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Reflection and Transmission of Electromagnetic Waves on the Boundary between Isotropic and Anisotropic Media.

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Abstract-The problems of reflection and transmission on a plane boundary between an isotropic medium and anisotropic medium are investigated in various cases with different directions of the static magnetic field relative to the orientation of the boundary surtace. Both plasma and ferrite media are included with a unified point of view. The coefficients of reflection and transmission are obtained by finding the input wave impedances of the anisotropic medium in the direction normal to the boundary.

I. Introduction

The propagation of electromagnetic waves in an homogeneous and inhomogeneous or discontinuous medium has been extensively investigated by many authors with solutions to the field equations and magnetodynamic equations of motion characterizing the medium subject to the proper boundary conditions. In most cases the results are very complicated without simplifying assumptions. The case of propagation of electromagnetic waves through an interface between an istropic medium and an anistropic medium has particular interest and applications in the field of study of plasmas and ferrite medium and renders a relatively easier method of attack in the case of plane boundary. The usual method of treating reflection and transmission can be extended to the case involving anisotropic medium and give a more physical insight into the problem by using the impedance concept in analogy with the transmission line circuit. Together with the application of Snell's law for the wave normals which also holds for anisotropic medium the boundary conditions at the interface are automatically satisfied. The input impedances of the anisotropic medium are found in various cases with different orientations of the interface relative to

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the static magnetic field and wave normal. Then the reflection and transmission coefficient are obtained in the usual manner and the condition of cutoff and transmission can be discussed.

In the microwave range there are two types of anisotropic medium namely, the plasma (or gyroelectric medium) and the ferrite (or gyromagnetic medium) with a static magnetic field applied in both cases. Both types of the anisotropic media have similar properties and lead to similar results. They satisfy the same wave equation except that in the gyroelectric medium the scalar permittvity ϵ_0 of the free space (or ϵ for the isotropic medium) is replaced by the permittivity tensor , the permeability remains approximately the same as the permeability μ_0 of the free space, while in the gyromagnetic medium the scalar permeability μ is replaced by the permeability tensor with the permittivity approximately equal to ϵ_0.

2. The Permittivity and Permeability Tensor

The permittivity tensor of a stationary cold plasma with an applied static magnetic field Bo in the direction of z-axis is given by the following expression

$$[\epsilon] = \epsilon_0 \begin{bmatrix} S & -jD & O \\ jD & S & O \\ O & O & L \end{bmatrix} = \epsilon_0 [k] \tag{1}$$

Where [K] is the dielectric tensor to denote matrix indicated by the brackets

$$S = 1 + \frac{\omega_p^2 (\omega + j\nu)}{\omega [\omega_c^2 - (\omega + j\nu)^2]}$$

$$D = \frac{\omega_c}{\omega} \cdot \frac{\omega_p^2}{\omega_c^2 - (\omega + j\nu)^2}$$

$$L = 1 - \frac{\omega_p^2}{\omega(\omega + j\nu)}$$

where $\omega_p^2 = \dfrac{n_e e^2}{m\epsilon_o}$,

$\omega_c = \dfrac{eB_o}{m}$,

$n_o =$ electron density,

$\omega_p =$ the electron plasma frequency,

$\omega_o =$ the cyclotron frequency,

$\nu =$ effective collision frequency.

When the effect of collisions is negleted

$$S = 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}, \quad D = \frac{\omega_c\, \omega_p^2}{\omega(\omega_c^2 - \omega^2)}, \quad L = 1 - \frac{\omega_p^2}{\omega^2}$$

For a hot plasma neglecting the effect of collisions, the components S and D remain unchanged, then L becomes

$$L = 1 - \frac{\omega_p^2}{\omega^2 - k^2 a^2}$$

where $a^2 = \dfrac{kT}{m}$

$k =$ Boltzmann constant, $T =$ electron temperature.

In the above expressions only electrons in the plasma are considered, additional terms due to ions in the plasma must be added to include their effects.

The permeability tensor of a ferrite medium with an applied static magnetic field B_o in the direction of z-axis is given by the following expression, in which the same letters have been used to denote the corresponding similar tensor element as those used for gyroelectric medium.

$$[\mu] = \mu_o \begin{bmatrix} S & -jD & 0 \\ jD & S & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mu_o [M]$$

where $S = 1 + \dfrac{\omega_M(\omega_o + j\omega\alpha)}{(\omega_o + j\omega\alpha) - \omega^2}$

$$D = \frac{\omega\, \omega_M}{\omega^2 - (\omega_o + j\omega\alpha)^2}$$

$\omega_o = \gamma H_o$, $H_o =$ total internal static field in the medium

$\omega_M = \gamma M_o$, $M_o =$ saturated magnetization

$\gamma =$ gyromagnetic ratio

$\alpha =$ damping factor

when damping is neglected,

$$S = 1 - \frac{\omega_o \omega_M}{\omega^2 - \omega_o^2} \qquad D = \frac{\omega\ \omega_M}{\omega^2 - \omega_o^2}$$

Note the remarkable similarities in the above expressions for the tensor elements compared with those for the permittivity tensor.

3. The Wave Equations

3.1 Gyro-electric medium. From the following Maxwell's equations

$$\nabla \times \underline{E} = -j\omega\mu_o \underline{H} \tag{3a}$$

$$\nabla \times \underline{H} = j\omega\epsilon_o [k] \cdot \underline{E} \tag{3b}$$

where the time dependent factor $e^{i\omega t}$ has been assumed in the field components wave equation can be obtained as follows.

$$\nabla \times \nabla \times \underline{E} = \omega^2 \mu_o \epsilon_o [k] \cdot \underline{E} \tag{4}$$

For a propagating plane wave with time and space variation according to the factor $e^{j(wt - \underline{k} \cdot \underline{r})}$, the operater ∇ can be replaced by the vector $-j\underline{k}$ and the wave equation becomes

$$\underline{k} \times \underline{k} \times \underline{E} = -k_o^2 [k] \cdot \underline{E} \tag{5}$$

where $k_o^2 = \omega_o^2 \mu_o \epsilon_o$ and $\underline{k}$ is the propagation vector.

3.2 Gyromagnetic Medium. The Maxwell's equations become

$$\nabla \times \underline{E} = -j\omega[\mu] \cdot \underline{H} \tag{6a}$$

$$\nabla \times \underline{H} = j\omega\epsilon_o \underline{E} \tag{6b}$$

and the corresponding wave equation is

$$\nabla \times \nabla \times \underline{H} = \omega^2 \epsilon_o \mu_o [M] \cdot \underline{H} \tag{7}$$

or

$$\underline{k} \times \underline{k} \times \underline{E} = -k_o^2 [M] \cdot \underline{H} \tag{8}$$

4. The Dispersion Equations

Either of the wave equations (5) and (8) represents a set of three linear homogeneous equations for the three field components (E_x, E_y, E_z) and (H_x, H_y, H_z) respectively.
Expanding k x k x E in Catesion coordinates. with

$\underline{k} = \underline{a}_x, k_x + \underline{a}_y k_y + \underline{a}_z k_z$, where $\underline{a}_x, \underline{a}_y, \underline{a}_z$ are the unit vector in x, y, z directions, we obtain

$$\underline{k}\times\underline{k}\times\underline{E}+\underline{k}_o^2\,[k]\cdot\underline{E}=\begin{bmatrix} Sk_o^2-k_y^2-k_z^2, & k_xk_y-jDk_o^2, & k_xk_z \\ k_yk_x+jDk_o^2, & Sk_o^2-k_z^2-k_x^2, & k_yk_z \\ k_zk_x, & k_zk_y, & Lk_o^2-k_x^2-k_y^2 \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \quad (10)$$

To simplify the above expression without loss of generality let us take the direction of propagation k to lie in the x-z plane and let θ be the angle that k makes with the direction of the static magnetic field which coincides with the z-axis as shown in Fig. 1.

With the above orientation of the coordinate axes we have $k_x = k\sin\theta$, $k_y = 0$, $k_z = k\cos\theta$ and equation (10) reduces to

$$\begin{bmatrix} S-n^2\cos^2\theta & -jD & n^2\sin\theta\cos\theta \\ jD & S^2-n^2 & O \\ n^2\sin\theta\cos\theta & O & L-n^2\sin^2\theta \end{bmatrix}\begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}=0 \quad (11)$$

where $n = k/k_o$ is the vector index of refraction.

These equations have a nontrivial solution only when the determinant of the coefficients is zero. Evaluating the determinant we obtain the following dispersion equation.

$$An^4-Bn^2+C=0 \quad (12)$$

where

$$A=S\sin^2\theta+L\cos^2\theta$$

$$B=(S^2-D^2)\sin^2\theta+LS(1+\cos^2\theta)$$

$$C=L(S^2-D^2)$$

From Eq.(12) we can solve for the refractive index of the anisotropic medium which depends on the direction of propagation of the wave as well as on the components of permittivity tensor.

Similar espressions can be obtained for the ferrite medium by setting L equal to unity and components of E replaced by components of H, thus

$$\begin{bmatrix} S-n^2\cos^2\theta, & -jD, & n^2\sin\theta\cos\theta \\ jD, & S-n^2, & O \\ n^2\sin\theta\cos\theta, & O, & 1-n^2\sin^2\theta \end{bmatrix}\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}=O \quad (13)$$

Two cases are of particular interest about the nature of the refractive index, namely, the case when the direction of propagation of the wave is parallel and the case when it is perpendicular to the direction of the static magnetic field. A brief description is given below.

4.1 Propagation along the Magnetic Field.

In this case $\theta = 0$ and Equation (12) reduces to

$$n^4 - 2Sn^2 + (S^2 - D^2) = 0$$

and there are two solutions for n

$$n_+^2 = S + D$$

$$n_-^2 = S - D$$

which are the refractive indices for a right-handed circularly polarized wave and a left-handed circularly polarized wave respectively as can be seen from Equation (11) which reduces to

$$\begin{bmatrix} S-n^2 & -jD & O \\ jD & S-n^2 & O \\ O & O & L \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = O \tag{14}$$

From the above equation we obtain,

$$\frac{E_x}{E_y} = \frac{S-n^2}{-jD} = \mp j$$

which verify the above statement

4.2 Propagation Perpendicular to the Magnetic Field.

In this case $\theta = \frac{\pi}{2}$ and Equation (12) gives following values of n^2.

$$n_{ord}^2 = L \tag{15a}$$

$$n_{ex}^2 = \frac{S^2 - D^2}{S} \tag{15b}$$

The former is the refractive index for the ordinary wave since it has the same value as that for propogation without a static magnetic field and Eq. (15b) gives the refractive index for the extraordinary wave.

5. Reflection of Electromagnetic wave by a Plane Boundary

5.1 Static Magnetic Field Perpendicular to the Plane of Incidence.

Consider a ferrite medium in the two cases of polarization.

(1a) Perpendicular Polarization - E perpendicular to the plane of incidence . The relative orientations of the fields and coordinate axes are shown in Fig. 2. In this case

$E = a_z E_Z$ has only z component.

For simplicity of symbols, the field components of the electromagnetic wave in the anisotropic medium are denoted by letters without subscripts.

The Maxwell curl equation for E is

$$\nabla \times \underline{E} = -j\omega\mu_o [M] \cdot \underline{H} \tag{16}$$

where [M] is given by Eq. (2)

or writing in the component form

$$-jk_y E_z = -j\omega\mu_o (SH_x - jDH_y) \tag{17a}$$

$$jk_x E_z = -j\omega\mu_o (jDH_x + SH_y) \tag{17b}$$

Solving the above equations simulatneously for Hx we have

$$Hx = \frac{Sk_y - jDk_x}{\omega\mu_o (S^2 - D^2)} E_z$$

The input impedance for the wave propagating in the y direction is therefore

$$Zy = \frac{E_z}{H_x} = \frac{\omega\mu_o (S^2 - D^2)}{Sk_y - jDk_x}$$

Substituting the values of $k_x = k \sin\theta_r, k_y = k \cos\theta_r$ and dividing by the wave impedance of free space in y direction i.e.

$$Z_{oy} = \frac{z_o}{\cos\theta_r} = \frac{\omega\mu_o}{k_o \cos\theta_i}$$

where $z_o = \sqrt{\frac{\mu_o}{\epsilon_o}}$ we obtain the normalized input impedance of ferrite medium

$$Z_{in} = \frac{(S^2 - D^2) k_o \cos\theta_i}{k(S\cos\theta_r - jD\sin\theta_r)}$$

Using the Snell's law for refraction, $n = \frac{k}{k_o} = \sin\theta i / \sin\theta_r$ which holds also for the anisotropic medium we can express the normalized input impedance in terms of the angle of incidence as follows

$$Z_{in} = \frac{(S^2 - D^2) k_o \cos\theta_i}{S(k^2 - k_o^2 \sin^2\theta_i)^{1/2} - jDk_o \sin\theta_i} \tag{18a}$$

or

$$Z_{in} = \frac{n^2 \cos\theta_i}{(n^2 - \sin\theta_i) - jD/S \sin\theta_i} \tag{18b}$$

where we have put $n^2 = k^2/k_o^2 = (S^2 - D^2)/S$ as the refractive index for the extraordinary wave for the propagation perpendicular to the magnetic field given by Eq (15). The reflection and transmission coefficients are then given by the usual expressions $(Zin - 1)/(Zin + 1)$ and $2Zin/(Zin + 1)$ respectively. Complete reflection corresponds to $Zin = 0$ or ∞. Neglecting losses in the ferrite this corresponds to $\omega = \omega_c + \omega_m$ and $\omega = \omega_o$. The transmission condition corresponds to $Z_{in} = 1$ for zero reflection coefficient.

(1b) Parallel Polarization -- E in the plane of incidence

In this case $H = a_z H_z$. The orientations of the fields relative to the coordinate axes are shown in Fig. 3.

From the curl equation $\nabla \times H = j\omega\epsilon_o E$

or

$$\begin{aligned} -jk_y H_z &= j\omega\epsilon_o E_x \\ jk_x H_z &= j\omega\epsilon_o E_y \end{aligned} \tag{19}$$

we may obtain the input impedance in the y direction of the ferrite medium directly from the first of Eqs. (19), thus we obtain

$$Z_y = -\frac{E_x}{H_z} = \frac{k_y}{\omega\epsilon_o} = \frac{k\cos\theta_r}{\omega\epsilon_o} \text{ or } Z_y = nZ_o\cos\theta_r$$

The normalized input impedance is then

$$Z_{in} = \frac{Z_y}{Z_o\cos\theta_i} = \frac{n\cos\theta_r}{\cos\theta_i} = \left(\frac{n^2 - \sin^2\theta_i}{1 - \sin^2\theta_i}\right)^{1/2} \tag{20}$$

It is interesting to note that the characteristics of the medium is not affected by the static magnetic field, or it acts as isotropic medium and complete reflection occurs for $\sin\theta i = n$ and complete transrnission occurs only when $n = 1$ which means that the bounary surface does not exist, this is of course a trivial result.

Similar expressions for the normalized input impedance and coefficients of reflection and transmission for a plasma medium may be obtained in the same way with the following results:

Perpendicular Polarization:

$$Z_{in} = \frac{Z_y\cos\theta_i}{Z_o} = \left(\frac{1 - \sin^2\theta_i}{n^2 - \sin^2\theta_i}\right)^{1/2} \tag{21}$$

Parallel Polarization:

$$Z_{in} = \frac{(n^2 - \sin^2\theta_i)^{1/2} - j\frac{D}{S}\sin\theta_i}{n^2\cos\theta_i} \tag{22}$$

It is interesting to note that Eqs. (21) and (22) are the reciprocals of Eqs. (20) and (18) for the ferrite medium.

5.2 Static Magnetic Field Perpendicular to the Plane Boundary

The direction of propagation and the direction of the static magnetic field relative to the coordinate axes are shown in Fig. 4. The boundary surface is the x-y plane and the anisotropic medium lies in the region $z > 0$, the static magnetic field along z-axix is perpendicular to the boundary surface. The orientations of the coordinate axes are the same as these used in Fig. 1, so the angle θ in Fig. 1 is the angle of refraction θ_r in Fig. 4.

From Eqs. (11) and (14) we can obtain the relative magnitudes between the components of the electric field (for plasma) and magnetic field (for ferrite). Taking the relative magnitude as unity we may identify these field components with the cofactors of the coefficients in Eq. (11) and (14), thus we find from Equation (11) for the plasma medium.

$$\begin{aligned} E_x &= (S-n^2)(L-n^2\sin^2\theta) \\ E_y &= jD(n^2\sin^2\theta - L) \\ E_z &= (n^2-S)n^2\sin\theta\cos\theta \end{aligned} \tag{23}$$

and from Equation (14) for the ferrite medium

$$\begin{aligned} H_x &= (S-n^2)(1-n^2\sin\theta) \\ H_y &= jD(n^2\sin^2\theta - L) \\ H_z &= (n^2-S)n^2\sin\theta\cos\theta \end{aligned} \tag{24}$$

Now considering a ferrite medium in region $Z > 0$, we have the Maxwell curl equation for H of a plane wave

$$k \times H = -\omega\epsilon_o E$$

which may be written as follows.

$$\begin{bmatrix} -k\cos\theta H_y \\ k\cos\theta H_x - k\sin\theta H_z \\ k\sin\theta H_y \end{bmatrix} = -\omega\epsilon_o \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

Using Eq. (26) we have

$$E_x = \frac{k}{\omega\epsilon_o} jD(n^2\sin^2\theta - 1)\cos\theta \tag{25}$$

$$E_y = -\frac{k}{\omega\epsilon_o}(S - n^2)\cos\theta$$

$$E_z = \frac{k}{\omega\epsilon_o} jD(n^2\sin^2\theta - 1)\sin\theta$$

From Eqs. (26) or Eqs. (27) we see that x and z components of H and E have $\frac{\pi}{2}$ radians phase difference with the y components, so waves propagating in the z and x directions are elliptically polarized. For the propagation in the z direction which we are interested in deriving the input impedance in the direction normal to the boundary, we have correspondingly two input impedances for the two 90° out of phase field components as follows:

$$Z_{1z} = \frac{E_x}{H_y} = Z_o n\cos\theta$$

$$Z_{2z} = -\frac{E_y}{H_x} = Z_o\frac{n\cos\theta}{1 - n^2\sin^2\theta}$$

The normalized input impedances are

$$Z_{1in} = \frac{Z_{1z}\cos\theta_1}{Z_o} = n\cos\theta\cos\theta_1 \tag{26}$$

$$= [(n^2 - \sin^2\theta_1)(1 - \sin^2\theta_1)]^{1/2}$$

$$Z_{2in} = \frac{Z_{2z}}{Z_o\cos\theta_1} = \frac{(n^2 - \sin^2\theta_1)^{1/2}}{\cos^3\theta_1} \tag{27}$$

Similar expression can be obtained for the plasma medium in the region $z > 0$, thus

$$Z_{1in} = \frac{(L - \sin^2\theta_1)\cos\theta_1}{L(n^2 - \sin^2\theta_1)^{1/2}} \tag{28}$$

$$Z_{2in} = [(n^2 - \sin^2\theta_1)(1 - \sin^2\theta_1)]^{-1/2} \tag{29}$$

The value of n in Eqs. (26) to (29) should be calculated from the dispersion equation (12)

6. Concluding Remarks

As mentioned before there exist the dual or reciprocal relations between the input impedances and therefore the reflection and transmission coeffcients or the ferrite and plasma media. For instance in the case of static magnetic field normal to the plane of incidence Eq (18) for the perpendicular polarization of a wave incident upon a ferrite medium is the dual of Eq. (22) for the parallel polarization of a wave incident upon a plasma medium. The reason is that the anisotropic property of a ferrite medium is caused by the component of a-c magnetic field normal to the static magnetic field, while the anisotropic property of a plasma is caused by the component of a-c electric field normal to the static magnetic field. In these cases the value of n is given by $n = [(S^2-D^2)/S]^{1/2}$, the refractive index for extraordinary wave. Similar dual relations exist between Eq. (20) for the parallel polarization of a wave incident upon a ferrite medium and Eq. (21) for the perpendicular polarization of a wave incident upon a plasma medium. These equations show no anisotropic property, since the a-c magnetic field in the ferrite case and the a-c electric field in the plasma case have no component normal to the static magnetic field. Eqs. (20) and (21) correspond to the reflection and transmission of ordinary waves. The index of refraction in Eq. (20) is given by $n = \sqrt{\frac{\mu}{\mu_o}}$ and the value of n in Eq. (21) may be given by $n = (1 - \omega_p^2/\omega^2)$ for a cold plasma without collisions.

The existence of two input impedances for the case of static magnetic field normal to the plane boundary corresponds to the two types of polarizations giving rise to the ordinary and extraordinary waves. Since the value of n is obtained from the dispersion equation (12) which in turn is a function of the angle θ, the numerical computation or trial and error method should be used to determine the value of n and Z_{in}.

The condition for the complete reflection corresponds to $Z_{in} = 0$ or $Z_{in} = \infty$. In the case of Eq (22) for the plasma medium complete reflection corresponds to the cut off condition n= 0 and the resonance condition $n = \infty$. Neglecting the effect of collision and temperature these conditions are given by

cut off:
$$\frac{\omega_p^2}{\omega^2} = 1 \pm \frac{\omega_c}{\omega}$$

and resonance:
$$\frac{\omega_p^2}{\omega^2} = 1 - \frac{\omega_c^2}{\omega^2}$$

For the ferrite case the conditions for the complete reflection corresponds to $\omega = \omega_o \quad \omega_m$ (cut-off) and $\omega = \omega_o$ (resonance)

For the cases given by Eqs. (20) and (21), the condition for the complete reflection is given by $\sin\theta_i = n$. The angle θ_i exists only when $n < 1$ which is possible only for the plasma medium.

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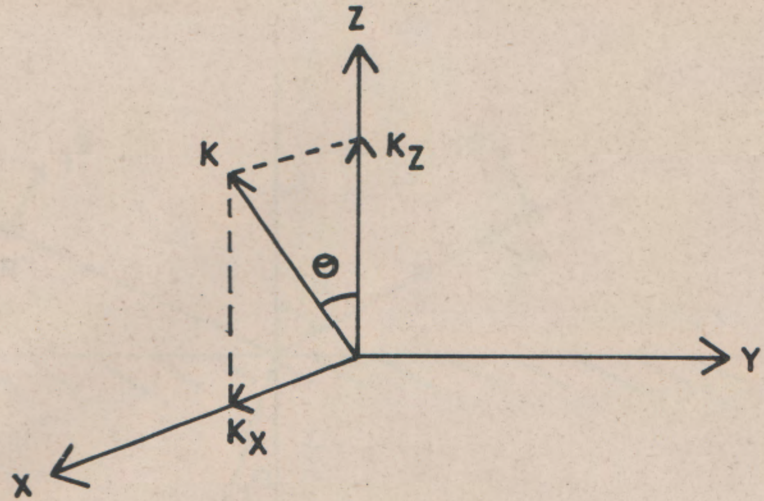


FIG·1

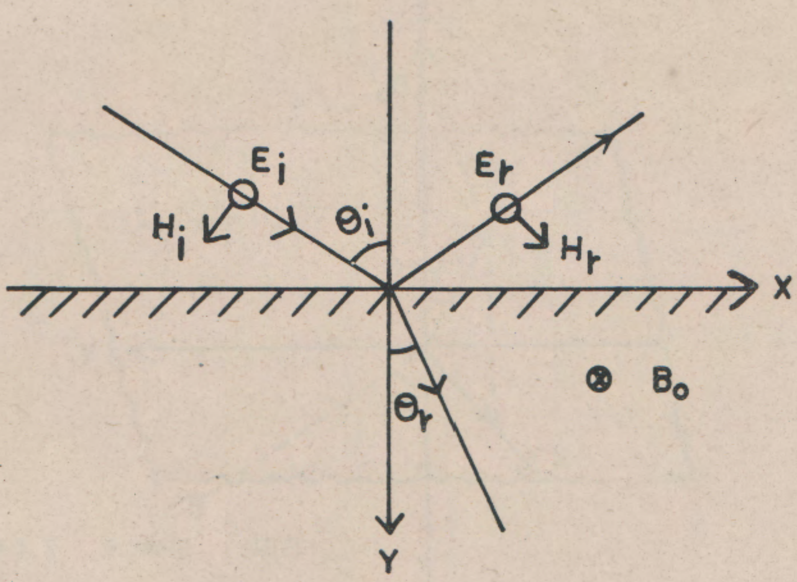


FIG · 2

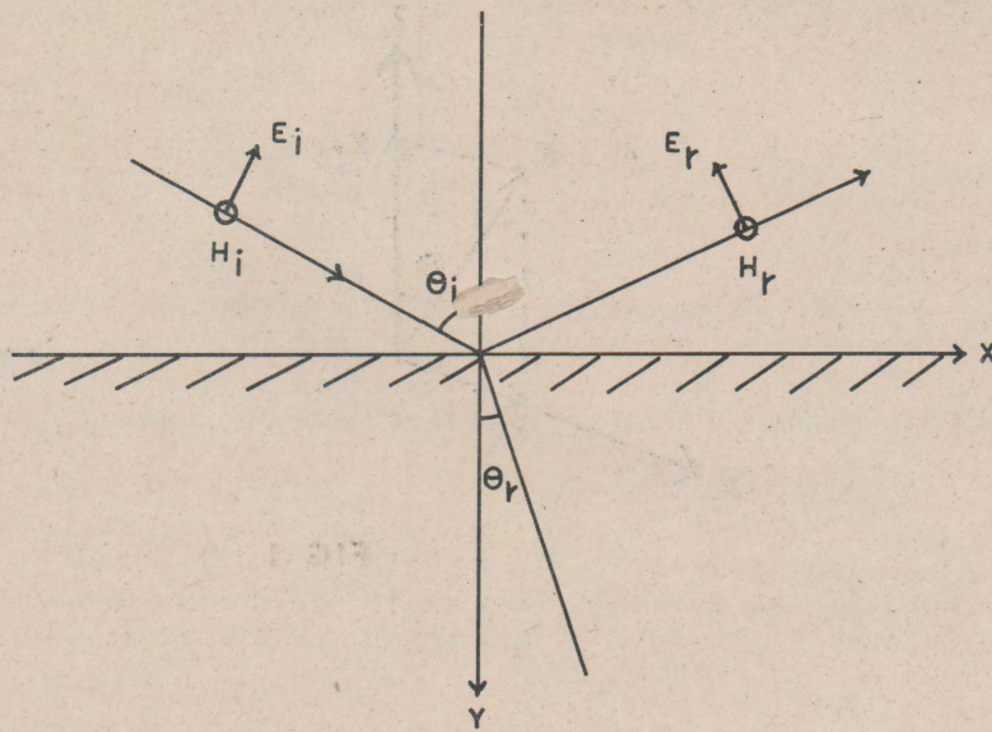
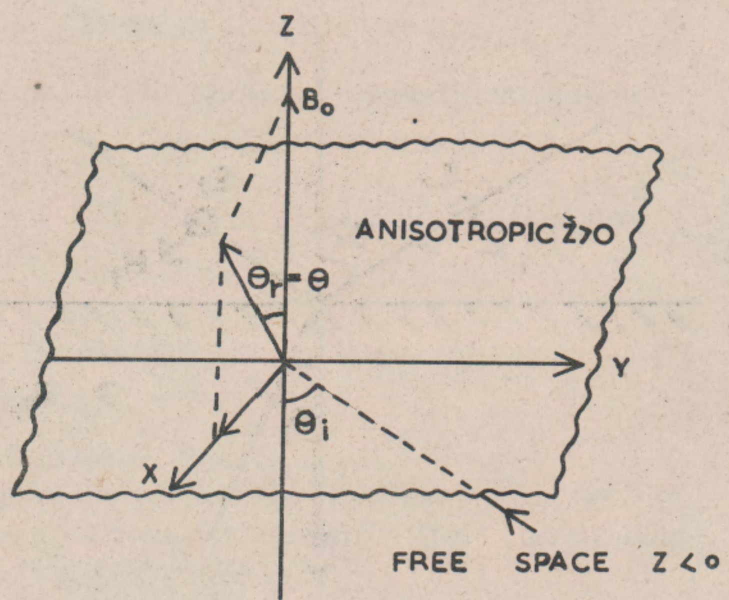


FIG. 3



B_o NORMAL TO THE BOUNDARY

FIG. 4