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A Study on the High Energy Ballistics

by

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1. Introduction

The purpose of this study is twofolds: to present systematically the development of such dynamics of motion as related to the external ballistics of long-range missiles, and to construct a gunnery table for the firing angle, angular range, altitude and flight time for firing velocities approaching the satellite region. The importance of this study is to treat the falling body problem as the motion which relates the transition from bound state to free state, and therefrom to calculate the firing angle and range as well as the time of flight in terms of the firing velocity. The scope of this study is limitted to the discussion under the following assumptions, that the burn-out time for energizing the projectile is small compared to flight time, that the dissipative effect is ignored, that the spheroidalsphape of the earth and its non-uniform density is neglected, and that the coriolis force is not considered.

2. Vertical Launching

The dynamics of motion of a body on earth's surface to escape the potential well follows the equation of motion

$$\frac{d}{dt}(mv) = -G\frac{Mm}{r^2} \tag{1}$$

where GM/r_o^2 is defined as the acceleration g and r_o the equatorial radius of the earth. Neglecting the relativistic effects we have the solution of this non-linear equation with the initial condition that at $t = 0$, velocity is v_o at the earth's surface where $r = r_o$,

$$\frac{1}{2}(v^2 - v_0^2) = GM\left(\frac{1}{r} - \frac{1}{r_o}\right) \tag{2}$$

Substituting $v \rightarrow 0$ at $r \rightarrow \infty$ yields the critical launding velocity

$$v^* = \sqrt{2r_o g} = \sqrt{2GM/r_o} \tag{3}$$

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or 25,000 miles/hr. The escape velocity from the earth surface. Defining α as the ratio v_o/v^* the maximum height and maximum altitude are calculated respectively,

$$\frac{1}{r^*} = \frac{1}{r_o}(1-\alpha^2) \qquad (4)$$

$$h^* = \frac{v_o^{\,2}}{2g}\,\frac{1}{1-\alpha^2} \qquad (5)$$

The low energy relation follows in the limit of $\alpha \rightarrow 0$, and $\alpha \rightarrow 1$ delineates the escape phenomenon as $h \rightarrow \infty$.

For low energy propulsion state Eq. (2) incorporate with Eq. (1) could be written as

$$\int_{v_o}^{v} \frac{dv}{(v^2+v_o^2)^2} = -\frac{1}{4GM}\int_{o}^{t} dt \qquad (6)$$

For the case $v^2 > 0, \alpha < 1$ Eq. (6) could be integrated [1] as

$$--\frac{1}{r_o}(1-\alpha^2)\,t = \frac{v}{v^2+v^{*2}(1-\alpha^2)} - \frac{\alpha}{v^*} + \frac{1}{v^*\sqrt{1-\alpha^2}}\left(\tan^{-1}\frac{v}{v^*\sqrt{1-\alpha^2}} - \sin^{-1}\alpha\right) \qquad (7)$$

and the time of flight t^* at $v = 0$ is

$$t^* = \frac{r_o}{v^*(1-\alpha^2)}\left(\alpha + \frac{1}{\sqrt{1-\alpha^2}}\sin^{-1}\alpha\right) \qquad (8)$$

Where $t^* \rightarrow \infty$ as $\alpha \rightarrow 1$ and for $\alpha \rightarrow 0$ t^* is v_o/g as expected. For the critical flight case $v^2 = 0$, $\alpha = 1$ Eq. (6) could be integrated as

$$\frac{v}{v^*} = 1 + \frac{v^* t}{2r_o} \qquad (9)$$

where $t \rightarrow \infty$ $v \rightarrow 0$ yields the case of floating into space with zero velocity. For the escape flight case $v^2 < 0$, $\alpha > 1$ Eq. (6) could be integrated as

$$-\frac{1}{r_o}(1-\alpha^2)t = \frac{v}{v^2-(v_0^2-v^{*2})} - \frac{\alpha}{v^*} \cdot - \frac{1}{2\sqrt{v_0^2-v^{*2}}} \ell n \frac{\sqrt{v_0^2-v^{*2}}+v}{\sqrt{v_0^2-v^{*2}}-v}$$

$$+\frac{1}{2\sqrt{v^2-v^{*2}}} \ell n \frac{\sqrt{v_0^2-v^{*2}}+v_o}{\sqrt{v_0^2-v^{*2}}-v_o} \tag{10}$$

where $v^2 = v^2 - v^{*2}$ is the asymptotic value for $t \to \infty$

3. Non-Vertical Launching

The equation of dynamics of motion in polar coordinates is characterized [2] by

$$m(\ddot{r} - r\theta^2 + \frac{GM}{r^2}) = 0 \tag{11}$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0 \tag{12}$$

Eq. (12) is the conservation of angular momentum which state the first Kepler's law. Letting $r = 1/u$ and changing timedependency to space-dependency Eq. (11) becomes

$$\frac{d^2u}{d\theta^2} + u = -\frac{mk}{p^2} \tag{13}$$

where $K = -GMm$, p is the angular momentum. The solution of Eq. (13) is

$$\frac{1}{r} = -\frac{mk}{p^2} \pm \sqrt{(\frac{mk}{p^2})^2 + \frac{2mE}{p^2}} \sin(\theta - \theta_0) \tag{14}$$

where E is the total energy

$$E = \frac{1}{2}m\mathring{r}^2 + \frac{1}{2mr^2} + \frac{k}{r} \tag{15}$$

and the coefficient of sine's is the turning points calculated from letting $\mathring{r} = 0$, θ_o determinates the orientation of the orbit. The semi-major, semi-minor axes and the eccentricity are respectively,

$$a = \frac{K}{2E} \tag{16}$$

$$b = \frac{p}{\sqrt{-2mE}} \tag{17}$$

$$\varepsilon = \sqrt{1 + \frac{2Ep^2}{mk^2}} \tag{18}$$

If the rooket is fired at $t = 0$, $r = r_o$ with an initial velocity v_o at a firing angle ψ then the following relations

$$\mathring{r} = v_0 \cos\psi \tag{19}$$

$$r\mathring{\theta} = v_0 \sin\psi \tag{20}$$

$$p = m r_0 v_0 \sin\psi \tag{21}$$

are established. Substituting in Eqs. (15) and (14) yields

$$E = \frac{1}{2} m v^{*2} (1 - \alpha^2) \tag{22}$$

$$\frac{1}{r} = \frac{1}{2 r_0 \alpha^2 \sin^2\psi} 1 + \sqrt{1 - 4\alpha^2 (1 - \alpha^2) \sin^2\psi} \sin(\theta - \theta_0) \tag{23}$$

As the rotation of elliptic orbit Q_o determinates the angular range ψ and is calculated at $t = 0$, $r = r_o$ and $\theta = \pi/2$ we have from Eq. (23)

$$\theta_0 = \cos^{-1} \left[\frac{2\alpha^2 \sin^2\psi - 1}{\sqrt{1 - 4\alpha^2 (1 - \alpha^2) \sin^2\psi}} \right] \tag{24}$$

Using the relation $\psi = 2\pi - 2\theta_o$ and taking the approximated value for α 0 in Eq. (24) yields the angular range

$$r_o\psi=\frac{v_o{}^2}{g}\sin 2\psi \tag{25}$$

which gives $r_o\psi \to \frac{v_o{}^2}{g}$ as $\psi \to \frac{\pi}{4}$. Calculating the max. ψ * yields the optimal firing angle,

$$\sin^2\psi^*=\frac{1}{2(1-\alpha^2)} \tag{26}$$

which states that $\psi^* \to \frac{\pi}{2}$ when $\pi^2 \to \frac{1}{2}$ and the satellite state occurs when $1>\alpha^2\geq\frac{1}{2}$. Using Eq. (26) we could calculate at ψ * the rotation of elliptic orbit the angular range the effective altitude as

$$\theta_o^*=\cos^{-1}\left[\frac{-\sqrt{1-2\alpha^2}}{1-\alpha^2}\right] \tag{27}$$

$$\psi^*=2\sin^{-1}\frac{2}{1-\alpha^2} \tag{28}$$

$$\frac{r_o}{r^*}=\frac{1-\alpha^2}{\alpha^2}\left[1+\sqrt{1-2\alpha^2}\right] \tag{29}$$

Eq. (28) states that $\psi^* \to \pi$ as $\alpha^2 \to \frac{1}{2}$.

4. Flight Time

Rewriting Eq. (15) as

$$t=-\int\frac{du}{u^2\sqrt{au^2+bu+c}} \tag{30}$$

where a, b and c are respectively

$$a=-r_o{}^2v_o{}^2\sin^2\psi \tag{31}$$

$$b=r_o v^{*2} \tag{32}$$

$$c=v^{*2}(\alpha^2-1) \tag{33}$$

and therefore could be integrated [1]

$$t = -\frac{r}{v^*(\alpha^2-1)}\sqrt{\frac{r_0}{r}-1+\alpha^2\left[1-\left(\frac{r_0}{r}\right)^2\sin^2\psi\right]}+\frac{r_0\alpha}{v^*(\alpha^2-1)}\cos\psi$$

$$+\frac{r_0}{2v^*(1-\alpha^2)^{3/2}}\sin^{-1}\frac{\frac{r_0}{r}-2(1-\alpha^2)}{\frac{r_0}{r}\sqrt{1-4\alpha^2(1-\alpha^2)\sin^2\psi}}$$

$$-\frac{r_0}{2v^*(1-\alpha^2)^{3/2}}\sin\frac{2\alpha^2-1}{\sqrt{1-4\alpha^2(1-\alpha^2)\sin^2\psi}} \tag{34}$$

Using Eq. (23) we could calculate the flight time by noting

$$\tau = \frac{T}{2\pi}\int_{\frac{\pi}{2}}^{\frac{\pi}{2}-\theta_0}\frac{r^2 d\theta}{ab} \tag{35}$$

where a and b are given by Eqs. (16) and (17)

$$a = \frac{r_0}{2(1-\alpha^2)} \tag{36}$$

$$b = \frac{r_0\alpha\sin\psi}{\sqrt{1-\alpha^2}} \tag{37}$$

and we have

$$\tau^* = \sqrt{2}\alpha^2\frac{r_0}{v_0}\frac{1}{(1-\alpha^2)^{3/2}}\left[\frac{1}{\sqrt{2}}\frac{1}{\alpha}\left(\sin^{-1}\frac{\cos 2\theta_0+b'}{1+b'\cos 2\theta_0}\right.\right.$$

$$\left.-\sin^{-1}\frac{\cos\theta_0+b'}{1+b'\cos\theta_0}\right)-b'\left(\frac{\sin 2\theta_0}{1+b'\cos 2\theta_0}-\frac{\sin\theta}{1+b'\cos\theta_0}\right)$$

(38)

where we have sued the relation expressed in Eq. (26) and b' is given by (39)

$$b' = \sqrt{1 - \alpha^2}. \tag{39}$$

5. Gunnery Table

The optimal firing angle ϕ*, angular range ϕ *, rotation of elliptic orbit θ_0^*, effective altitude $\frac{r_o}{r^*}$ and flight time τ* have been calculated respectively in Eqs. (25), (28), (27), (29) and (38). The following is the gunnery table for idealized flight.

α	α^2	ϕ*(degrees)	ϕ*(deg.)	θ_0*(degrees)	$\frac{ro}{r*}$	τ*(hours)
0.10	0.01	45.29	1.17	179.4	0.995	0.045
0.15	0.0225	45.66	2.64	178.7	0.989	0.069
0.20	0.04	46.19	4.78	177.6	0.980	0.094
0.25	0.0625	46.92	7.64	176.2	0.969	0.121
o.30	0.09	47.84	11.35	174.3	0.955	0.148
0.35	0.123	49.01	16.05	172.0	0.939	0.183
0.40	0.160	50.49	21.96	169.0	0.921	0.220
0.45	0.203	52.35	29.42	165.3	0.900	0.263
0.49	0.240	54.21	26.84	161.6	0.883	0.303
0.55	0.303	57.85	51.40	154.3	0.857	0.375
0.60	0.360	62.11	69.03	145.8	0.837	0.452
0.65	0.423	68.51	94.0	133	0.829	0.552
0.695	0.483	79.6	183.2	110.9	0.873	- -
0.70	0.49	81.95	147.8	106.1	0.894	0.682

The table shows that the remotest objectives on the earth are within an hour flight time. The $\frac{r_o}{r^*}$ shows a mimum in the neighborhood of $\frac{r_o}{r^*} = 0.65$ corresponding to the maximum altitude of $\frac{r_o}{r^*} = 0.830$ at $\frac{v_o}{v^*} = 0.644$.

6. Conclusion

The dynamics of motion as related to the external ballistics of long-range missiles has been systematically presented. The gunnery table for the optional firing angle, the angular range, the rotation of elliptic orbit, the effective altitude, and the flight time has been constructed.

One interesting finding may commend as follows. Using ´the Kepler's third law.

$$\tau^2 = \frac{4\pi^2}{GM} a^3 \tag{40}$$

and the relation (36) we find the flight time for an elliptic orbit as

$$\tau = -\frac{\pi}{\sqrt{2}}\sqrt{\frac{r_0}{g}}\frac{1}{(1-\alpha^2)^{3/2}} \tag{41}$$

and when $\alpha^2 \rightarrow \frac{1}{2}$ the time required for circular orbit $\alpha^2 = \frac{1}{2}$

$$\tau = 2\pi\sqrt{\frac{r_0}{g}} \tag{42}$$

to be 90 minutes, the shortest period for a satellite and Eq. (42) is the same as a simple pendulum. Furthermore v_0 for circular orbit could be calculated as 17,700 miles/hr, and $\alpha = 0.707$. The elliptic orbit results if $1 > \alpha > 0.707$, circular orbit results if $\alpha = 0.707$, and intercontinental ballistic missile results if $\alpha < 0.707$ and of course escape results if $\alpha > 1$.

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References

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2. K. R. Symon, "Mechanics", 1953, Addison-Weley Co.