M_2=10$

Fig. 6