Matric Analysis of the Basic Uniformly Distributed R-C Networks

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Abstract: In this paper the distributed R-C network is discussed with emphasis on its matrix analysis and three different mathematical methods applying to infinitesimal section of a basic, uniformly distributed, R-C thin-film network are presented. The resultant matric forms are explicitly different. However, they can be found to be the same by putting some numerical check points into them.

1. INTRODUCTION

In the past few years, considerable emphasis has been placed on the field of "Microelecrronics". The main reason for considering this method of circuit construction is the reliability and the extreme size reduction it affords. The distributed R-C network offers a method of realizing a number of useful circutit functions by essentially a single integrated component.

A distributed R-C network is analogous to a uniform transmission line. An intuitive approach to a distributed R-C network is made by starting with the two-element lumped parameter R-C integrating circuit. If the number of series resistors and shunt capacitors is increased without limit in a ladder network such that the total series resistance and shunt capacitance remain invariant, the result is a network that consists of a large number of the same lumped R-C networks in series, and is a close approximation to a distributed R-C network.

The matric method used in this paper is essentially applied for the purpose of simplifying the manipulation of several systems of linear transformations in complicated duodinode networks. In particular, the transfermatrix is the most effective tool for the analysis of distributed

cascade networks.

In this paper, three different methods of matric analysis are applied to an infinitesimal section of a basic, uniformly distributed, R-C thin-film network, (i. e. one section of lumped R-C ladder network.) The resultant matric forms that are separately derived by "transformation to a digonal matrix", "Chebyshev polynomials", and "homogeneous matrix", are explicitly different. However, they can be found to be the same by putting some numerical check points into them.

2. ANALYSIS OF BASIC DISTRIBUTED R-C NETWORKS BY TRANSFORMATION TO DIAGONAL MATRICES

As mentioned in the introduction, the basic uniformly distributed R-C network can be analyzed by three different matric methods. The first one is a method by applying matric algebra to transform the transmission matrix of single section of the distributed R-C network to a diagonal form, and from which the formula for the gain of $\underline{n}$ identical sections of R-C network can easily be derived as a matric product. If the value of $\underline{n}$ is assumed to be a large number, it means that the distributed R-C ladder is very much like the thin-film distribsted R-C network. The mathematic procedure is presented as follows.

Assumming $\underline{n}$ sections cascade L type R-C network, as shown in Figure 2-1, where R_1 and C_1 are elements of single L type section, $\underline{n}$ is a definite positive large number.

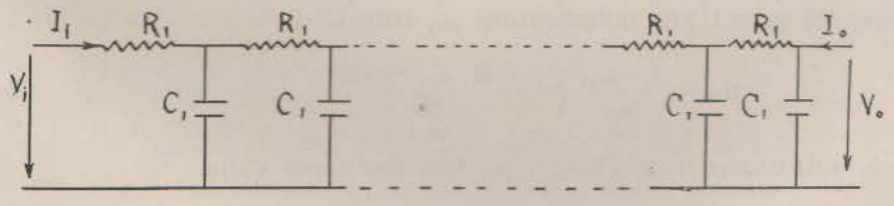


Figure 2-1. Basic Distributed R-C Ladder

In single section, the transmission matrix is

$$[{}^1a]=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix}1 & R_1\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ CS & 1\end{bmatrix}=\begin{bmatrix}1+R.C.S & R_1\\ C.S & 1\end{bmatrix} \tag{2-1}$$

and $$|[{}^1a]| = 1 \tag{2-2}$$

Because $[a^1]$ has two linearly independent invariant vectors, $[{}^1a]$ can be transformed to a diagonal matrix as

$$[P]^{-1}\,[{}^1a][P]=\text{Diag.}(\lambda_1\ \lambda_2) \tag{2-3}$$

Now, the roots of the characteristic equation are given

by $$\left|\begin{bmatrix}1+R_1C_1S-\lambda & R_1\\ C_1S & 1-\lambda\end{bmatrix}\right|=0 \tag{2-4}$$

$$\lambda_1=\frac{2+R_1\ C_1\ S+R'}{2} \tag{2-5}$$

$$\lambda_2=\frac{2+R_1\ C_1\ S-R'}{2} \tag{2-6}$$

where $$R'=\sqrt{4R_1\ C_1\ S+(R_1C_1S)^2}$$

Let $$[{}^1a]\ [P]=[P]\ [\text{Diag.}]$$

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}=\begin{bmatrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix} \tag{2-7}$$

$$\begin{bmatrix}a_{11}\ p_{11}+a_{12}\ p_{21} & a_{11}\ p_{12}+a_{22}\ p_{22}\\ a_{21}\ p_{11}+a_{22}\ p_{21} & a_{21}\ p_{12}+a_{12}\ p_{22}\end{bmatrix}=\begin{bmatrix}p_{11}\ \lambda_1 & p_{12}\ \lambda_2\\ p_{21}\ \lambda_1 & p_{22}\ \lambda_2\end{bmatrix} \tag{2-8}$$

which gives $$a_{11}\ p_{11}+a_{12}\ p_{21}=p_{11}\ \lambda_1 \tag{2-9}$$

$$a_{11}\ p_{12}+a_{12}\ p_{22}=p_{12}\ \lambda_2 \tag{2-10}$$

$$a_{21}\ p_{11}+a_{22}\ p_{21}=p_{21}\ \lambda_1 \tag{2-11}$$

$$a_{21}\ p_{12}+a_{22}\ p_{22}=p_{22}\ \lambda_2 \tag{2-12}$$

Now let $p_{11}=1$ and substituting p_{11} into (2-9)

$$p_{21}=\frac{\lambda_1-a_{11}}{a_{12}}=\frac{-R_1\ C_1\ S+R'}{2R_1} \tag{2-13}$$

If p_{11} is substituted into (2-11), p_{21} has the same value

$$p_{21}=\frac{a_{21}}{\lambda_1-a_{22}}=\frac{-R_1\ C_1\ S+R'}{2R_1}$$

Let $p_{12}=1$ and substituting p_{12} into (2-10)

$$p_{22}=\frac{\lambda_2-a_{11}}{a_{12}}=\frac{-R_1\,C_1\,S-R'}{2R_1} \tag{2-14}$$

Returning to the matrix equation (2-7)

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}=\begin{bmatrix}1 & 1\\ p_{21} & p_{22}\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}\begin{bmatrix}1 & 1\\ p_{21} & p_{22}\end{bmatrix}^{-1}$$

$$=\frac{1}{p_{22}-p_{21}}\begin{bmatrix}p_{22}\,\lambda_1-p_{21}\,\lambda_2 & \lambda_2-\lambda_1\\ p_{21}\,p_{22}\,(\lambda_1-\lambda_2) & p_{22}\,\lambda_2-p_{21}\,\lambda_1\end{bmatrix} \tag{2-15}$$

In $\underline{n}$ sections

$$[{}^0a]=[{}^1a]\,[{}^1a]\cdots[{}^1a]=[{}^1a]^n \tag{2-16}$$

By linear transformation (similar method as applying in single section)

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}^n=\begin{bmatrix}1 & 1\\ p_{21} & p_{22}\end{bmatrix}\begin{bmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{bmatrix}^n\begin{bmatrix}1 & 1\\ p_{21} & p_{22}\end{bmatrix}^{-1}$$

$$=\begin{bmatrix}1 & 1\\ p_{21} & p_{22}\end{bmatrix}\begin{bmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{bmatrix}\begin{bmatrix}1 & 1\\ p_{21} & p_{22}\end{bmatrix}^{-1}$$

$$=\frac{1}{p_{22}-p_{21}}\begin{bmatrix}p_{22}\,\lambda_1^n-p_{21}\,\lambda_2^n & \lambda_2^n-\lambda_1^n\\ p_{21}\,p_{22}\,(\lambda_1^n-\lambda_2^n) & p_{22}\,\lambda_2^n-p_{21}\,\lambda_1^n\end{bmatrix} \tag{2-17}$$

Substituting p_{22}, p_{21}, λ_1 and λ_2 into (2-17)

$$\begin{bmatrix}1+R_1\,C_1\,S, R_1\\ SC_1 \quad 1\end{bmatrix}^n=\frac{-R_1}{R'}$$

$$\begin{bmatrix}-\frac{R_1\,C_1\,S+R'}{2R_1}\left(\frac{2+R_1\,C_1\,S+R'}{2}\right)^n+\frac{R_1\,C_1\,S-R'}{2R_1}\left(\frac{2+R_1\,C_1\,S-R'}{2}\right)^n & \left(\frac{2+R_1\,C_1\,S-R'}{2}\right)^n-\left(\frac{2+R_1\,C_1\,S+R'}{2}\right)^n\\ \left(\frac{R_1\,C_1\,S-R'}{2R_1}\right)\left(\frac{R_1\,C_1\,S+R'}{2R_1}\right)\left\{\left(\frac{2+R_1\,C_1\,S+R'}{2}\right)^n-\left(\frac{2+R_1\,C_1\,S-R'}{2}\right)^n\right\} & -\frac{R_1\,C_1\,S+R'}{2R_1}\left(\frac{2+R_1\,C_1\,S+R'}{2}\right)^n+\frac{R_1\,C_1\,S-R'}{2R_1}\left(\frac{2+R_1\,C_1\,S-R'}{2}\right)^n\end{bmatrix}$$

(2-18)

For special case S=0, it means that a zero-frequency or D. C. source is applied to the distributed network

$$R'=0,$$

$$p_{22}=p_{21}=0$$

$$\lambda_1=\lambda_2=1$$

Substituitng all these values, gives the transmission matrix at zero frequency

$$\begin{bmatrix} 1+R_1\ C_1\ S & R_1 \\ SC_1 & 1 \end{bmatrix}=\begin{bmatrix} 1 & nR_1 \\ 0 & 1 \end{bmatrix}_{s=0} \tag{2-19}$$

3. ANALYSIS OF BASIC DISTRIBUTED R-C NETWORKS BY CHEBYSHEV POLYNOMIALS

This is a second method that is applied to analyze distributed R-C networks by using matric techniques and Chebyshev functions. First assume that the network consists of n identical R-C sections. The formula for the overall gain of the distributed R-C network can be derived in matric form in which the matric elements are in summation series form. If the value of n is a finite integer, the elements of the matric formula can easily be solved as the sum of the finite series. The mathematical procedure is presented as follows.

Consider the R-C chain of n identical L section, as shown in Figure 2-1. Each section has the transfer matrix, from equation (2-1)

$$[{}^1a]=\begin{bmatrix} 1+R_1\ C_1\ S_1 & R_1 \\ SC_1 & 1 \end{bmatrix} \tag{3-1}$$

Hence the whole chain has the transfer matrix

$$[{}^0a]=[{}^1a]^n=f([{}^1a]) \tag{3-2}$$

By Sylvester's interpolation formula, (8) (9), if a square matrix $[{}^1a]$ has order m and distinct latent roots $\lambda_1\ \lambda_2 \cdots \lambda_m$,

then

$$f([{}^1a])=\sum_{i=1}^{m} f(\lambda_i) \prod_{\substack{1\le j\le m \\ j\neq i}} \frac{[{}^1a]-\lambda_j I}{\lambda_i-\lambda_j} \tag{3-3}$$

where I is the unit matrix of of order m. Thus the transfer matrix $[^0a]$ of the R-C chain has the representation:

$$[^0a]=[^1a]^n=f([^1a])=f(\lambda_1)\frac{[^1a]-\lambda_2 I}{\lambda_1-\lambda_2}+f(\lambda_2)\frac{[^1a]-\lambda_1 I}{\lambda_2-\lambda_1}$$

$$=\frac{-\lambda_1^n([^1a]-\lambda_2 I)+\lambda_2^n([^1a]-\lambda_1 I)}{\lambda_2-\lambda_1}$$

$$=\frac{[^1a](\lambda_2^n-\lambda_1^n)-\lambda_1\lambda_2(\lambda_2^{n-1}-\lambda_1^{n-1})\ I}{\lambda_2-\lambda_1} \qquad (3\text{-}4)$$

The trace of $[^1a]$ is $(1+R_1\ C_1\ S)$ and the determinant of $[^1a]$ is unity. Both the trace and $|[^1a]|$ are preserved under a similarity transformation which converts $[^1a]$ into the diagonal matrix with elements λ_1 and λ_2.

Thus $$\lambda_1+\lambda_2=2+R_1\ C_1\ S \qquad (3\text{-}5)$$

$$\lambda_1\ \lambda_2=1 \qquad (3\text{-}6)$$

Let $$\lambda_1=e^{j\theta} \quad \text{and} \quad \lambda_2=e^{-j\theta} \qquad (3\text{-}7)$$

Substituting (3-7) into (3-4) gives

$$[^0a]=\frac{[^1a]\ (e^{-jn\theta}-e^{jn\theta})-e^{j\theta}\ e^{-j\theta}\ (e^{-j(n-1)}-j^{(n-1)\theta})\ I}{e^{-j\theta}-e^{j\theta}}$$

$$=\frac{[^1a]\ \mathrm{Sin}\ (n\theta)-I\mathrm{Sin}(n-1)\theta}{\mathrm{Sin}\theta} \qquad (3\text{-}8)$$

Substituting (3-7) into(3-5) gives

$$2+R_1\ C_1\ S=2\mathrm{Cos}\theta \qquad (3\text{-}9)$$

placing $$\mu=\mathrm{Cos}\theta \qquad (3\text{-}10)$$

then $$\frac{\mathrm{Sin}(n\theta)}{\mathrm{Sin}\theta}=\frac{\mathrm{Sin}(n\theta)}{\sqrt{1-\mathrm{Cos}^2\theta}}=\frac{\mathrm{Sin}(n\ \mathrm{Cos}^{-1}\ \mu)}{\sqrt{1-\mu^2}}=U_{n-1}(\mu) \qquad (3\text{-}11)$$

where $U_{n-1}(\mu)$ is the Chebyshev function of the second kind (10). Thus equation (3-8) has the representation

$$[^0a]=[^1a]\ U_{n-1}(\mu)-IU_{n-2}(\mu)$$

$$[{}^{0}a]=\begin{bmatrix}(1+R_1 C_1 S)\,U_{n-1}(\mu)-U_{n-2}(\mu) & R_1 U_{n-1}(\mu)\\ C_1 S U_{n-1}(\mu) & U_{n-1}(\mu)-U_{n-2}(\mu)\end{bmatrix} \tag{3-12}$$

Since $$\mu=1+\frac{R_1 C_1 S}{2} \tag{3-13}$$

the Chebyshev function can be represented in the form:
(see appendix)

$$U_{n-1}(\mu)=\sum_{r=0}^{n-1}\binom{n+r}{2r+1}2^r(\mu-1)^r \tag{3-14}$$

Substituting (3-13) into (3-14), gives

$$U_{n-1}(\mu)=\sum_{r=0}^{n-1}\binom{n+r}{2r+1}R_1^r(C_1 S)^r \tag{3-15}$$

All elements of matrix equation (3-12) can be arranged in the series form as equation (3-15).

Where follows

$$\begin{aligned}U_{n-1}(\mu)-U_{n-2}(\mu)&=\sum_{r=0}^{n-1}\binom{n+r}{2r+1}R_1^r(C_1 S)^r\\ &\quad-\sum_{r=0}^{n-2}\binom{n+r-1}{2r+1}R_1^r(C_1 S)^r\\ &=\sum_{r=0}^{n-2}\left\{\binom{n+r}{2r+1}-\binom{n+r-1}{2r+1}\right\}R_1^r(C_1 S)^r\\ &\quad+R_1^{n-1}(C_1 S)^{n-1}\\ &=\sum_{r=0}^{n-2}\binom{n+r-1}{2r}R_1^r(C_1 S)^r+R_1^{n-1}(C_1 S)^{n-1}\end{aligned}$$

$$= \sum_{r=0}^{n-1} \binom{n+r-1}{2r} R_1^r (C_1 S)^r \tag{3-16}$$

Furthermore

$$(1+R_1 C_1 S)\, U_{n-1}(\mu) - U_{n-2}(\mu) = \sum_{r=0}^{n-1} \binom{n+r-1}{2r} R_1^r (C_1 S)^r + \sum_{r=0}^{n-1} \binom{n+r}{2r+1} R_1^{r+1} (C_1 S)^{r+1}$$

$$= 1 + R_1^n (C_1 S)^n + \sum_{r=0}^{n-1} \binom{n+r-1}{2r} R_1^r (C_1 S)^r + \sum_{r=0}^{n-2} \binom{n+r}{2r+1} R_1^{r+1} (C_1 S)^{r+1}$$

$$= 1 + R_1^n (C_1 S)^n + \sum_{r=0}^{n-2} \binom{n+r+1}{2r+2} R_1^{r+1} (C_1 S)^{r+1}$$

$$= \sum_{r=0}^{n} \binom{n+r}{2r} R_1^r (C^1 S)^r \tag{3-17}$$

By substituting equations (3-15), (3-16), and (3-17) into (3-12), the explicit form of the transfer matrix of $\underline{n}$ identical bilateral L section of R-C network is obtained as

$$[{}^0a] = \begin{bmatrix} \sum_{r=0}^{n} \binom{n+r}{2r} R^r (C_1 S)^r & \sum_{r=0}^{n-1} \binom{n+r}{2r+1} R_1^{r+1} (C_1 S)^r \\ \sum_{r=0}^{n-1} \binom{n+r}{2r+1} R_1^r (C_1 S)^{r+1} & \sum_{r=0}^{n-1} \binom{n+r-1}{2r} R_1^r (C_1 S)^r \end{bmatrix} \tag{3-18}$$

where

$$ {}^{0}a_{11}=\sum_{r=0}^{n}\frac{(n+r)!}{(2r)!\ (n-r)!}R_1^{r}\ (C_1\ S)^{r} $$

$$ =1+\frac{(n+1)!}{2!\ (r-1)!}R_1\ C_1\ S+\frac{(n+2)!}{4!\ (n-2)!}(R_1\ C_1\ S)^{2}+\cdots $$

$$ =1+\frac{(n+1)\ n}{2!}R_1\ C_1\ S+\frac{(n+2)(n+1)\ n\ (n-1)}{4!}(R_1\ C_1\ S)^{2}+\cdots $$

$$ {}^{0}a_{12}=\sum_{r=0}^{n-1}\frac{(n+r)!}{(2r+1)!\ (n-r-1)!}R_1^{r+1}\ (C_1\ S)^{r} $$

$$ =n\ R_1+\frac{(n+1)\ n\ (n-1)}{3!}R_1^{2}\ (C_1\ S) $$

$$ +\frac{(n+2)\ (n+1)\ n\ (n-1)\ (n-2)}{5!}R_1^{3}(C_1\ S)^{2}+\cdots $$

$$ {}^{0}a_{21}=\sum_{r=0}^{n-1}\frac{(n+r)!}{(2r+1)!\ (n-r-1)!}R_1^{r}\ (C_1\ S)^{r+1} $$

$$ =n\ C_1\ S+\frac{(n+1)\ n\ (n-1)}{3!}R_1\ (C_1\ S)^{2} $$

$$ +\frac{(n+2)\ (n+1)\ n\ (n-1)\ (n-2)}{5!}R_1^{2}\ (C_1\ S)^{3}+\cdots $$

$$ {}^{0}a_{22}=\sum_{r=0}^{n-1}\frac{(n+r-1)!}{(2r)!\ (n-r-1)!}R_1^{r}\ (C_1\ S)^{r} $$

$$ =1+\frac{n\ (n-1)}{2!}R_1C_1\ S+\frac{(n+1)\ n\ (n-1)\ (n-2)}{4!}(R_1\ C_1\ S)^{2}+\cdots $$

with this matrix form in a special case when $S\to 0$

$$ [{}^{0}a]=\begin{bmatrix}{}^{0}a_{11} & {}^{0}a_{12}\\ {}^{0}a_{21} & {}^{0}a_{22}\end{bmatrix}=\begin{bmatrix}1 & nR_1\\ 0 & 1\end{bmatrix} \qquad (3\text{-}19) $$

4. ANALYSIS OF BASIC DI STRIBUTED R-C NETWORKS BY HOMOGENEOUS MATRICES

4. 1 ANALYSIS OF BASIC NETWORK

At first, a more general basic network, as shown in Figure 4-1, is considered. That is the equivalent circuit of an elemental section of the basic R-C-NR network. The differential equations of line to line at the two ends of an elemental section can be written as

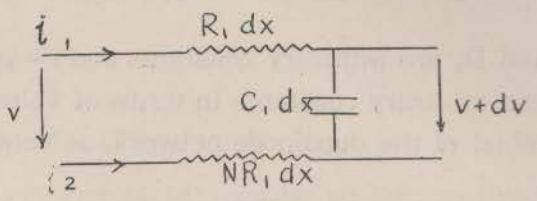


Figure 4-1. Elemental Section of the Basic R-C-NR Network

$$-\frac{dv}{dx}=i_1 R_1-i_2 NR_1 \tag{4-1}$$

differential (4−1)

$$\frac{d^2v}{dx^2}=NR_1\frac{di_2}{dx}-R_1\frac{di_1}{dx} \tag{4-2}$$

since
$$-\frac{\partial i_1}{\partial x}dx=C_1\,dx\frac{\partial v}{\partial x}=j\omega\,C_1\,vdx \tag{4-3}$$

$$\frac{di_1}{dx}=-j\omega\,C_1\,v \tag{4-4}$$

and
$$\frac{di_2}{dx}=\frac{di_1}{dx}=j\omega\,C_1\,v \tag{4-5}$$

Substituting (4-4), and (4-5) into (4-2), gives

$$\frac{d^2v}{dx^2}=j\omega\,R_1\,C_1\,(1+N)\,v \tag{4-6}$$

where ω is the angular frequency of the impressed sinusoid, R_1 and C_1 are

the resistances and capacitances respectively of distributed (thin films) R-C network per unit length and per unit width, It should be noted that equations (4-4) and (4-6) are diffusion equations in the thin films, so that the voltage and current diffuse into the network from the input terminals.

The solutions to equations (4-4) and (4-6) are

$$v = Ae^{rx} + Be^{-rx} \tag{4-7}$$

$$i_1 = \frac{-j\omega C_1}{r}(Ae^{rx} - Be^{-rx}) + D_1 \tag{4-8}$$

and

$$i_2 = \frac{j\omega C_1}{r}(Ae^{rx} - Be^{-rx}) + D_2 \tag{4-9}$$

where A, B, D_1, and D_2 are arbitrary constants and $r = \sqrt{j\omega (1+N) R_1 C_1}$

Evaluating these arbitrary constants in terms of voltages and currents applied to the terminal of the duodinode network, as shown in Figure 4-2,

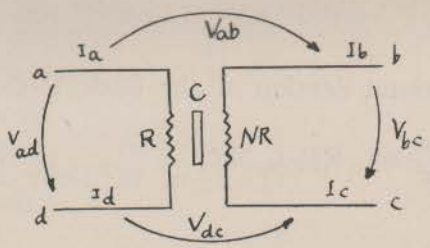


Figure 4-2. General Duodinode R-C-NR Network on the Y−Basis

putting $x=0$, gives

$$V_{dc} = A + B \tag{4-10}$$

putting $x=L$, gives

$$V_{ab} = Ae^{\theta} + Be^{-\theta} \tag{4-11}$$

where $e = rL = \sqrt{j\omega (1+N) R_1 C_1}$, L is the total length of the films,
$R = R_1 L$ is the total resistance of the resistive films; and
$C = C_1 L$ is the total capacitance between resistive films.

From equation (4-1)

$$V_{ad} = \int_L^0 i_1 R_1 \, dx = \frac{-j\omega C}{r} \int_L^0 R_1 (Ae^{rx} - Be^{-rx}) \, dx + R_1 D_1 \int_L^0 dx$$

$$=\frac{1}{1+N}\{A\ (e^{\theta}-1)+B(e^{-\theta}-1)\}+G_1 \qquad (4\text{-}12)$$

where $G_1=-R_1\ D_1\ L=-RD$, is an arbitrary constant.

Similarly

$$V_{bc}=\int_{L}^{0} i_2\,NR_1\,dx=\frac{j\omega\,C_1R_1N}{r}\int_{L}^{0}(Ae^{rx}-Be^{-rx})dx+NR_1D_2\int_{L}^{0}dx$$

$$=\frac{N}{1+N}\{A\ (1-e^{\theta})+B(1-e^{-\theta})\}+G_2 \qquad (4\text{-}13)$$

where $G_2=-R_1\ D_2\ NL=-D_2\ NR$

$$\text{If } \frac{dv}{dx}\to 0 \quad \frac{i_1}{i_2}=N \quad \text{i. e. } \frac{D_1}{D_2}=N$$

so $G_1=G_2=G$

Equations (4-12) and (4-13) can be written as the forms

$$V_{ad}=\frac{1}{1+N}\{A(e^{\theta}-1)+B(e^{-\theta}-1)\}+G \qquad (4\text{-}12.\ 1)$$

$$V_{bc}=\frac{N}{1+N}\{A\ (1-e^{\theta})+B(1-e^{-\theta})\}+G \qquad (4\text{-}13.\ 1)$$

The same method can be used to get the the terminal currents

at $x=L,\ I_a=-i_1=\frac{j\omega\,C_1}{r}(Ae^{rL}-Be^{-rL})-D_1$

$$=\frac{\theta}{(1+N)\ R}(Ae^{\theta}-Be^{-\theta})+\frac{G}{R} \qquad (4-14)$$

at $x=L,\ I_b=-i_2=\frac{-j\omega\ C_1}{r}(Ae^{rL}-Be^{-rL})-D_2$

$$=\frac{-\theta}{(1+N)\ R}(Ae^{\theta}-Be^{-\theta})+\frac{G}{NR} \qquad (4\text{-}15)$$

at $x=0,\ I_c=i_2=\frac{j\omega\ C_1}{r}(A-B)+D_2$

$$=\frac{\theta}{(1+N)\ R}(A-B)-\frac{G}{NR} \qquad (4\text{-}16)$$

at $$x=0,\ I_d=i_1=-\frac{j\omega C_1}{r}(A-B)+D_1$$

$$=\frac{-\theta}{(1+N)R}(A-B)-\frac{G}{R} \tag{4-17}$$

4. 2 HOMOGENEOUS MATRIX FORM OF SOLUTION

Equations (4-10) to (4-17) give the general solution for the basic duodinode networks in terms of the constants A, B, and G. All the constants can be eliminated to obtain the terminal currents in terms of the terminal voltages.

From equations (4-10) and (4-11)

$$A=\frac{V_{ab}-e^{-\theta}V_{dc}}{2\sinh\theta} \tag{4-18}$$

and $$B=\frac{e^{\theta}V_{dc}-V_{ab}}{2\sinh\theta} \tag{4-19}$$

From equation (4-12)

$$G=V_{ad}-\frac{1}{1+N}\left\{\frac{(V_{ab}-e^{-\theta}V_{dc})(e^{\theta}-1)}{2\sinh\theta}+\frac{(e^{\theta}V_{dc}-V_{ab})(e^{-\theta}-1)}{2\sinh\theta}\right\}$$

$$=\frac{NV_{ad}+V_{bc}}{1+N} \tag{4-20}$$

Substituting all arbitrary constants A, B, and G into (4-14), (4-15), (4-16), and (4-17), there results

$$I_a=\frac{\theta}{(1+N)R}\left\{\frac{V_{ab}e^{\theta}-V_{dc}}{2\sinh\theta}-\frac{V_{dc}-e^{-\theta}V_{ab}}{2\sinh\theta}\right\}+\frac{NV_{ad}+V_{bc}}{(1+N)R}$$

$$=\frac{\theta}{(1+N)R}\left\{\frac{V_{ab}(e^{\theta}+e^{-\theta})-2V_{dc}}{2\sinh\theta}\right\}+\frac{NV_{ad}+V_{bc}}{(1+N)R}$$

$$=\frac{1}{(1+N)R}\left\{\frac{\theta V_{ab}}{\tanh\theta}-\frac{\theta V_{dc}}{\sinh\theta}+NV_{ad}+V_{bc}\right\}$$

$$=\frac{1}{(1+N)R}\left\{\left(\frac{\theta}{\tanh\theta}+N\right)V_a+\left(1-\frac{\theta}{\tanh\theta}\right)\right.$$

$$\left.V_b+\left(\frac{\theta}{\sinh\theta}-1\right)V_c+\left(\frac{-\theta}{\sinh\theta}-N\right)V_d\right\} \tag{4-21}$$

Similarly,

$$I_b=\frac{1}{(1+N)R}\left\{\left(1-\frac{\theta}{\tanh\theta}\right)V_a+\left(\frac{\theta}{\tanh\theta}+\frac{1}{N}\right)V_b\right.$$

$$+\left(\frac{-\theta}{\sinh\theta}-\frac{1}{N}\right)V_c+\left(\frac{\theta}{\sinh\theta}-1\right)V_d\Big\}$$

(4-22)

$$\cdot_c=\frac{1}{(1+N)R}\Big\{\left(\frac{\theta}{\sinh\theta}-1\right)V_a+\left(\frac{-\theta}{\sinh\theta}-\frac{1}{N}\right)V_b$$

$$+\left(\frac{\theta}{\tanh\theta}+\frac{1}{N}\right)V_c+\left(\frac{-\theta}{\tanh\theta}+1\right)V_d\Big\}$$

(4-23)

$$I_d=\frac{1}{(1+N)R}\Big\{\left(\frac{-\theta}{\sinh\theta}-N\right)V_a+\left(\frac{\theta}{\sinh\theta}-1\right)V_b$$

$$+\left(\frac{-\theta}{\tanh\theta}+1\right)V_c+\left(\frac{\theta}{\tanh\theta}+N\right)V_d\Big\}$$

(4-24)

Compact equations (4-21), (4-22), (4-23), and (4-24) in the homogeneous admittance matrix equation as (4-25)

$$\begin{pmatrix} I_a \\ I_b \\ I_c \\ I_d \end{pmatrix}=\frac{1}{(1+N)R}\begin{pmatrix} \frac{\theta}{\tanh\theta}+N & 1-\frac{\theta}{\tanh\theta} & \frac{\theta}{\sinh\theta}-1 & -\frac{\theta}{\sinh\theta}-N \\ 1-\frac{\theta}{\tanh\theta} & \frac{\theta}{\tanh\theta}+\frac{1}{N} & -\frac{\theta}{\sinh\theta}-\frac{1}{N} & \frac{\theta}{\sinh\theta}-1 \\ \frac{\theta}{\sinh\theta}-1 & -\frac{\theta}{\sinh\theta}-\frac{1}{N} & \frac{\theta}{\tanh\theta}+\frac{1}{N} & -\frac{\theta}{\tanh\theta}+1 \\ \frac{-\theta}{\sinh\theta}-N & \frac{\theta}{\sinh\theta}-1 & 1-\frac{\theta}{\tanh\theta} & \frac{\theta}{\tanh\theta}+N \end{pmatrix}\begin{pmatrix} V_a \\ V_b \\ V_c \\ V_d \end{pmatrix}$$

(4-25)

The advantage of the homogeneous admittance matrix is that it makes it possible to treat all terminals of the duodinode on an equal basis. It also can simplify the derivation of parameters for particular configurations.

For certain configurations an impedence basis rather than an admittance basis facilitates the derivation. The homogeneous impedance matrix equation can be derived from equations (4-10), (4-11), (4-12), and (4-13). All the currents and voltages coordinates are shown in Figure 4-3.

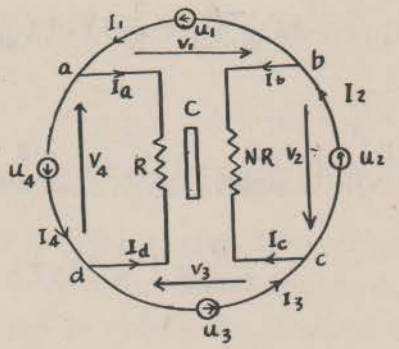


Figure 4-3. General Duodinode R-C-NR Network on the Z-Basis

Comparing the coordinates in Figure 4-2 and Figure 4-3, the relations of voltages and relations of currents are:

$$V_1=V_{ab},\quad V_2=V_{bc},\quad V_3=V_{cd},\quad V_4=V_{da} \tag{4-26}$$

$$\left.\begin{aligned} I_a&=I_1-I_4 & I_c&=I_3-I_2\\ I_b&=I_2-I_1 & I_d&=I_4-I_3 \end{aligned}\right\} \tag{4-27}$$

From the production of equation (4-6) and $e^{-\theta}$, the result added to equation (4-15). Thus

$$A=\frac{-(1+N)R}{2\theta\ \sinh\theta}\left\{-I_1+I_2(1-e^{-\theta}-\frac{1-e^{-\theta}}{1+N})+I_3e^{-\theta}+I_4\frac{(1-e^{-\theta})}{1+N}\right\} \tag{4-28}$$

From the production of equation (4-16) and e^{θ}, the result added to equation (4-15), there is obtained

$$B=\frac{-(1+N)R}{2\theta\ \sinh\theta}\left\{-I_1+I_2(1-e^{\theta}-\frac{1-e^{\theta}}{1+N})+I_3e^{\theta}+I_4\frac{(1-e^{\theta})}{1+N}\right\} \tag{4-29}$$

Let equation (4-16) and to equation (4-17)

then $$G=\frac{-RN}{1+N}(I_c+I_d)=\frac{RN}{1+N}(I_2-I_4) \tag{4-30}$$

Now, substituting all arbitrary constants A, B, and G in terms of currents I_1, I_2, I_3, and I_4 into equations (4-26) and (4-10) to (4-13)

$$V_1=V_{ab}=Ae^{\theta}+Be^{-\theta}$$

$$=\frac{-(1+N)R}{2\theta\sinh\theta}\left\{-I_1(e^{\theta}+e^{-\theta})+I_2(-2+e^{\theta}+e^{-\theta}-\frac{e^{\theta}+e^{-\theta}-2}{1+N})\right.$$

$$\left.+2I_3+I_4\frac{e^{\theta}+e^{-\theta}-2}{1+N}\right\}$$

$$=\frac{R}{\theta}\left\{\frac{1+N}{\tanh\theta}I_1-(N\tanh\frac{\theta}{2})I_2-\frac{1+N}{\sinh\theta}I_3-(\tanh\frac{\theta}{2})I_4\right\}$$

(4-31)

Simiiarly

$$V_2=\frac{R}{\theta}\left\{-(N\tanh\frac{\theta}{2})I_1+\frac{N}{1+N}(\theta+2N\tanh\frac{\theta}{2})I_2\right.$$

$$\left.-(N\tanh\frac{\theta}{2})I_3-\frac{N}{1+N}(\theta-2\tanh\frac{\theta}{2})I_4\right\} \quad (4\text{-}32)$$

$$V_3=\frac{R}{\theta}\left\{-\frac{1+N}{\sinh\theta}I_1-(N\tanh\frac{\theta}{2})I_2+\frac{1+N}{\tanh\theta}I_3-(\tanh\frac{\theta}{2})I_4\right\}$$

(4-33)

$$V_4=\frac{R}{\theta}\left\{-(\tanh\frac{\theta}{2})I_1-\frac{N}{1+N}(\theta-2\tanh\frac{\theta}{2})I_2-(\tanh\frac{\theta}{2})I_3\right.$$

$$\left.+\frac{1}{1+N}(N\theta+2\tanh\frac{\theta}{2})I_4\right\} \quad (4\text{-}34)$$

Compact equations (4-31) to (4-34) in the homogeneous impedance matrix equation as (4-35)

$$\begin{pmatrix}V_1\\V_2\\V_3\\V_4\end{pmatrix}=\frac{R}{\theta}\begin{pmatrix}\frac{1+N}{\tanh\theta} & -N\tanh\frac{\theta}{2} & -\frac{1+N}{\sinh\theta} & -\tanh\frac{\theta}{2}\\ -N\tanh\frac{\theta}{2} & \frac{N}{1+N}(\theta+2N\tanh\frac{\theta}{2}) & -N\tanh\frac{\theta}{2} & -\frac{N}{1+N}(\theta-2\tanh\frac{\theta}{2})\\ -\frac{1+N}{\sinh\theta} & -N\tanh\frac{\theta}{2} & \frac{1+N}{\tanh\theta} & -\tanh\frac{\theta}{2}\\ -\tanh\frac{\theta}{2} & -\frac{N}{1+N}(\theta-2\tanh\frac{\theta}{2}) & -\tanh\frac{\theta}{2} & \frac{1}{1+N}(N\theta+2\tanh\frac{\theta}{2})\end{pmatrix}\begin{pmatrix}I_1\\I_2\\I_3\\I_4\end{pmatrix}$$

(4-35)

4. 3 BASIC DISTRIBUTED R-C NETWORK

Now, applying the homogeneous impedance matrix equation (4-35) to

derive the Z matrix equation of basic uniformly distributed R-C network, as shown in Figure 4-4 (a), (b), and (c).

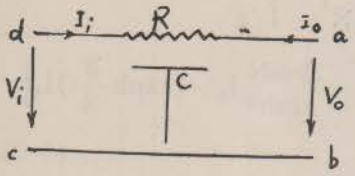


Figure 4-4 (a) A Simple R-C Network of Thin-film

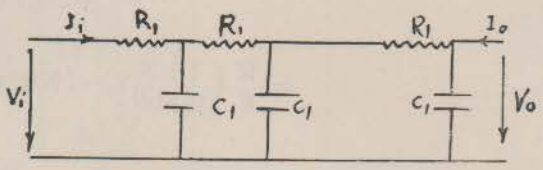


Figure 4-4 (b) An Equivalent Network of Figure 4-4 (a)

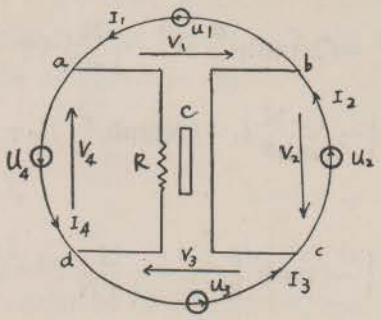


Figure 4-4(c) A Common-mesh Symmetic Network as Figure 4-3 but N=0

Comparing Figure 4-4(a) and Figure 4-4(c), the connection matrix between the voltages of Figure 4-4 (a)'s network and Figure 4-4(c)'s network is

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} \tag{4-36}$$

Similarly the connection matrix for currents is

$$\begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_i \\ I_o \end{pmatrix} \tag{4-37}$$

From equation (4-35), and putting N=0

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = \frac{R}{\theta} \begin{pmatrix} \frac{1}{\tanh\theta} & 0 & -\frac{1}{\sinh\theta} & -\tanh\frac{\theta}{2} \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\sinh\theta} & 0 & \frac{1}{\tanh\theta} & -\tanh\frac{\theta}{2} \\ -\tanh\frac{\theta}{2} & 0 & -\tanh\frac{\theta}{2} & 2\tanh\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} \tag{4-38}$$

So, by transformations, the matrix equation can be written as the following expressions:

$$\begin{pmatrix} V_1 \\ V_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \frac{R}{\theta} \begin{pmatrix} \frac{1}{\tanh\theta} & 0 & -\frac{1}{\sinh\theta} & -\tanh\frac{\theta}{2} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\sinh\theta} & 0 & \frac{1}{\tanh\theta} & -\tanh\frac{\theta}{2} \\ -\tanh\frac{\theta}{2} & 0 & -\tanh\frac{\theta}{2} & 2\tanh\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_1 \\ I_0 \end{pmatrix}$$

$$= \frac{R}{\theta} \begin{pmatrix} \frac{1}{\tanh\theta} & \frac{1}{\sinh\theta} \\ \frac{1}{\sinh\theta} & \frac{1}{\tanh\theta} \end{pmatrix} \begin{pmatrix} I_1 \\ I_0 \end{pmatrix} \tag{4-39}$$

where $\theta = \sqrt{j\omega RC} = \sqrt{SRC}$

Hence the Z matrix of the distributed R-C network is

$$[Z]=\frac{R}{\theta}\begin{bmatrix}\dfrac{1}{\tanh\theta} & \dfrac{1}{\sinh\theta}\\ \dfrac{1}{\sinh\theta} & \dfrac{1}{\tanh\theta}\end{bmatrix} \tag{4-40}$$

Applying Strecker-Feldtkeller duodinode matric equalities to transform [Z] to transfer matrix [a]

$$[a]=\begin{bmatrix}\dfrac{Z_{11}}{Z_{21}} & \dfrac{|Z|}{Z_{21}}\\ \dfrac{1}{Z_{21}} & \dfrac{Z_{22}}{Z_{21}}\end{bmatrix}=\begin{bmatrix}\cosh\theta & \dfrac{R\sinh\theta}{\theta}\\ \dfrac{\theta\sinh\theta}{R} & \cosh\theta\end{bmatrix} \tag{4-41}$$

For a special case, if S→0, $\theta\to 0$ and $\cosh\theta=1$

$$\lim_{\theta\to 0}\frac{\sinh\theta}{\theta}=\lim_{\theta\to 0}\frac{1}{\theta}\left(\theta+\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}+\cdots+\frac{\theta^{2n-1}}{\lfloor 2n-1}\right)=1$$

Equation (4-41) reduces to

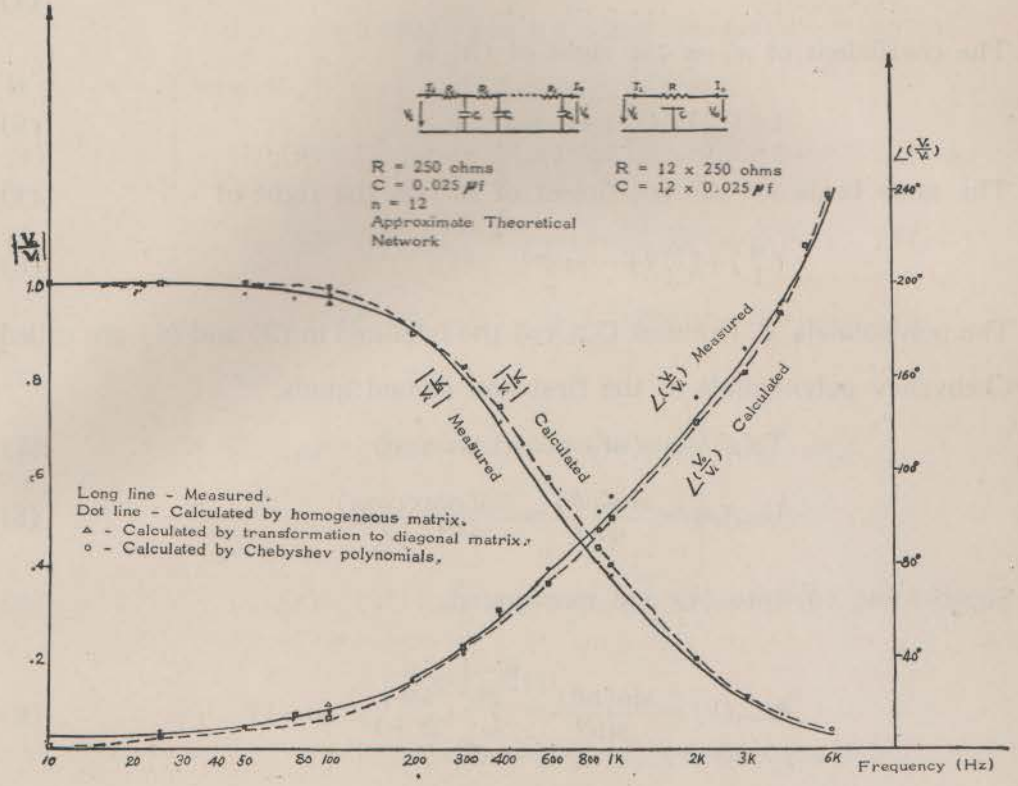
$$[a]=\begin{bmatrix}1 & R\\ 0 & 1\end{bmatrix}_{s\to 0}, \tag{4-42}$$

a well-known elementary [a]-matrix expression, as it should be.

5. CONCLUSION

In this paper, as mentioned in the introduction, three different methods of matric analysis were applied to analyse the basic uniformly distributed R-C ladder. These different resultant forms have been calculated with zero frequency, and all have the same result described in the equations (2-19), (3-19) and (4-42). The check point at f=100 hertz, the results were obtained in the closed values for open circuit voltage transfer function, $(0.965/-19^{\circ}$ by transformation to diagonal matrix; $0.970\ /-17.2^{\circ}$ by chebyshev polynomials: $0.975\ /-16^{\circ}$ by homogeneous matrix.) Another check points calculated by computer and the measured values of $(\frac{V_0}{Vi})$ are shown as figure below. So, all these results serve to show that the approximate theoretrical distributed R-C network, with increasing L type R-C sections up to twelve,

have much approximate characteristics as the thin film distributed R-C network. This is an important result, since the theory and design of lumped parameter R-C networks is much simpler than that required for distributed R-C networks, and thus lumped-network theory and experimental results can thereby be employed as a close predictor of microelectronic circuitry.



APPENDIX

Chebyshev polynomials

Consider de Moivre' s formula,

$$(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$$

snd putting $\cos\theta=\mu$ (1)

$$\sin\theta=\sqrt{1-\mu^2}$$

then $$\cos(n\theta)+i\sin(n\theta)=(\mu+i\sqrt{1-\mu^2})^n \quad (2)$$

Separating real and imaginary parts:

$$\cos(n\theta)=\mu^n-\binom{n}{2}\mu^{n-2}(1-\mu^2)+\binom{n}{4}\mu^{n-4}(1-\mu^2)^2-\cdots\cdots \quad (3)$$

$$\frac{\sin(n\theta)}{\sin\theta}=\binom{n}{1}\mu^{n-1}-\binom{n}{3}\mu^{n-3}(1-\mu^2)+\binom{n}{5}\mu^{n-5}(1-\mu^2)^2-\cdots\cdots \quad (4)$$

The coefficient of μ^n on the right of (3) is

$$1+\binom{n}{2}+\binom{n}{4}+\cdots=2^{n-1} \quad (5)$$

The same holds for the coefficient of μ^{n-1} on the right of (4)

$$\binom{n}{1}+\binom{n}{3}+\cdots=2^{n-1} \quad (6)$$

The polynomials $T_n(\mu)$ and $U_{n-1}(\mu)$ thus defined in (3) and (4) are called Chebyshev polynomials of the first and second kinds,

$$T_n(\mu)=\cos(n\theta)=\cos(\arccos\mu) \quad (7)$$

$$U_{n-1}(\mu)=\frac{\sin(n\theta)}{\sin\theta}=\frac{\sin(\arccos\mu)}{\sqrt{1-\mu^2}} \quad (8)$$

Substituting (6) into (4) and rearranged,

$$U_{n-1}(\mu)=\frac{\sin(n\theta)}{\sin\theta}=\sum_{r=0}^{n-1}\binom{n+r}{2r+1}2^r(\mu-1)^r \quad (9)$$

The induction method can be used to prove that (9) is the same as (4).

If $n=1$

(4) $U_{n-1}(\mu)=1$

(9) $U_{n-1}(\mu)=1$

If $n=2$

(4) $U_{n-1}(\mu)=\binom{2}{1}\mu=2\mu$

(9) $U_{n-1}(\mu)=\binom{2}{1}+\binom{3}{3}2(\mu-1)=2+2\mu-2=2\mu$

If $n=3$

(4) $$U_{n-1}(\mu)=\binom{3}{1}\mu^2-\binom{3}{3}(1-\mu^2)=3\mu^2-1+\mu^2=4\mu^2-1$$

(9) $$U_{n-1}(\mu)=\binom{3}{1}+\binom{4}{3}2(\mu-1)+\binom{5}{5}2^2(\mu-1)^2$$

$$=3+8(\mu-1)+4(\mu^2-2\mu+1)=4\mu^2-1$$

If $n=4$

(4) $$U_{n-1}(\mu)=\binom{4}{1}\mu^3-\binom{4}{3}\mu(1-\mu^2)=4\mu^3-4\mu(1-\mu^2)=8\mu^3-4\mu$$

(9) $$U_{n-1}(\mu)=\binom{4}{1}+\binom{5}{3}2(\mu-1)+\binom{6}{5}2^2(\mu-1)^2+\binom{7}{7}2^3(\mu-1)^3$$

$$=4+20(\mu-1)+24(\mu-1)^2+8(\mu-1)^3=8\mu^3-4\mu$$

If $n=5$

(4) $$U_{n-1}(\mu)=\binom{5}{1}\mu^4-\binom{5}{3}\mu^2(1-\mu^2)+\binom{5}{5}\mu^0(1-\mu^2)^2$$

$$=5\mu^4-10\mu^2(1-\mu^2)+(1-2\mu^2+\mu^4)=16\mu^4-12\mu^2+1$$

(9) $$U_{n-1}(\mu)=\binom{5}{1}+\binom{6}{3}2(\mu-1)+\binom{7}{5}2^2(\mu-1)^2+\binom{8}{7}2^3(\mu-1)^3$$

$$+\binom{9}{9}2^4(\mu-1)^4$$

$$=5+40(\mu-1)+84(\mu-1)^2+64(\mu-1)^3+16(\mu-1)^4$$

$$=16\mu^4-12\mu^2+1$$

If $n=6, 7, \cdots r$, substitute them into (4) and (9), the same results will be obtained. Hence equation (9) can be represented the equation (4).

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