SCIENCE BULLETIN
CHIAO-TUNG UNIV.
Vol.3,No.2(1969)

A Generalized Technique in The Solutions of Some Diffraction Problems concerning Dielectric Wedges

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Abstract-A modified technique in dealing with the generalized Wiener-Hopf equations is presented. The basic procedure is to apply the topological concept, by which a transform equation is reduced without performing product factorization to a set of matrix equations. Moreover, the theory of distribution is applied to simplify the problem formulations. A new theorem concerning additvie decomposition is deduced and used to verify the authenticity of the procedure. In addition, some advantages and disadvantages of the method are discussed. In this paper an example, diffraction of a plane wave by a right-angled dielectric wedge, is included to illustate the applicatin of the method.

I. Introduction

The main object of this paper is to show that considerable simplification can be achieved by formulating diffraction problems in the sense of theory of distributions and that there is a procedure of solving the Wiener-Hopf equations without performing a product factorization. The analysis, an extension of the earlier work of the auther, is in essence a generalization of the Wiener-Hopf techinque from one to two complex variables.

There are some basic approaches for the problems which can be solved by the Wiener-Hopf technique: the Jones method, the integral equation method and so on. We give pride of place to Jones' method, because it provides a routine procedure. However, for a class of problems such as a quarter-plane, a right-angled wedge, etc., it involves somewhat lengthy manipulation in obtaining the transform equations. This tediousness can be avoided by utilizing the concept of distribution. This paper shows an approach

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to formulate a given problem into a partial differential equation with an inhomogeneous term which contains the imformation of boundary condition. The transform equation can then be obtained by simply taking the transform of the differential equation.

In Section 3 we give the mathematically trivial problem, a dielectric wedge of angle π, to demonstrate the utility of the present approach. To fully appreciate the whole picture of this approach we give another problem of a right-angled dielectric wedge in Section 4. The formulation of this complicated problem is just a straightforward generalization of that of the π -angled wedge from one to two space variables. Application of the Laplace transform converts this formulation into a transform equation which relates four unknown functions of two complex variables. In Section 5 we introduce a general procedure of solving the transform equation without performing a factorization.The fundamental steps of this methods are:(i) By additive decomposition the transform equation is separated into a set of four integral equations over the respective tubes; (ii) Using the basis functions of the topological space we reduce the integral equations to the matrix equations. This approach is somewhat similar to the method of moments. Further reduction of the solution to a form suitable for numerical computation is also made in this section. In Section 6 we provide a rigorous justification of the method described in Section 5. We show that the solution obtained does satisfy the wave equation together with the boundary and radiation conditions. The relevant mathematical arguments and notations are summarized in Section 2. A new theorem of additive decompesition together with its detailed proof is also included. In the final section the advantages and disadvantages of the present method are well discussed.

2. The Transform Theory, Notations and Analytic Function Theory

This section is primarily concerned with those aspects of this vast subject which are relevant to the meterial to be used in the analysis.

First introduce the one-dimensional bilateral transform,

$$F(s)=\mathfrak{L}_x\ f(x)=\int_{-\infty}^{\infty} f(x)\exp(-sx)\,dx, \tag{2.1}$$

where $s=u+iv$. Define the functions $f_+(x)$ and $f_-(x)$ as

$$f_+(x)=\begin{cases} f(x), x>0 \\ 0, x<0 \end{cases}; \quad f_-(x)=\begin{cases} 0, x>0, \\ f(x), x<0. \end{cases} \tag{2.2}$$

Applying the transform yields

$$F_+(s)=\mathcal{L}_x f_+(x), F_-(s)=\mathcal{L}_x f_-(x). \tag{2.3}$$

If $|f_+(x)| < c_1\exp(u^+x)$ as $x\to\infty$, $F_+(s)$ is analytic in $u> u^+$;
If $|f_-(x)| < c_2\exp(u^-x)$ as $x\to-\infty$, $F_-(s)$ is analytic in $u < u^-$.

A refinement of the general decomposition theorem essentially contains the basic idea of our verification which will be carried out in Section 6 and is thus stated in some detail as follows.

Theorem. Let F(s) be analytic function of one complex variable s, regular in the strip $u^+<u<u^-$, such that $F(s) = o(|s|^{-2})$ as $|s|\to\infty$, uniformly for all arguments of s in the strip. Then

$$F(s)=F_+(s)+F_-(s), \tag{2.4a}$$

$$\lim_{s\to\infty} sF_+(s)=\lim_{s\to-\infty}(-s)F_-(s), \tag{2.4b}$$

$$\lim_{s\to\infty}(-s^2)\frac{d}{ds}\{sF_+(s)\}=\lim_{s\to-\infty}s^2\frac{d}{ds}\{sF_-(s)\} \tag{2.4c}$$

where $F_+(s)$ is analytic for all $u>u^+$ and $F_-(s)$ is analytic for all $u<u^-$.

Proof. To prove (2.4a) apply Cauchy's theorem to F(s). Thus, for $u^+< c<u<d<u^-$,

$$F(s)=-\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{F(z)}{z-s}dz+\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty}\frac{F(z)}{z-s}dz. \tag{2.5}$$

The first integral denoted by $F_+(s)$ is uniformly convergent and therefore analytic in $u>u^+$. Similarly, the second integral denoted by $F_-(s)$ is analytic in $u<u^-$.

To prove (2.4b, c) we may regard the contribution from a branch point of F(s) to the integrals in (2.5) as that from a continuous line of simple poles situated along two sides of the branch cut, since the contours of the integrals can be deformed to the left or right half of the s-plane by pushing out the contours to the branch cut. Thus, without loss of generality, we assume hereupon that F(s) contains only simple poles and no branch points.

Consider

$$\lim_{s\to\infty} sF_+(s) = -\frac{1}{2\pi i}\lim_{s\to\infty}\int_{c-i\infty}^{c+i\infty}\frac{sF(z)}{z-s}dz \tag{2.6}$$

Deform the contour of the above integral to the left half-plane so that the limit and integral operators are interchangeable. This gives

$$\lim_{s\to\infty} sF_+(s) = \frac{1}{2\pi i}\int_{C_-} F(z)dz = R[F;S_-] \tag{2.7}$$

where C_- is the contour consisting of the line u=c and the left infinite semicircle, S_- the left half of the s-plane and $R(F;S_-)$ the sum of the residues of F in S_-. With the arguments similar to those above, we have

$$\lim_{s\to-\infty} (-s)F_-(s) = -R[F;S_+] \tag{2.8}$$

where S_+ is the right half of the s-plane and $R(F;S_+)$ the sum of the residues of F in S_+. By the analytic continutaion, the fact that $F(s) = o(|s|^{-2})$ as $|s|\to\infty$ for all s in the strip implies that $F(s) = o(|s|^{-2})$ as $|s|\to\infty$ for all s in the whole s-plane. Consequently, the integration of F(s) along the circle of infinite radius with the center at s=0 vanishes. From the residue theorem it follows that

$$R[F;S_-]+R[F;S_+]=0 \tag{2.9}$$

Thus, the asymtotic relation (2.4b) is immediately established by inserting (2.7) and (2.8) into (2.9).

The other relation (2.4c) may be verified by similar means together with the residue relation,

$$R[sF;S_-]+R[sF;S_+]=0 \tag{2.10}$$

where $R[sF;S_-]$, $R[sF;S_+]$ denote the sums of the residues of sF in S_-, S_+ respectively. This completes the proof of the theorem.

Next, consider the two-dimensional transform defined by

$$F(\underline{s}) = \mathcal{L}_{\underline{x}} f(\underline{x}) = \int_{\underline{x}} f(\underline{x})\exp(-\underline{s}\cdot\underline{x})d^2x \tag{2.11}$$

where $\underline{x}=(x_1, x_2)$, $\underline{s}=(s_1,s_2)$, $\underline{s}\cdot\underline{x}=s_1x_1+s_2x_2$, $s_j=u_j+iv_j$ $(j=1,2)$ and the integration is over the whole x_1x_2-plane X. Intreducing the functions $f_n(\underline{x})$; $(n=1,2,3,4)$,

$$f_n(\underline{x}) = \begin{cases} f(\underline{x}), & \underline{x}\in Q_n, \\ 0, & \underline{x}\in(\chi-Q_n) \end{cases} \tag{2.12}$$

where Q_n denotes the nth quadrant of X, we have

$$F_n(\underline{s}) = \mathcal{L}_{\underline{x}}\, f_n(\underline{x}) \tag{2.13}$$

If the function f_n (x) exp(-u·x) is square integrable over X for every $\underline{u}$ in (D,n) i.e.

$$\int_X |\, f_n(\underline{x}) \exp(-\underline{u}\cdot\underline{x})\,|^2\, d^2x < \infty, \tag{2.14}$$

then (i) $F_n(\underline{s})$ is of bounded L_2-norm for some $\underline{u}$ in (D,n), where the L_2-norm is defined by

$$||F_n(\underline{u}+i\underline{v})||_2 = \{\int |F_n(\underline{u}+i\underline{v})|^2\, d^2v\}^{1/2}; \tag{2.15}$$

(ii) $F_n(\underline{s})$ is analytic for all $\underline{s}$ in the tube T(D,n)with the basis(D,n) defined by

$$\begin{aligned} &(D,1): u_1 > d_1^+,\ u_2 > d_2^+, \\ &(D,2): u_1 < d_1^-,\ u_2 > d_2^+, \\ &(D,3): u_1 < d_1^-,\ u_2 < d_2^-, \\ &(D,4): u_1 > d_1^+,\ u_2 < d_2^-. \end{aligned} \tag{2.16}$$

Conversely, if $F_n(\underline{s})$ is of bounded L_2-norm for some u in (D,n), there exists a unique inverse $f_n(x)$ independent of $\underline{u}$.

All the functions which satisfy the conditions (i) and (ii) stated in the previous paragraph form the Banach space H(D,n) which is complete in a sense of the metric defined by

$$d[F_n(\underline{s}) - G_n(\underline{s})] = ||F_n(\underline{s}) - G_n(\underline{s})||_2. \tag{2.17}$$

A decomposition theorem of two complex variables [2] is restated here without proof as follows: If F(s) is an element of the Banach space of the functions, say H(T), which are analytic and of bounded L_2-norm in a tube T(D) with the basis D:

$$d_j^+ < u_j < d_j^-,\ (j=1,2), \tag{2.18}$$

then F(s) can be uniquely decomposed into a sum of four functions $F_n(s) \triangleq [F(s)]_n$, n=1,2,3,4, which belong to the Banach spaces, say H(T,n), of analytic and bounded L_2-norm functions in the respective tubes T(D,n). Moreover, $F_n(s)$ has the asymtotic behavior, for $s \in T(D,n)$

$$F_n(\underline{s}) = o(1) \text{ as } |s_j| \to \infty,\ j=1,2. \tag{2.19}$$

$F_n(s)$ is expansible in a series of functions χ^n_{jk}, which are given by

$$\chi^n_{jk}(\underline{s}) = \Psi_{\sigma_n j}^{\sigma_n}(s_1 - b_1^{\sigma_n}; a_1)\Psi_{\tau_n k}(s_2 - b_2^{\tau_n}; a_2)$$

$$j, k = 1, 2, \cdots\cdots \qquad (2.20)$$

where $d_m^+ < b_m^+ < 0 < b_m^- < d_m^-$, $(m = 1, 2)$;

$$\Psi_m(s, a) = \frac{(s-a)^{m-1}}{(s+a)^m}, \Psi_{-m}(s, a) = \frac{(s+a)^{m-1}}{(s-a)^m}, a > 0; \qquad (2.21)$$

α_n, τ_n are the sign symbols:

$$\sigma_n = \begin{cases} +, n=1,4, \\ -, n=2,3, \end{cases} \quad \tau_n = \begin{cases} +, n=1,2, \\ -, n=3,4. \end{cases} \qquad (2.22)$$

With these mathematical arguments we are ready to study some diffraction problems in the subsequent sections. A time factor $\exp(i\omega t)$ is used and the m.k.s. system of units is employed throughout this paper. Let x_1, x_2, x_3 or x, y, z denote a right-handed Cartesian coordinate system. The permittivity and permeability of the dielectric will be denoted by ε_d, μ_0, and the corresponding constants of the medium exterior to the wedges by ε_a, μ_0. ε_d and ε_a usually have small imaginary parts. The propagation constants are given by

$$k_d^2 = \omega^2 u_0 \varepsilon_d \qquad \text{and} \qquad k_a^2 = \omega^2 u_0 \varepsilon_a .$$

The branches of k must be chosen so that Im $k < 0$.

3. A Dielectric Wedge of Angle π

The dielectric half-space is specified by $x \geq 0, -\infty < y < \infty, -\infty < z < \infty$. If a plane wave whose electric intensity is given by $\hat{z}\exp(-ik_a x)$ is normally incident on the dielectric half-space, a transmitted and a reflected waves are excited inside and outside the dielectric. Let $f_+(x)$, $f_-(x)$ denote the electric intensities of these excited waves. Then $f_+(x)$, $f_-(x)$ satisfy the time-harmonic wave equations:

$$\left(\frac{d^2}{dx^2} + k_a^2\right) f_-(x) = 0, x < 0; \qquad (3.1a)$$

$$(\frac{d^2}{dx^2} + k_d^2)f_+(x) = 0,\ x > 0. \tag{3.1b}$$

The tangential component of the total electric intensity is continuous across the dielectric surface, so is that of the total magnetic intensity. These imply that the solutions of (3.1) must be subject to the boundary condition:

$$f_+(0+) = f_-(0-) + 1, \tag{3.2a}$$

$$\frac{df_+}{dx}(0+) = \frac{df_-}{dx}(0-) - ik_a. \tag{3.2b}$$

As these scattered waves are outgoing, $f_+(x)$ and $f_-(x)$ are also required to satisfy the radiation condition.

Using the concept of distribution, we can reformulate the problem into an alternative form. Let the distribution f(x) represent a discontinuous function,

$$f(x) = \begin{cases} f_+(x), & x > 0; \\ f_-(x), & x < 0. \end{cases} \tag{3.3}$$

From (3.2) we see that both f and its derivative have finite jumps across the dielectric surface. Such discontinuities, which may be designated as secondary sources, are due to the electric and magnetic intensities of the primary wave on the dielectric surface. According to theory of distribution, the derivative of the distribution f is equal to the distribution defined by the derivative of f as a function whenever it exists, augmented by a Dirac distribution at the point at which f is discontinuous with a coefficient equal to the jump of f at this point. Thus, with (3.2) we have

$$\frac{df}{dx} = \frac{df_-}{dx} + \frac{df_+}{dx} + \delta(x), \tag{3.4}$$

$$\frac{d^2f}{dx^2} = \frac{d^2f_-}{dx^2} + \frac{d^2f_+}{dx^2} + \delta'(x) - ik_a\delta(x) \tag{3.5}$$

where $\delta(x)$ denotes the Dirac delta function. Add $[k_a^2 + (k_d^2 - k_a^2)1(x)]f(x)$, where 1(x) denotes the Heaviside unit step function, to both sides of (3.5) and then with (3.1) simplify the right side of the resultant equation to give

$$\left[\frac{d^2}{dx^2}+k_a^2+(k_d^2-k_a^2)1(x)\right]f(x)=\delta'(x)-ik_a\delta(x). \tag{3.6}$$

Summerizing, we have the boundary-value problem, that is to solue (3.6) subject to the asymtotic behavior:

$$f(x)\rightarrow\begin{cases} c_1\exp(d^+x) \text{ as } x\rightarrow\infty, d^+<0; & (3.7a)\\ c_2\exp(d^-x) \text{ as } x\rightarrow-\infty, d^->0; & (3.7b)\\ c_3 \quad \text{ as } x\rightarrow 0^+ \quad ; & (3.7c)\\ c_4 \quad \text{ as } x\rightarrow 0- \quad + & (3.7d)\end{cases}$$

Note that the conditions (3.7a,b) are implied by the radiation condition.

The solution of the above problem can be easily carried out by the usual Wiener-Hopf procedure:
Applying $\mathcal{L}_x$ to (3.6) yields

$$\frac{s^2+k_d^2}{s^2+k_a^2}F_+(s)+F_-(s)=\frac{1}{s+ik_a}\ . \tag{3.8}$$

This is a typical Wiener-Hopf equation, in which $F_+(s)$ is analytic in $u>d+$ whereas $F_-(s)$ is analytic in $u<d^-$. Perform product and then additive decompesitions to give

$$\frac{s+ik_d}{s+ik_a}F_+(s)-\frac{2k_a/(k_a+k_d)}{s+ik_a}=J(s), \tag{3.9a}$$

$$\frac{s-ik_a}{s-ik_d}F_-(s)-\frac{(k_d-k_a)/(k_d+k_a)}{s-ik_d}=-J(s). \tag{3.9b}$$

These equations determine $F_+(s)$, $F_-(s)$ to within a bounded entire function J(s). By Liouville's theorem J(s) is a constant; moreover, (3.7c,d) together with (3.9) imply that $J(s)\rightarrow 0$ as $s\rightarrow\infty$. This follows that $J(s)\equiv 0$ and thus the solutions F_+, F_- are uniquely obtained from (3.9). An inverse transformation immediately gives the transmitted and reflected fields as

$$f_+(x)=\frac{2k_a}{k_d+k_a}\exp(-ik_dx),\ f_-(x)=\frac{k_d-k_a}{k_d+k_a}\exp(ik_ax).$$

In fact, these solutions can be easily obtained through the concept of transimission line. However, we went through these details.

in order to illustrate the application of theory of distribution and review the essential idea of the Wiener-Hopf method. A straightforward generalization of the above arguments would enable us to tackle a sophisticated problem which will be described in the next section.

4. A Dielectric Wedge of Angle $\pi/2$

The problem considered is that of determining the scattered fields inside and outside a right-angled dielectric wedge which is specified by $x_1 \geq 0$, $x_2 \geq 0$, $-\infty < x_3 < \infty$. The primary field is a linearly E-polarized plane wave whose electric intensity is specified by $\hat{x}_3 \exp(-k_1x_1 - k_2x_2)$ where $k_1 = ik_a \cos\theta_0$, $K_2 = ik_a \sin\theta_0$, and $0 < \theta_0 < \pi/2$.

This is a two-dimensional problem as all field components are independent of x_3. Let the electric intensity of the secondary field be denoted by $f(x) \triangleq f(x_1,x_2)$, which defines the refracted and diffracted fields inside the wedge as well as the reflected and diffracted fields outside the wedge. This electric field f(x) certainly satisfy the inhomogeneous Helmholtz's wave equation together with the integrability condition ensured by the outgoing property of electromagnetic waves. We have, in sum, the boundary-value problem.

$$\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + k_a^2 + (k_d^2 - k_a^2)1(x_1)1(x_2)\right]f(\underline{x})$$
$$= [\delta'(x_1) - k_1\delta(x_1)]1(x_2)\exp(-k_2x_2) + [\delta'(x_2)$$
$$-k_2\delta(x_2)]1(x_1)\exp(-k_1x_1) \tag{4.1}$$

which is to be solved subject to the requirement,

$$\int_X |f(\underline{x})|^2 d^2x < \infty . \tag{4.2}$$

Note that the formulation (4.1) is a straightforward generalization of (3.6) from one to two space variables.

Application of a double Laplace transform formally reduces the above problem to an equivalent problem, that of solving a transform equation, which relates four unknown functions of two complex variables in the Banach space H(T).

5. A Modified Procedure

The fundamental steps in the complex variable procedure are conveniently summerized in this paragraph: Introduce $f_n(x)$ as the function, which vanishes outside the nth quadrant of the x_1x_2-plane, so that

$$f(x)=\sum_{n=1}^{\infty} f_n(\underline{x}) \cdot \qquad (5.1)$$

Application of the transform $\pounds\underline{x}$ to (4.1) and then simplification immediately convert the formulation into an alternative form,

$$K(\underline{s})F_1(\underline{s})+\sum_{n=1}^{4} F_n(s)=G_1(\underline{s}) \qquad (5.2)$$

where $F_n(\underline{s})=\pounds\underline{x}f_n(\underline{x})$, $K(\underline{s})=(k_d^2-k_a^2)/(s_1^2+s_2^2+k_a^2)$ and $G_1(s)=1/[(s_1+k_1)(s_2+k_2)]$.

In addition, by Parseval's formula and the existence condition of $F_n(s)$ we obtain the following properties: $F_n(s)$ belongs to the Banach space H(T,n) and so does $K(s)F_1(s)$ to H(T). Decompose the first term of (5.2) additively and then apply analytic continuation. Being governed by the functional behavior at infinity, (5,2) is thus equivalently separated into a set of four independent equations:

$$[K(\underline{s})F_1(\underline{s})]_1+F_1(\underline{s})=G_1(\underline{s}), \qquad (5.3a)$$

$$F_n(s)=-[K(s)F_1(s)]n, n=2,3,4. \qquad (5.3b,c,d)$$

(5.3a) is an inhomogeneous integral equation of second kind over the tube (T,1). It is conveniently expressed in the language of operators:

$$P(F_1)+U(F_1)=G_1 \qquad (5.4)$$

where P is a linear operator and U a unit operator on H(T, 1). Since a solution of (5.2) exists and is unique for the given G_1[2], an inverse operator $(P+U)^{-1}$ exists such that

$$F_1=(P+U)^{-1}(G_1). \qquad (5.5)$$

$F_n(s)$, N = 2,3,4, can then be found through substituting (5.5) into (5.3b,c,d), respectively. Finally, $F_n(s)$, (n = 1,2,3,4), is of bounded L_2-norm for some u in (D,n) so that there exists a unique inverse,

$$f(\underline{x})=\sum_{n=1} \pounds_{\underline{x}}^{-1}F_n(\underline{s}) \qquad (5.6)$$

The practical details of manipulation will be given in the next section.

There is a method for dealing with (5.3), which reduces the functional equations over the tubes to matrix equations over the complex field. Let F_n and G_1 be expanded in serieses of basis functions χ^n_{jk} in H(T,n) as

$$F_n = \sum_{j,k=1}^{\infty} c^n_{jk} \chi^n_{jk} \quad n=1,2,3,4. \tag{5.7}$$

$$G1 = \sum_{\ell,m=1}^{\infty} c^o_{\ell m} \chi^1_{\ell m} \tag{5.8}$$

Substituting (5.7) and (5.8) into (5.3), and using the linearity of the operators as well as linear independence of the basis functions, we can write (5.3) equivalently in the matrix forms as

$$[A^1(\ell+m, j+k)]\{C^1(j+k)\} = \{C^o(\ell+m)\} \tag{5.9a}$$

$$\{C^n(\ell+m)\} = -[A^n(\ell+m, j+k)]\{C^1(j+k)\}, n=2,3,4. \tag{5.9b,c,d}$$

where $[A^n]$ are $2\infty \times 2\infty$ square matrices with entries $A^n(\mu, \upsilon)$ and $\{C^n\}$ are $2\infty \times 1$ column matrices with entries $C^n(\mu)$. Note that in (5.9) the sequences of columns of $[A^n]$ must be consistent with those of rows of the factor matrices $\{C^n\}$; so must be the sequences of rows of $[A^n]$ with those of the product matrices $\{C^n\}$. The resultant entries are

$$C^n(\ell+m) = c^n_{\ell m}, \quad n=0,1,2,3,4; \tag{5.10}$$

$$c^o_{\ell m} = 4a_1a_2 \frac{(-b_1^+ - k_1 + a_1)^{\ell-1}}{(-b_1^+ - k_1 - a_1)^{\ell}} \cdot \frac{(-b_2^+ - k_2 + a_2)^{m-1}}{(-b_2^+ - k_2 - a_2)^{m}}; \tag{5.11}$$

$$A^n(\ell+m, j+k) = \frac{a_1a_2}{\pi^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\underline{z}_o)\chi^1_{jk}(\underline{z}_o)\Psi_{-\sigma_n\ell}(it_1; a_1)\Psi_{-\tau_n m}(it_2; a_2)\,dt_1dt_2 + \delta_{1n}\delta_{j\ell}\delta_{km},$$

$$n=1,2,3,4 \tag{5.12}$$

where $z_o = (z_1, z_2) = (b_1^{\sigma n} + it_1, b_2^{\tau n} + it_2)$ and $\delta_{\ell m} \triangleq$ Kronecker delta. All the entries are purely numerical constants which proceed to determine in the next paragraph.

The double integrals in (5.12) converge absolutely; in practice the integrals may be computed by integrating first with respect to t_1 and then with respect to t_2 (or vice versa).
Define

$$\gamma(z_2)=\sqrt{z_2^2+k_a^2}. \tag{5.13}$$

That branch of γ is taken which reduces to k_a if $z_2 = 0$. It follows that $\operatorname{Im}\gamma < 0$ for all z_2 on the integration paths given in (5.12). By deforming the contours to the left or right half of the s_1-plane, the iterated integrals can be evaluated by residues. Thus we can reduce the double integrals to the single integrals as follows:

$$A^n(\ell+m, j+k)=\frac{a_1a_2(k_d^2-k_a^2)}{\pi}\int_{-\infty}^{\infty} W_{j\ell}^{\sigma_n}(b_2^{\tau_n}+it_2)\Psi_k(b_2^{\tau_n}+it_2 - b_2^{+}; a_2)\cdot\Psi_{-\tau_n m}(it_2; a_2)\,dt_2+\delta_{1n}\delta_{j\ell}\delta_{km},$$

$$n=1,2,3,4 \tag{5.14}$$

with

$$W_{j\ell}^{+}(z_2)=\begin{cases}\dfrac{1}{a_1[(-b_1^{+}+a_1)^2+\gamma^2]}+\dfrac{1}{i\gamma[(b_1^{+}+\gamma)^2-a_1^2]} & \text{for } j=\ell, \\[2ex] \dfrac{(-b_1^{+}-a_1+i\gamma)^{j-2}}{i\gamma(-b_1^{+}+a_1+i\gamma)^{j}} & \text{for } j>\ell; \\[2ex] \dfrac{(-b_1^{+}+a_1-i\gamma)^{\ell-2}}{(-b_1^{+}-a_1-i\gamma)^{\ell}} & \text{for } j<\ell; \end{cases}$$

$$W_{j\ell}^{-}(z_2)=\frac{(-b_1^{+}-a_1+i\gamma)^{j-1}(-b_1^{-}-a_1+i\gamma)^{\ell-1}}{i\gamma(-b_1^{+}+a_1+i\gamma)^{j}(-b_1^{-}+a_1+i\gamma)^{\ell-1}} \quad \text{for all } j, \ell.$$

A knowledge of the original function KF_1 is fully contained in the matrix entries (5.14), which are convergent and well suited for machine computation. The unknown constant c_{jk}^{n}, $n=1,2,3,4$ are then determined through (5.9) by the simple matrix operations: inversion and product. The solution is exact if the matrix of infinite order can be inverted.

To complete the solution use (5.7) and the Laplace inversion formulae to find

$$f_n(\underline{x}) = \sum_{j,k=1}^{\infty} c^n_{jk} \sum_{\mu=1}^{j-1} \frac{\sigma_n \Gamma(j)}{\Gamma(\mu+1)[\Gamma(j-\mu)]^2} (-\sigma_n 2a_1 x_1)^{j-u-1}$$

$$\exp[(b_1^{\sigma_n} - \sigma_n a_1)x_1] \sum_{j=0}^{k-1} \frac{\tau_n \Gamma(k)}{\Gamma(\nu+1)[\Gamma(k-\nu)]^2} (-\tau_n 2a_2 x_2)^{k-\nu-1}$$

$$\exp[(b_2^{\tau_n} - \tau_n a_2)x_2], n=1,2,3,4 \qquad (5.15)$$

6. Verification

In order to show that the solution obtained is the only solution satisfying all the conditions of the problem, we should provide here a rigorous justification of the procedure described in the previous sections.

(5.2) is equivalent to (4.1), as the Laplace transform merely changes the differential operator over the real field with the functional linear transformation over the complex field. Thus the secondary field given in (5.15) does satisfy the steady state wave equation with the space-varying coefficient (4.1). Moreover, the bounded L_2-norm of $F_n(s)$ implies that f(x) is square integrable and therefore outgoing, or equivalently the radiation condition can be proved by applying a generalization of the final-value theorem in one variable,

$$\lim_{x\to\infty} f(x) = \lim_{s\to 0} sF(s). \qquad (6.1)$$

It remains to prove the boundary condition: The electric and magnetic intensities of the total field are continuous across the dielectric surface.

To justify the boundary condition, besides the ordinary initial-value theorem for the Laplace transform we require such additional relations as

$$\lim_{x\to 0+} \frac{df}{dx} = \lim_{s\to\infty} (-s^2) \frac{d}{ds} (sF_+(s)) \qquad (6.2a)$$

and

$$\lim_{x\to 0-} \frac{df}{dx} = \lim_{s\to -\infty} s^2 \frac{d}{ds} (sF_-(s)) \qquad (6.2b)$$

which can be readily established.

From (5.2) we observe that

$$K(s)F_1(s)=o(|s_1|^{-2}) \text{ as } |s_1|\to\infty \quad \text{for } \underline{s}\,\varepsilon T.$$

Application of the theorem of Section 2 gives

$$\lim_{s_1\to\infty} s_1[KF_1]_1 = \lim_{s_1\to-\infty}(-s_1)[KF_1]_2 \tag{6.3a}$$

and

$$\lim_{s_1\to\infty}(-s_1{}^2)\frac{\partial}{\partial s_1}(s_1[KF_1]_{\circ}) = \lim_{s_1\to-\infty} s_1^2\frac{\partial}{\partial s_1}(s_1[KF_1]_2). \tag{6.3b}$$

In accordance with (5.3a) and (5.3b), replace $[KF_1]_1$ by $G_1 - F_1$ and $[KF_1]_2$ by F_2. From the initial value theorem and (6.2) it may be then concluded that for $x_2 > 0$

$$f_1(o+,x_2) = f_2(o-,x_2)+\exp(-k_2x_2); \tag{6.4a}$$

$$\frac{\partial f_1}{\partial x_1}(o+,x_2) = \frac{\partial f_2}{\partial x_1}(o-,x_2)-k_1\exp(-k_2x_2). \tag{6.4b}$$

By a similar argument we may also conclude that f_1 and $f_4 + \exp(-k_1x_1 - k_2x_2)$ are equal at the surface $x_1 > 0$, $x_2 = 0$; so are their partial derivatives with respect to x_2. This completes the proof of the boundary condition.

7. Discussion

One of the main reasons for presenting the numerical details here is to show that the solution in terms of the basis functions χ^n_{jk} converges very slowly. Because of computational time we only find an approximate solution of the infinite system (5.9a) by solving the first nine equations in nine unknowns, assuming $c^1_{jk} = 0$, j, $k \geq 4$. The following result, $c^n_{jk} \times 10^5$, is given for $\varepsilon_a = 10.0 - 10.06$, $\varepsilon_b = 1.0 - i0.01$ (relative constants), $f = 3 \times 10^9$ c/s and $\theta = 45^{\circ}$ by choosing $b_j^+ = -0.2$, $b_j^- = 0\ 2$ ($j=1,2$) and $a_1 = a_2 = 0.1$:

j	k \ n	1	2	3	4
1	1	-20.636 i1.428	1.786-i8.499	0.5094 i0.1410	1.747-i8.484
1	2	-20.589 i0.5106	2.158-i8.390	1.393 -i2.593	1.928-i8.484
1	3	-20.533-i0.4057	1.636-i5.514	1.473 -i2.543	2.099-i8.449
2	1	-20.638 i0.5052	1.320-i5.618	0.5017 i0.1638	2.128-i8.385
2	2	-20.55 -i0.4073	2.432-i8.314	0.4967 i0.1785	2.306-i8.401
2	3	-20.45 -i0.1318	2.786-i8.177	0.4895 i0.1969	2.475-i8.358

j	k \ n	1	2	3	4
3	1	-20.61 -i0.4208	2.381-i8.353	0.4968 i0.1785	2.502-i8.287
3	2	-20.48 -i0.1327	2.684-i8.266	0.489 i0.1968	2.679-i8.297
3	3	-20.35 -i0.2232	3.041-i8.077	0.4812 i0.2314	2.844-i8.167

As $A^n(\mu,\nu)$ decay very slowly with increasing μ and ν , very poor accuracy of the result is expected. We observe that choosing arbitrary values of a_1 and a_2 has little effect on the slowness of the convergent serieses $A^n(\mu,\nu)$. However, it does effect the series c^o_{jk} especially if the medium outside the wedge has a large loss. The best choice is $a_j = -\mathrm{Im}(k_j) + b_j^+$, $j = 1,2$. Therefore, the present technique is only applicable to the medium with large loss.

There are a great number of dense sets of basis functions in H(T,n). It would appear that double series $A^n(\mu,\nu)$ in terms of a dense set may converge less slowly than that in terms of another dense set. Thus by using a suitable set of basis functions, an improvement in the present technique is possible. This will be investigated in a sequel to this paper.

Note in passing that a knowledge concerning attenuation constants of propageting waves is mainly contained in the numbers $(b_j^+ - a_j)$ and $(b_j^- + a_j)$, $j = 1,2$; whereas that concerning phase concerning is mainly contained in c^n_{jk}. From (5.15) we see that $f_n(x)$ are expansible in integral powers of x_1, x_2 for any finite x in the respective quadrants. This implies that the total field remains finite as x approaches 0 along any ray.

Only those highly idealized problems can be solved exactly. Furthermore, the success of applying the usual Wiener - Hopf procedure to solve a transform equation of one complex variable depends on whether it is practical to perform a product factorization; a transform equation in connection with a somewhat complecated geometry, boundaries parpendicular to each other as well as parallel to each other, cannot be solved exactly or even approximately by the usual procedure, since branch points occur in certain functions of the equation. For example,

$$A(s)F_+(s)+B(s)F_-(s)+C(s)F_+(-s)+D(s)F_-(-s)+E(s)=0 \tag{7.1}$$

where A, ..., E are known, F_+, F_-, G_+, G_- are unknown. The present technique provides a routine procedure to deal with the equations of the type given in (7.1) as well as those equations whose solutions are handicaped by product factorization.

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