GROUP VELOCITY AND ENERGY VELOCITY IN LOSSLESS, ISOTROPIC, AND HOMOGENEOUS DIELECTRIC SLAB GUIDES

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1. Introduction

When a uniform E. M. Plane wave travels in a dense perfect dielectric incident on the boundary of a less dense perfect dielectric, total reflection will occur, if the incident angle is greater than the critical angle. If both dielectrics are lossless, isotropic, and homogeneous, then the reflected plane wave in the denser medium will still be uniform and the transmitted wave in the less dense material will be nonuniform[1].

Let us consider a slab guide immersed in air. The geometry of the slab is assumed to be infinitely wide, but with finite thickness. The dielectric in the slab is lossless, isotropic, and homogeneous where its permittivity is larger than that in air and its permeability is not less than that in air. Thus the uniform plane waves propagating in the slab possess the property of total reflection as stated in the preceeding paragraph.

The investigations of dielectric slab guide have been largely based on the graphical method[1] while in this paper, the analytic method is adopted. The field distribution is described first, and then the phase, group and energy velocity are derived, finally the relationship among them is established.

II. The Field Distribution in the Slab Guide[2]

The coordinate system in which the slab guide is depicted is shown in Fig. 1. Regions (I) and (III) are air with permeability μ_o and permittivity ϵ_o. Region (II) is dense dielectric whose permeability and permittivity are μ and ϵ where $\mu \geq \mu_o$ and $\epsilon > \epsilon_o$

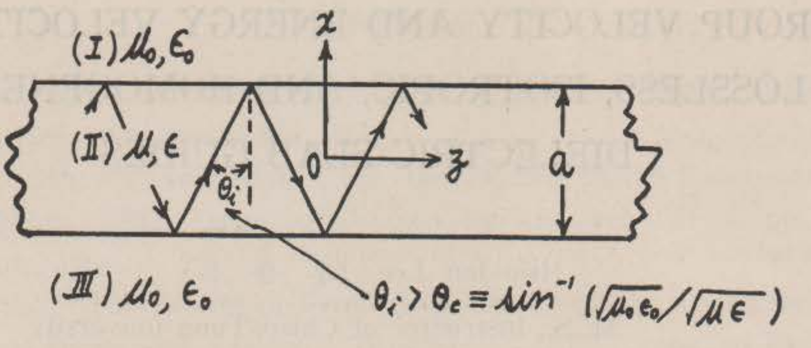

Fig. 1. Dielectric slab and the coordinate system

Before describing the field distribution in the slab, one assumes the wave existing in the slab has the following properties:

(a) The time variation of the wave is given by $e^{j\omega t}$, where ω is the angular frequency.

(b) Propagation of the wave is in the z-direction with a propagation factor $e^{-j\beta_z z}$, where β_z is the phase constant in the z-direction.

(c) No variation in y-direction, i.e. consider the waves of TM_{no} and TE_{no} only.

(d) The field decays away from the surface according to a factor $e^{-\alpha_x|x|}$, where α_x is the attenuation constant along the x-direction in air region.

A. TM_{no} modes

For the TM_{no} modes it is assumed that the magnetic field intensity vector H is parallel to the y-axis. In solving the wave equation the solution for H_y in region (II) may be of the symmetrical type (even) or antisymmetric type (odd), where "even" and "odd" refer to the way that H_y varies with x about the symmetry plane x=O.

1. Even solution

For the even solution H_y in region (II) will be the following form:

$$H_y = H_o \cos\beta_x x \; e^{-j\beta_z z}, \quad |x| \leqq \frac{a}{2} \tag{1}$$

where a=the thickness of the slab, H_o=amplitude constant, β_x=the phase constant in the x-direction in the region (II) and has relation to β_z by the following equation

$$\beta_z^2=\omega^2\mu\epsilon-\beta_x^2 \tag{2}$$

In the air region H_y may be represented by

$$H_y=H_o' \, e^{-\alpha_x|x|-j\beta_z z}, \quad |x|\geq\frac{a}{2} \tag{3}$$

where α_x and β_z have a relationship as follows:

$$\beta_z^2=\omega^2\mu_0\epsilon_0+\alpha_x^2 \tag{4}$$

At $x=\pm\frac{a}{2}$, H_y must be continuous from Eqs. (1) and (3) yields.

$$H_o'=H_o\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}} \tag{5}$$

From Maxwell's equations and Eqs. (1), (3), (5), the field of even TM_{no} modes will be:

for $x\geq\frac{a}{2}$

$$H_{y1}=H_o\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_z z} \tag{6-a}$$

$$E_{x1}=H_o\cdot\frac{\beta_z}{\omega\epsilon_o}\cdot\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_z z} \tag{6-b}$$

$$E_{z1}=-H_o\cdot\frac{\alpha_x}{j\omega\epsilon_o}\cdot\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_z z} \tag{6-c}$$

for $-\frac{a}{2}\leq x\leq\frac{a}{2}$

$$H_{y2}=H_o\cos(\beta_x x)e^{-j\beta_z z} \tag{7-a}$$

$$E_{x2}=H_o\cdot\frac{\beta_z}{\omega\epsilon}\cdot\cos(\beta_x x)e^{-j\beta_z z} \tag{7-b}$$

$$E_{z2}=-H_o\cdot\frac{\beta_x}{j\omega\epsilon}\cdot\sin(\beta_x x)e^{-j\beta_z z} \tag{7-c}$$

for $x\leq-\frac{a}{2}$

$$H_{y3}=H_o\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{8-a}$$

$$E_{x3}=H_o\cdot\frac{\beta_z}{\omega\epsilon_o}\cdot\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{8-b}$$

$$E_{z3}=H_o\cdot\frac{\alpha_x}{j\omega\epsilon_o}\cdot\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{8-c}$$

At $x=\pm\frac{a}{2}$, E_z must be continuous from Eqs. (7-c) and either (6-c) or (8-c) yield the following transcendental equation for even Tm_{no} modes:

$$\alpha_x=\frac{1}{\epsilon_r}\cdot\beta_x\cdot\tan(\frac{\beta_x a}{2}) \tag{9}$$

wherh $\epsilon_r=\frac{\epsilon}{\epsilon_o}$: the relative permittivity of dielectric slab. Since the value of α_x must be positive real and finite. Hence $(\beta_x a/2)$ will be in the ranges as follows:

$$\frac{n\pi}{2}\leq(\frac{\beta_x a}{2})<\frac{(n+1)\pi}{2},\qquad n=0,2,4,\cdots\cdots \tag{10}$$

2. Odd solution

For odd solution, H_y in region (II) will have the following form:

$$H_y=H_o\sin\beta_x x\ e^{-j\beta_z z},\qquad |x|\leq\frac{a}{2} \tag{11}$$

Since H_y must continuous at $x=\pm\frac{a}{2}$, by conbining Eqs. (3) and (11), one can obtain

$$H_o'=\pm\sin(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}}.H_o \tag{12}$$

where "+" sign for $x\geq\frac{a}{2}$ and "−" sign for $x\leq-\frac{a}{2}$. Similarly, from Eqs. (3), (11), (12) and Maxwell's equations, the field of odd TM_{no} modes will be:

for $x\geq\frac{a}{2}$

$$H_{y1}=H_o\sin(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_z z} \tag{13-a}$$

$$E_{x1}=H_o\cdot\frac{\beta_z}{\omega\epsilon_o}\cdot\sin(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_z z} \tag{13-b}$$

$$E_{z1}=-H_o\cdot\frac{\alpha_x}{j\omega\epsilon_o}\cdot\sin(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_z z} \tag{13-c}$$

for $-\frac{a}{2}\leq x\leq\frac{a}{2}$

$$H_{y2}=H_o\sin(\beta_x x)\,e^{-j\beta_z z} \tag{14-a}$$

$$E_{x2}=H_o\cdot\frac{\beta_z}{\omega\epsilon}\cdot\sin(\beta_x x)e^{-j\beta_z z} \tag{14-b}$$

$$E_{z2}=H_o\cdot\frac{\beta_x}{j\omega\epsilon}\cdot\cos(\beta_x x)e^{-j\beta_z z} \tag{14-c}$$

for $x\leq-\frac{a}{2}$

$$H_{y3}=-H_o\sin(\frac{\beta_x x}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{15-a}$$

$$E_{x3}=-H_o\cdot\frac{\beta_z}{\omega\epsilon_o}\cdot\sin(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{15-b}$$

$$E_{z3}=-H_o\cdot\frac{\alpha_x}{j\omega\epsilon_o}\cdot\sin(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{15-c}$$

E_z must be continuous at $x=\pm\frac{a}{2}$, by use of Eqs. (14-c) and either (13c) or (15-c) yields

$$\alpha_x=-\frac{1}{\epsilon_r}\cdot\beta_x\cdot\cot(\frac{\beta_x a}{2}) \tag{16}$$

the ranges of $(\frac{\beta_x a}{2})$ will be

$$\frac{n\pi}{2}\leq\frac{\beta_x a}{2}<\frac{(n+1)\pi}{2},\qquad n=1,3,5,\cdots\cdots \tag{17}$$

B. TE_{no} modes

Since the derivation for the TE_{no} modes is similar to that for the TM_{no} modes it will suffice to list the equations below.

1. Even TE_{no} modes

for $x\geq\frac{a}{2}$

$$E_{y1}=E_o\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_z z} \tag{18-a}$$

$$H_{x1}=-E_o\cdot\frac{\beta_z}{\omega\mu_o}\cdot\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_z z} \tag{18-b}$$

$$H_{z1}=E_o\cdot\frac{\alpha_x}{j\omega\mu_o}\cdot\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_x z} \tag{18-c}$$

for $-\frac{a}{2}\leq x\leq\frac{a}{2}$

$$E_{y2}=E_o\cos(\beta_x x)e^{-j\beta_z z} \tag{19-a}$$

$$H_{x2}=-E_o\cdot\frac{\beta_z}{\omega\mu}\cdot\cos(\beta_x x)e^{-j\beta_z z} \tag{19-b}$$

$$H_{z2}=E_o\cdot\frac{\beta_x}{j\omega\mu}\cdot\sin(\beta_x x)e^{-j\beta_z z} \tag{19-c}$$

for $x\leq-\frac{a}{2}$

$$E_{y3}=E_o\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{20-a}$$

$$H_{x3}=-E_o\cdot\frac{\beta_z}{\omega\mu_o}\cdot\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{20-b}$$

$$H_{z3}=-E_o\cdot\frac{\alpha_x}{\omega\mu_o}\cdot\cos(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{20-c}$$

and

$$\alpha_x=\frac{1}{\mu_r}\cdot\beta_x\cdot\tan(\frac{\beta_x a}{2}) \tag{21}$$

$$\frac{n\pi}{2}\leq\frac{\beta_x a}{2}<\frac{(n+1)\pi}{2},\qquad n=0,2,4,\cdots\cdots \tag{22}$$

where $\mu_r=\frac{\mu}{\mu_o}$: the relative permeability of dielectric slab.

2. Odd TE_{no} modes

for $x\geq\frac{a}{2}$

$$E_{y1}=E_o\sin(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_z z} \tag{23-a}$$

$$H_{x1}=-E_o\cdot\frac{\beta_z}{\omega\mu_o}\cdot\sin(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_z z} \tag{23-b}$$

$$H_{z1}=E_o\cdot\frac{\alpha_x}{j\omega\mu_o}\cdot\sin(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}-\alpha_x x-j\beta_z z} \tag{23-c}$$

for $-\frac{a}{2}\leq x\leq\frac{a}{2}$

$$E_{y2}=E_o\sin(\beta_x x)e^{-j\beta_z z} \tag{24-a}$$

$$H_{x2}=-E_o\cdot\frac{\beta_z}{\omega\mu}\cdot\sin(\beta_x x)e^{-j\beta_z z} \tag{24-b}$$

$$H_{z2}=-E_o\cdot\frac{\beta_x}{j\omega\mu}\cdot\cos(\beta_x x)e^{-j\beta_z z} \tag{24-c}$$

for $x\leq-\frac{a}{2}$

$$E_{y3}=-E_o\sin(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{25-a}$$

$$H_{x3}=E_o\cdot\frac{\beta_z}{\omega\mu_o}\cdot\sin(\frac{\beta_x a}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{25-b}$$

$$H_{z3}=E_o\cdot\frac{\alpha_x}{j\omega\mu_z}\cdot\sin(\frac{\beta_y a}{2})e^{\frac{\alpha_x a}{2}+\alpha_x x-j\beta_z z} \tag{25-c}$$

and

$$\alpha_x=-\frac{1}{\mu_r}\cdot\beta_x\cdot\cot(\frac{\beta_x a}{2}) \tag{26}$$

$$\frac{n\pi}{2}\leq\frac{\beta_x a}{2}<\frac{(n+1)\pi}{2},\qquad n=1,3,5,\cdots\cdots \tag{27}$$

Because of the similarity among the four types of modes, detailed discussion is given to the even TM_{no} modes only.

III. The critical Frequency of the Even TM_{no} Modes

At the critical frequency, the angle of incident wave, i.e. $\theta_i=\sin^{-1}(\beta_z/\omega\sqrt{\mu\epsilon})$, is equal to the critical angle, i.e. $\theta_c=\sin^{-1}(\sqrt{\mu_o\epsilon_o}/\sqrt{\mu\epsilon})$. By use of this condition and Eq. (4), the attenuation constant α_x is then equal to zero.

From Eqs. (9) and (10) the value of β_x at the condition of critical frequency will be

$$\beta_{xc}=\frac{n\pi}{a},\qquad n=0,2,4,\cdots\cdots \tag{28}$$

Combining Eqs. (2) and (4), the other relationship between α_x and β_x can be obtained.

$$\alpha_x=\left\{\omega^2(\mu\epsilon-\mu_o\epsilon_o)-\beta_x^2\right\}^{1/2} \tag{29}$$

Setting $\alpha_x=0$, then the critical frequency will be

$$\omega_c = \frac{\beta_{xc}}{\sqrt{\mu\epsilon - \mu_o \epsilon_o}} \tag{30}$$

substitution of (28) into (30) gives the critical frequency of even TM_{no} modes as follows:

$$\omega_c = \frac{n\pi}{a\sqrt{\mu\epsilon - \mu_o \epsilon_o}}, \qquad n=0,2,4,\cdots\cdots \tag{31}$$

When n=0 in Eq. (31), leading to $\omega_c=0$. Thus very low frequency waves may be guided by a dielectric slab. If n=2,4,6,……in Eq. (31), then any lower frequency will bring the incident angle below critical, and there will be no basis for the waves to the slab.

From Eq. (6) and (8), the field in the air region, for the critical condition will be:

for $x \geq \frac{a}{2}$

$$H_{y1} = H_o e^{-j\beta_z z} \tag{32}$$

$$E_{x1} = \sqrt{\frac{\mu_o}{\epsilon_o}} \cdot H_o e^{-j\beta_z z} \tag{33}$$

$$E_{z1} = 0 \tag{34}$$

for $x \leq -\frac{a}{2}$

$$H_{y3} = H_o e^{-j\beta_z z} \tag{35}$$

$$E_{x3} = \sqrt{\frac{\mu_o}{\epsilon_o}} \cdot H_o e^{-j\beta_z z} \tag{36}$$

$$E_{z3} = 0 \tag{37}$$

These equations indicate that the field extends uniformly to infinits outside the slab, in air region, at the critical frequency, and the field will become a uniform plane wave traveling along z-direction.

IV. The Phase Velocity of Even TM_{no} Modes

The phase velocity[1] is the velocity of propagation of the surfaces of constant phase for a single frequency wave. It can be expressed by

$$V_p = \frac{\omega}{\beta_z} \tag{38}$$

Combining Eqs. (9) and (29) yield the relationship between ω and β_x as follows:

$$\frac{1}{\epsilon_r}\cdot\beta_x\cdot\tan\left(\frac{\beta_x a}{2}\right)=\left\{\omega^2(\mu\epsilon-\mu_o\epsilon_o)-\beta_x^2\right\}^{1/2} \tag{39}$$

From Eq. (39) the angular frequency may be expressed in terms of β_x

$$\omega=\frac{1}{\sqrt{\mu\epsilon-\mu_o\epsilon_o}}\cdot\beta_x\cdot\left\{1+\frac{1}{\epsilon_r^2}\cdot\tan^2\left(\frac{\beta_x a}{2}\right)\right\}^{1/2} \tag{40}$$

or

$$\omega=\frac{2}{\pi}\cdot\omega_{c1}\cdot\left(\frac{\beta_x a}{2}\right)\cdot\left\{1+\frac{1}{\epsilon_r^2}\cdot\tan^2\left(\frac{\beta_x a}{2}\right)\right\}^{1/2} \tag{41}$$

where $\omega_{c1}=\frac{\pi}{a\sqrt{\mu\epsilon-\mu_o\epsilon_o}}$: the critical frequency of mode 1, and then, substitution of Eq. (40) into (2) yields.

$$\beta_z=\frac{1}{\sqrt{\mu_r\epsilon_r-1}}\cdot\beta_x\cdot\left\{1+\frac{\mu_r}{\epsilon_r}\cdot\tan^2\left(\frac{\beta_x a}{2}\right)\right\}^{1/2} \tag{42}$$

or

$$\left(\frac{\beta_z a}{2}\right)=\frac{1}{\sqrt{\mu_r\epsilon_r-1}}\cdot\left(\frac{\beta_x a}{2}\right)\cdot\left\{1+\frac{\mu_r}{\epsilon_r}\cdot\tan^2\left(\frac{\beta_x a}{2}\right)\right\}^{1/2} \tag{43}$$

Substituting Eqs. (40) and (42) into (38) then the phase velocity of even TM_{no} modes will be

$$V_p=\left\{\frac{1+\frac{1}{\epsilon_r^2}\cdot\tan^2\left(\frac{\beta_x a}{2}\right)}{1+\frac{\mu_r}{\epsilon_r}\cdot\tan^2\left(\frac{\beta_x a}{2}\right)}\right\}^{1/2}\frac{1}{\sqrt{\mu_o\epsilon_o}} \tag{44}$$

$$\text{Since } V_p=\frac{\omega}{\beta_z}=\frac{1}{\sqrt{\mu\epsilon}}\cdot\frac{1}{\sin\theta_i} \tag{45}$$

By comparison of (45) and (44), the sine of incident angle of even TM_{no} modes may be written as

$$\sin\theta_i=\left\{\frac{1+\frac{\mu_r}{\epsilon_r}\cdot\tan^2\left(\frac{\beta_x a}{2}\right)}{\mu_r\epsilon_r+\frac{\mu_r}{\epsilon_r}\tan^2\left(\frac{\beta_x a}{2}\right)}\right\}^{1/2} \tag{46}$$

The curve for V_p versus ω can be plotted by use of Eqs. (41) and (44) with the parameter $(\beta_x a/2)$ varing in the range of of Eq. (10).

$$\frac{n\pi}{2}\leq\left(\frac{\beta_x a}{2}\right)<\frac{(n+1)\pi}{2}, \qquad n=0,2,4,\cdots\cdots \tag{47}$$

when

$$\left(\frac{\beta_x a}{2}\right)=\frac{n\pi}{2} \tag{48}$$

then

$$\omega=n\omega_{c1} \tag{49}$$

and

$$V_p=\frac{1}{\sqrt{\mu_o\epsilon_o}} \tag{50}$$

when

$$\left(\frac{\beta_x a}{2}\right)\longrightarrow\frac{(n+1)\pi}{2} \tag{51}$$

then

$$\omega\longrightarrow\infty \tag{52}$$

and

$$V_p\longrightarrow\frac{1}{\sqrt{\mu\epsilon}} \tag{53}$$

So that, when

$$\frac{n\pi}{2}<\left(\frac{\beta_x a}{2}\right)<\frac{(n+1)\pi}{2} \tag{54}$$

then

$$n\omega_{c1}<\omega<\infty \tag{55}$$

and

$$\frac{1}{\sqrt{\mu_o\epsilon_o}}>V_p>\frac{1}{\sqrt{\mu\epsilon}} \tag{56}$$

The curves of V_p versus ω for even TM_{no} modes are shown in Fig. 2.

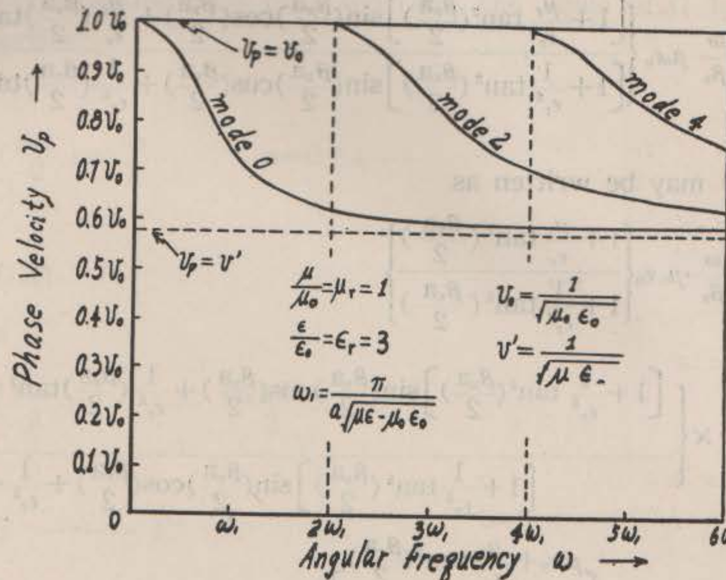

Fig. 2. Phase velocity versus frquency ω for even TM_{no} mode.

V. The Group Velocity of Even TM_{no} Modes

The group velocity[3,4] of a wave packet, i.e. a group of waves con tained within a narrow frequency band about the center frequency ω_o, is given by definition as:

$$V_g = \left(\frac{d\beta_z}{d\omega}\right)^{-1}_{\omega=\omega_o} \tag{57}$$

Taking the derivative of Eq. (2) with respect to ω, we obtain:

$$\frac{d\beta_z}{d\omega} = \frac{\omega}{\beta_z}\cdot\mu\epsilon - \frac{\beta_x}{\beta_z}\,\frac{d\beta_x}{d\omega} \tag{58}$$

where $\frac{d\beta_x}{d\omega}$ can be found by taking the derivative of Eq. (39) with respect to ω

$$\frac{d\beta_x}{d\omega} = \frac{\frac{\omega}{\beta_x}\cdot(\mu\epsilon - \mu_o\epsilon_o)\cdot\sin\left(\frac{\beta_x a}{2}\right)\cdot\cos\left(\frac{\beta_x a}{2}\right)}{\left[1+\frac{1}{\epsilon_r^2}\tan^2\left(\frac{\beta_x a}{2}\right)\right]\cdot\sin\left(\frac{\beta_x a}{2}\right)\cos\left(\frac{\beta_x a}{2}\right)+\frac{1}{\epsilon_r^2}\cdot\frac{\beta_x a}{2}\cdot\tan^2\left(\frac{\beta_x a}{2}\right)} \tag{59}$$

Substitution of (59) into (58) gives

$$\frac{d\beta_z}{d\omega}=\frac{\omega}{\beta_z}\mu_0\epsilon_0\left\{\frac{\left[1+\frac{\mu_r}{\epsilon_r}\tan^2(\frac{\beta_x a}{2})\right]\sin(\frac{\beta_x a}{2})\cos(\frac{\beta_x a}{2})+\frac{\mu_r}{\epsilon_r}(\frac{\beta_x a}{2})\tan^2(\frac{\beta_x a}{2})}{\left[1+\frac{1}{\epsilon_r^2}\tan^2(\frac{\beta_x a}{2})\right]\sin(\frac{\beta_x a}{2})\cos(\frac{\beta_x a}{2})+\frac{1}{\epsilon_r^2}(\frac{\beta_x a}{2})\tan^2(\frac{\beta_x a}{2})}\right\} \tag{60}$$

Eq. (60) may be written as

$$\frac{d\beta_z}{d\omega}=\frac{\omega}{\beta_z}\cdot\mu_0\,\epsilon_0\left\{\frac{1+\frac{\mu_r}{\epsilon_r}\tan^2(\frac{\beta_x a}{2})}{1+\frac{1}{\epsilon_r^2}\tan^2(\frac{\beta_x a}{2})}\right\}$$

$$\times\left\{\frac{\left[1+\frac{1}{\epsilon_r^2}\tan^2(\frac{\beta_x a}{2})\right]\sin(\frac{\beta_x a}{2})\cos(\frac{\beta_x a}{2})+\frac{1}{\epsilon_r^2}(\frac{\beta_x a}{2})\tan^2(\frac{\beta_x a}{2})\cdot\frac{\left[\frac{\mu_r\epsilon_r+\frac{\mu_r}{\epsilon_r}\tan^2(\frac{\beta_x a}{2})}{1+\frac{\mu_r}{\epsilon_r}\tan^2(\frac{\beta_x a}{2})}\right]}{\tan^2(\frac{\beta_x a}{2})}}{\left[1+\frac{1}{\epsilon_r^2}\tan^2(\frac{\beta_x a}{2})\right]\sin(\frac{\beta_x a}{2})\cos(\frac{\beta_x a}{2})+\frac{1}{\epsilon_r^2}\cdot(\frac{\beta_x a}{2})\cdot}\right\} \tag{61}$$

Set

$$\frac{\left[1+\frac{1}{\epsilon_r^2}\tan^2(\frac{\beta_x a}{2})\sin(\frac{\beta_x a}{2})\cos(\frac{\beta_x a}{2})+\frac{1}{\epsilon_r^2}(\frac{\beta_x a}{2})\tan^2(\frac{\beta_x a}{2})\cdot\left[\frac{\mu_r\epsilon_r+\frac{\mu_r}{\epsilon_r}\tan^2(\frac{\beta_x a}{2})}{1+\frac{\mu_r}{\epsilon_r}\tan^2(\frac{\beta_x a}{2})}\right]\right.}{\left[1+\frac{1}{\epsilon_r^2}\tan^2(\frac{\beta_x a}{2})\right]\sin(\frac{\beta_x a}{2})\cos(\frac{\beta_x a}{2})+\frac{1}{\epsilon_r^2}\cdot(\frac{\beta_x a}{2})\tan^2(\frac{\beta_x a}{2})}=G \tag{62}$$

By means of Eqs. (44), (38), and (62), then Eq. (61) may be simplified to:

$$\frac{d\beta_z}{d\omega}=\frac{\beta_z}{\omega}\bullet\frac{1}{G} \tag{63}$$

Substituting (63) into (57), the group velocity will be:

$$V_g=V_p\bullet\frac{1}{G} \tag{64}$$

Eq. (62) may be written as:

$$G=\frac{\left[1+\frac{1}{\epsilon_r^2}\tan^2(\frac{\beta_x a}{2})\right]\cos(\frac{\beta_x a}{2})+\frac{1}{\epsilon_r^2}(\frac{\beta_x a}{2})\sin(\frac{\beta_x a}{2})/\cos^2(\frac{\beta_x a}{2})\sin^2\theta_i\right]}{\left[1+\frac{1}{\epsilon_r^2}\tan^2(\frac{\beta_x a}{2})\right]\cos(\frac{\beta_x a}{2})+\frac{1}{\epsilon_r^2}(\frac{\beta_x a}{2})\sin(\frac{\beta_x a}{2})/\cos^2(\frac{\beta_x a}{2})} \tag{65}$$

By investegating Eqs. (65) and (46), the valus of G has following properties.

when

$$(\frac{\beta_x a}{2})=\frac{n\pi}{2}, \qquad n=0,2,4,\cdots\cdots \tag{66}$$

then

$$G=1 \tag{67}$$

when

$$(\frac{\beta_x a}{2}) \longrightarrow \frac{(n+1)\pi}{2} \tag{68}$$

then

$$G \longrightarrow 1 \tag{69}$$

when

$$\frac{n\pi}{2}<(\frac{\beta_x a}{2})<\frac{(n+1)\pi}{2} \tag{70}$$

then

$$G>1 \tag{71}$$

By use of the properties of G, the group velocity will be:

$$V_g=V_p,\ \text{when}\ \frac{\beta_x a}{2}=\frac{n\pi}{2}\ \text{or}\ \omega=n\omega_{c1} \tag{72}$$

$$V_g\rightarrow V_p,\ \text{when}\ \frac{\beta_x a}{2} \longrightarrow \frac{(n+1)\pi}{2}\ \text{or}\ \omega\rightarrow\infty \tag{73}$$

$$V_g<V_p,\ \text{when}\ \frac{n\pi}{2}<\frac{\beta_x a}{2}<\frac{(n+1)\pi}{2}\ \text{or}\ n\omega_{c1}<\omega<\infty \tag{74}$$

From Eqs. (64), (65), (44), and (41), the curves of V_g versus ω may be plotted with the parameter $(\beta_x a/2)$ in the range of Eq. (10)

$$\frac{n\pi}{2}\leq\frac{\beta_x a}{2}<\frac{(n+1)\pi}{2}, \qquad n=0,2,4,\cdots\cdots \tag{75}$$

The curves of V_g versus ω for even TM_{no} modes are shown in Fig. 3.

Till now, the properties of phase velocity and group velocity of the even TM_{no} modes in the slab guide, have been discussed. In the next section the properties of energy velocity will be discussed.

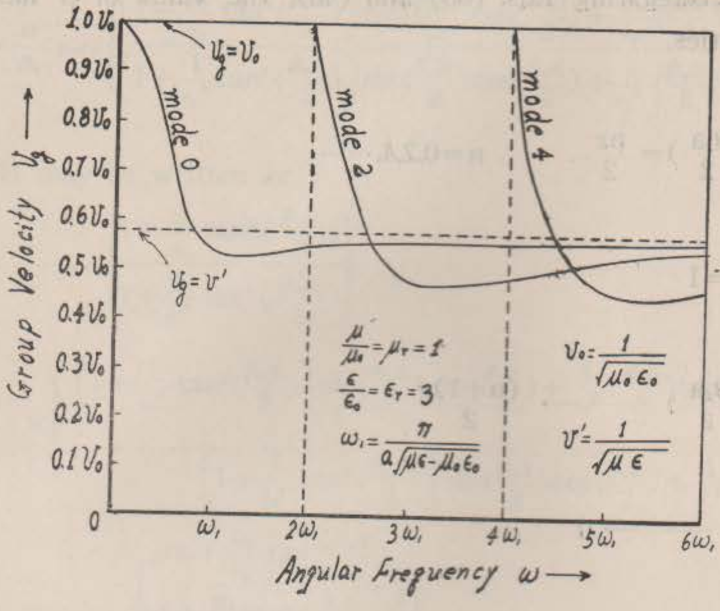

Fig. Group velocity versus frequency ω for even TM_{no} mode

VI. The Energy Velocity of Even TM_{no} Modes

From the field distribution as stated in the section II, we know that the real power flows in z-direction only. Thus the point relation of energy velocity[5] by definition may be written as:

$$Ve = \frac{\langle S_z \rangle}{\langle w_e \rangle + \langle w_m \rangle} \tag{76}$$

where:

$$\langle S_z \rangle = \frac{1}{2} R_e \ (E_x H_y^*) \tag{77}$$

the time-average tower density for TM_{no} modes

$$\langle w_e \rangle = \frac{\epsilon}{4}(E_x E_x^* + E_z E_z^*) \tag{78}$$

the time-average electric energy density for TM_{no} modes

$$\langle w_m \rangle = \frac{\mu}{4} H_y H_y^* \tag{79}$$

the time-average magnetic energy density for TM_{no} modes.

Now we discuss the even TM_{no} modes only.

For $x \geq \frac{a}{2}$, from Eqs. (6), (77), (78), and (79), then

$$\langle S_{z1}\rangle=\frac{H_o^2}{2}\cdot\frac{\beta_z}{\omega\epsilon_o}\cdot\cos^2\left(\frac{\beta_x a}{2}\right)e^{\alpha_x a-2\alpha_x x} \tag{80}$$

$$\langle w_{e1}\rangle=\frac{H_o^2}{4}\cdot\frac{(\beta_z^2+\alpha_x^2)}{\omega^2\epsilon_o}\cdot\cos^2\left(\frac{\beta_x a}{2}\right)e^{\alpha_x a-2\alpha_x x} \tag{81}$$

$$\langle w_{m1}\rangle=\frac{H_o^2}{4}\cdot\mu_o\cdot\cos^2\left(\frac{\beta_x a}{2}\right)e^{\alpha_x a-2\alpha_x x} \tag{82}$$

Combining (81) and (82) and using the relation

$\beta_z^2=\omega^2\mu_o\epsilon_o+\alpha_x^2$, we obtain

$$\langle w_{e1}\rangle+\langle w_{m1}\rangle=\frac{H_o^2}{2}\cdot\frac{\beta_z^2}{\omega^2\epsilon_o}\cdot\cos^2\left(\frac{\beta_x a}{2}\right)e^{\alpha_x a-2\alpha_x x} \tag{83}$$

Substituting Eqs. (80) and (82) into (76), then the energy velocity in region (I) will be:

$$V_{e1}=\frac{\omega}{\beta_z} \tag{84}$$

By comparison of (84) and (38), we find

$$V_{e1}=V_p \tag{85}$$

For $-\frac{a}{2}\leq x\leq\frac{a}{2}$, from Eqs. (7). (77), (78), and (79), we have:

$$\langle S_{z2}\rangle=\frac{H_o^2}{2}\ \frac{\beta_z}{\omega\epsilon}\cdot\cos^2(\beta_x x) \tag{86}$$

$$\langle w_{e2}\rangle=\frac{H_o^2}{4}\ \frac{1}{\omega^2\epsilon}\cdot\left[\beta_z^2\cos^2(\beta_x x)+\beta_x^2\sin^2(\beta_x x)\right] \tag{87}$$

$$\langle w_{m2}\rangle=\frac{H_o^2}{4}\cdot\mu\cdot\cos^2(\beta_x x) \tag{88}$$

Combining (87) and (88) and using the relation

$\beta_z^2=\omega^2\mu\epsilon-\beta_x^2$

gives

$$\langle w_{e2}\rangle+\langle w_{m2}\rangle=\frac{H_o^2}{2}\cdot\frac{\beta_z^2}{\omega^2\epsilon}\cdot\left[\cos^2(\beta_x x)+\frac{\beta_x^2}{2\beta_z^2}\right] \tag{89}$$

Substitution of (86) and (89) into (76), the energy velocity in region (II) willb e:

$$V_{e2}=\frac{\omega}{\beta_z}\ \frac{[\cos^2(\beta_x x)]}{\left[\cos^2(\beta_x x)+\frac{\beta_x^2}{2\beta_z^2}\right]} \tag{90}$$

Since $\frac{\beta_x^2}{\beta_z^2}$ is always positive,

therefore

$$V_{e2}<V_p \tag{91}$$

For $x\leq-\frac{a}{2}$, it is similar to that for $x\geq\frac{a}{2}$, it will suffice to list the equatioos below.

$$\langle S_{z3}\rangle=\frac{H_o^2}{2}\cdot\frac{\beta_z}{\omega\epsilon_o}\cdot\cos^2(\frac{\beta_x a}{2})e^{\alpha_x a+2\alpha_x x} \tag{92}$$

$$\langle w_{e3}\rangle=\frac{H_o^2}{4}\cdot\frac{(\beta_z^2+\alpha_x^2)}{\omega^2\epsilon_o}\cdot\cos^2(\frac{\beta_x a}{2})e^{\alpha_x a+2\alpha_x x} \tag{93}$$

$$\langle w_{m3}\rangle=\frac{H_o^2}{4}\cdot\mu_o\cdot\cos^2(\frac{\beta_x a}{2})e^{\alpha_x a+2\alpha_x x} \tag{94}$$

$$\langle w_{e3}\rangle+\langle w_{m3}\rangle=\frac{H_o^2}{2}\cdot\frac{\beta_z^2}{\omega^2\epsilon_o}\cdot\cos^2(\frac{\beta_x a}{2})e^{\alpha_x a+2\alpha_x x} \tag{95}$$

$$V_{e3}=V_p \tag{96}$$

From Eqs. (85), (91) and (96), we may find that the energy velocity in region (I) and (III) are equal to the phase velocity while smaller than the phase velocity in region (II).

But when we consider the energy velocity as the ratio of the spacial-average of $\langle S_z\rangle$ and $[\langle w_e\rangle+\langle w_m\rangle]$ along the x-direction, it follows:

$$V_e=\frac{\int_{-\infty}^{\infty}\langle S_z\rangle dx}{\int_{-\infty}^{\infty}[\langle w_e\rangle+\langle w_m\rangle]dx} \tag{97}$$

Both $\langle S_z\rangle$ and $[\langle w_e\rangle+\langle w_m\rangle]$ have different values in air and dielectric slab region, then

$$V_e=\frac{\int_{-\infty}^{-\frac{a}{2}}\langle S_{z1}\rangle dx+\int_{-\frac{a}{2}}^{\frac{a}{2}}\langle S_{z2}\rangle dx+\int_{\frac{a}{2}}^{\infty}\langle S_{z3}\rangle dx}{\int_{-\infty}^{-\frac{a}{2}}[\langle w_{e1}\rangle+\langle w_{m1}\rangle]dx+\int_{-\frac{a}{2}}^{\frac{a}{2}}[\langle w_{e2}\rangle+\langle w_{m2}\rangle]dx+\int_{\frac{a}{2}}^{\infty}[\langle w_{e3}\rangle+\langle w_{m3}\rangle]dx} \tag{98}$$

Substituting Eqs. of power density and energy density into Eq. (98) and integrating we obtain:

$$V_e=\frac{\omega}{\beta_z}\cdot\frac{(\frac{\alpha_x}{\beta_x})^2\left[(\frac{\beta_x a}{2})+\sin(\frac{\beta_x a}{2})\mathrm{con}(\frac{\beta_x a}{2})\right]+\epsilon_r\cdot(\frac{\alpha_x}{\beta_x}\cdot\cos^2\frac{\beta_x a}{2})}{(\frac{\alpha_x}{\beta_x})^2\left[(1+\frac{\beta_x^2}{\beta_z^2})(\frac{\beta_x a}{2})+\sin(\frac{\beta_x a}{2})\cos(\frac{\beta_x a}{2})\right]+\epsilon_r\cdot\frac{\alpha_x}{\beta_x}\cdot\cos^2(\frac{\beta_x a}{2})} \tag{99}$$

Substitution of Eqs. (9) and (42) into (99) and simplification yields.

$$V_e=\frac{\omega}{\beta_z}$$

$$\times\frac{\left[1+\frac{1}{\epsilon_r^2}\tan^2(\frac{\beta_x a}{2})\right]\sin(\frac{\beta_x a}{2})\cos(\frac{\beta_x a}{2})+\frac{1}{\epsilon_r^2}\cdot(\frac{\beta_x a}{2})\cdot\tan^2(\frac{\beta_x a}{2})}{\left[1+\frac{1}{\epsilon_r^2}\tan^2(\frac{\beta_x a}{2})\right]\sin(\frac{\beta_x a}{2})\cos(\frac{\beta_x a}{2})+\frac{1}{\epsilon_r^2}\cdot(\frac{\beta_x a}{2})\cdot\tan^2(\frac{\beta_x a}{2})\left[\frac{\mu_r\epsilon_r+\frac{\mu_r}{\epsilon_r}\tan^2(\frac{\beta_x a}{2})}{1+\frac{\mu_r}{\epsilon_r}\tan^2(\frac{\beta_x a}{2})}\right]} \tag{100}$$

By means of Eqs. (38) and (62), the Eq. (100) may be written as:

$$V_e=V_p\cdot\frac{1}{G} \tag{101}$$

By comparison of (101) and (64), we obtain:

$$V_e=V_g \tag{102}$$

From this result we may say that the total time-average power flowing along the slab guide is equal to the product of group velocity and the total time-average energy per unit length of the slab guide.

Since the energy velocity of the wave in the waveguide is to represent the energy propagation of the wave along the guide. Then Eq. (76) has no significance, when the energy velocity possesses different value in different regions. Thus the Eq. (98) will be the correct definition of energy velocity of a guided wave.

VII. Conclusion

The purpose of this paper according to Dr. L. J. Chu, is to prove whether the energy velocity is equal to the group velocity or not. The author carried out this work by computer programming with IBM 1620. The result was quite satisfactory. Later on, the author found a mathematical treatment of this problem. After a lengthy calculation it is shown that the energy velocity is certainly identical with group velocity.

During the process of the work, some properties of group velocity and phase velocity are worth being discussed: i.e. (1) the group velocity is less than or equal to the phase velocity, (2) the phase velocity is less than or equal to the velocity of light in air.

VIII. Acknowledgment

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