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An Extension of the Dieudonne's Mean Value Theorem

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Abstract

In this paper we extend the Dieudonne's mean value theorem[1] to the f(x) which is a mapping of [a,b] into a topological linear space and the $\phi(x)$ which is a continuous realvalued function on [a,b]. One of the sufficient conditions is the existence of the right derivative of f. Particulary, if f(x) is a mapping of [a,b] into [c,d], the sufficient condition can be relaxed to the existence of the Dini derivative.

1. Introduction

It is well known that the classical mean value theorem for real valued functions is written as

$$f(b) - f(a) = f(c)[b-a]$$

and has been extended by J, Dieudonne as follows[1]:

If f(x) is a continuous mapping of [a,b] into a Banach space F, and $\phi(x)$ a continuous mapping of [a,b] into R, and if there is a denumerable subset D such that, for each $x \in [a,b]-D$, f and ϕ have both a derivative at x with respect to [a,b], and that

$$|| f(x) || < \phi' (x)$$

then $|| f(b)-f(a) || \leq \phi(b)-\phi(a)$.

This theorem holds valid even if D is an empty set.

In section 3 the extension of the Dieudonne's theorem are presented and proved in detail, and the Dieudonne's mean value theorem becomes a special case of the extended theorem. From the extension, some useful theorems regarding the inequalities are given in section 4.

2. Definitions and Notations

In order to state the theorem more precisely, first we introduce some main definition. Notations being used are standard[3].

Definition 1: If f is a real-valued function defined on the set of real numbers R, then the four Dini derivates at x are defined as follows:

$$D^+f(x) = \lim_{h\to 0^+} \sup \frac{f(x+h) - f(x)}{h}$$

$$D_+f(x) = \lim_{h\to 0^+} \inf \frac{f(x+h) - f(x)}{h}$$

$$D^-f(x) = \lim_{h\to 0^-} \sup \frac{f(x+h) - f(x)}{h}$$

$$D_-f(x) = \lim_{h\to 0^-} \inf \frac{f(x+h) - f(x)}{h}$$

Definition 2: If $D^+f(x)=D_+f(x)$, we say that there is a right derivative at x, and denote it by $D_rf(x)$. Similarly, denote the left derivative by $D_1f(x)$.

Definition 3: A nonnegative real-valued function p defined on a vector space E is called a pseudonorm if

$$p(x+y) \leq p(x) + p(y)$$

and

$$p(ax) \leq a\ p(x)$$

Definition 4: A norm in a vector space E is a mapping N of E into R, having the following properties:

(a)	$N(x) \geq 0$	for every $x \varepsilon E$
(b)	$N(x) = 0$	if and only if x=0
(c)	$N(\lambda x) = \lvert\lambda\rvert N(x)$	for any $x \varepsilon E$ and scalar λ
(d)	$N(x+y) \leq N(x) + N(y)$	for any pair x,y in E.

Definition 5: A normed space is a vector space E with a given norm on E. While a complete normed space is called a Banach space.

Definition 6: A topological linear space is a linear space E with a topology such that each of the following mappings is continuous:

(a) the mapping of the product, E x E, with the product topology, into E, which is given by

$$(x,y) \rightarrow x + y, \qquad \text{for x and y in E}$$

(b) the mapping of the product, K x E, of the scalar field on E, with the product topology, into E, which is given by

$$(a,x) \rightarrow ax \qquad \text{for a in K and x in E.}$$

3. The Main Theorems

Theorem 1. If f(x) and g(x) are continuous mappings of an interval [a, b] into an interval [c,d] with $f(a) \leq g(a)$, and suppose that there exists a denumerable subset E of [a,b) such that. for all x in [a,b) - E,

$$D^+f(x) \neq +\infty, \qquad\qquad D^+g(x) \neq -\infty$$

and

$$D^+f(x) \leq D^+g(x) \tag{1}$$

then

$$f(x) \leq g(x) \qquad \text{for all } x \text{ in } [a,b] \tag{2}$$

Proof:

Suppose that the inequality (2) is not true. Then there exists a point t in (a,b) such that $f(t) > g(t)$. Thus, the continuities of f and g imply that there exists an ε, $0 < \varepsilon < 1$, such that $g(t) + \varepsilon(b-a+1) < f(t)$.

Let $\{\rho_n \mid n=1,2,3,\ldots\ldots\ldots\}$ be the exceptional set E and define the nondecreasing function

$$\sigma(x) = \sum_{\rho_n < x} 2^{-n}$$

for $a < x \leq b$ and $\sigma(a) = 0$. Then, $0 \leq \sigma(x) \leq 1$. Now we define

$$G(x) = g(x) + \varepsilon(x-a) + \varepsilon\sigma(x).$$

It follows that

$$f(t) > G(t). \tag{3}$$

Next we define A to be the set of all x in $[a,t)$ such that $f(x) \leq G(x)$. Since $f(a) \leq g(a) = G(a)$, A is nonempty and t is the upper bound of A. Therefore, there exists an α, such that $a \leq \alpha \leq t$ and such that $\alpha = \sup A$. We need to show that

$$f(\alpha) = G(\alpha).$$

Since $\alpha = \sup A$, there exists a nondecreasing sequence Sn, $a \leq Sn \leq \alpha$, such that $Sn \to \alpha$ and $Sn \in A$. This follows that

$$f(Sn) \leq G(Sn) \leq g(Sn) + \varepsilon(Sn-a) + \varepsilon\sigma(\alpha).$$

By continuity of f and g, we have

$$f(\alpha) \leq G(\alpha). \tag{4}$$

By (3), (4), we see that $\alpha \neq t$. Therefore $a \leq \alpha < t$. Now suppose $f(\alpha) < G(\alpha)$. Then by continuity of f and g there is a nonempty interval $(\alpha, \alpha+\delta)$ contained in (a,t) such that

$$f(x) < g(x) + \varepsilon(x-a) + \varepsilon\sigma(\alpha)$$

for all x in $(\alpha, \alpha+\delta)$.

But since $\sigma(x)$ is nondecreasing $f(x) < G(x)$ for all x in $(\alpha, \alpha+\delta)$ which contradicts the definition of α as sup A. Thus $f(\alpha) = G(\alpha)$ is proved.

Now it remains to prove $\alpha = b$. Suppose $\alpha < b$. In the above analysis we have established that $a \leq \alpha < t < b$, $f(\alpha) = G(\alpha)$, and $f(x) \geq G(x)$ on $(\alpha, t]$.

Thus, $$D^+f(\alpha) \geq D^+G(\alpha). \tag{5}$$

There are two exclusive cases under consideration: Either α belongs to E or it does not.

Case 1. $\alpha \notin E$. Then, (1) holds at $x = \alpha$

i.e. $D^+f(\alpha) \leq D^+g(\alpha)$

since

$$D^+g(\alpha) \leq D^+g(\alpha) + \varepsilon + \varepsilon D_+\sigma(\alpha) \leq D^+G(\alpha),$$

Thus

$$D^+f(\alpha) \leq D^+G(\alpha)$$

this is contrary to the inequality (5).

Case 2. $\alpha \varepsilon E$, say $\alpha = \rho_m$. Then, by continuity of f and g there exists a δ, $0 < \delta < t-\alpha$, such that for all $x \varepsilon (\alpha, \alpha+\delta)$,

$$|f(x) - f(\alpha)| \leq \frac{\varepsilon}{2} \cdot 2^{-m}$$

and

$$|g(x) - g(\alpha)| \leq \frac{\varepsilon}{2} \cdot 2^{-m}$$

Thus for all $x \varepsilon (\alpha, \alpha+\delta)$

$$\begin{aligned} f(x) \leq f(\alpha) + \frac{\varepsilon}{2} \cdot 2^{-m} &= G(\alpha) + \frac{\varepsilon}{2} \cdot 2^{-m} \\ &= g(\alpha) + \varepsilon(\alpha - a) + \varepsilon\sigma(\alpha) + \frac{\varepsilon}{2} \cdot 2^{-m} \\ &\leq g(x) + \varepsilon\{2^{-m} + \sigma(\alpha)\} + \varepsilon(x-a) \\ &\leq G(x), \end{aligned}$$

which contradicts the definition of α as sup A. Hence $f(x) \leq g(x)$ for all x in $[a,b]$ and the theorem is proved.

Similary we may extend the Dieudonne's theorem for the left Dini Derivative as follows:

Theorem 2. Let $f(x)$ and $g(x)$ be continuous mappings of an interval $[a,b]$ into an interval $[c,d]$ with $f(a) \geq g(a)$, and suppose that there exists a denumerable subset E of $(a,b]$ such that, for all x in $(a,b] - E$,

$$D^-f(x) \neq +\infty \qquad\qquad D^-g(x) \neq -\infty$$

and

$$D^-f(x) \leq D^-g(x).$$

then

$$f(x) \geq g(x) \qquad\qquad \text{for all } x \text{ in } (a,b).$$

The proof of theorem 2 is completely analogous to that of theorem 1, hence it is omitted.

Theorem 3. If $u(x)$ is a continuous mapping of $[a,b]$ into a topological linear space E, if there is a denumerable subset B of $[a,b)$ such that $u(x)$ possesses a vector right derivative $D_r u(x)$ at all x in $[a,b)-B$, and if $v(x)$ is a continuous realvalued function on $[a,b]$ such that

$$P(D_r u(x)) \leq D^+ v(x)$$

for all x in (a, b)-B where p is any continuous pseudonorm on E, then

$$p(u(b) - u(a)) \leq v(b) - v(a).$$

Proof:

We let $f(x) \equiv p(u(x)-u(a))$ and $g(x) \equiv v(x)-v(a)$ in theorem 1. Since $D^+ f(x) \leq p(D_r u(x))$ whenever $D_r u(x)$ exists, hence the relation

$$p(u(b) - u(a)) \leq v(b) - v(a)$$

is true by theorem 1.

The Dieudonne's mean-value theorem[1] follows from theorem 3 Where E is a Banach space with norm $\|\cdot\|$, $p \equiv \|\cdot\|$, and u(x) has a vector derivative at all but a countable number of points of [a, b).

4. Corollaries

Further, by theorem 1,2 and theorem 3, we infer the following corollaries:

Corollary 1. If there is a denumerable subset D of [a, b] such that, for each $x \in [a,b]-D$, f has at x a vector right derivative $D_r f(x)$ with respect to [a, b] such that $p(D_r f(x)) \leq M$, then

$$p(f(b) - f(a)) \leq M(b-a).$$

This corollary follows immediately from theorem 3, where $f \equiv u$ $v(x) = M(x-a)$.

Corollary 2. Let D denote any one of the four Dini derivates. If f(x) is continuous on [a, b] and $Df(x) \geq 0$ on [a, b] except at a countable number of points, then f(x) is nondecreasing on [a, b]. On the other hand, if $Df(x) \leq 0$ on [a, b] except at a countable number of points, then f(x) is nonincreasing on [a, b].

This corollary has been proved by R.P. Boas, Jr[2]. However, the proof follows from theorem 1 if D is D^+ or D_+ and from theorem 2 if D is D^- or D_-.

Corollary 3. If f(x) and g(x) are continuous real-value functions on an interval [a, b], if D denote any one of the Dini derivates and if Df(x) and Dg(x) are finite and equal on [a, b] except at a countable number of points, then f(x) and g(x) differ only by a constant on [a, b].

Proof:

If $D = D^+$ or D_+ we appeal to theorem 1, while if $D = D^-$ or D_- we appeal to theorem 2. we indicate the argument only in the case $D = D^+$, since the other three cases are entirely analogous.

Define

$$G(x) \equiv g(x) + f(a) - g(a).$$

Then

$$f(a) = G(a)$$

and

$$D^{+}f(x) = D^{+}G(x) = \text{finite}$$

except possibly on a countable set. Thus theorem 1 gives both $f(x) \leq G(x)$ and $G(x) \leq f(x)$ on $[a,b]$. These in turn imply that $f(x) = G(x) = g(x) + f(a) - g(a)$ on $[a,b]$.
Therefore,

$$f(x) - g(x) = f(a) - g(a) \text{ on } [a,b].$$

A similar corolly has been mentioned by R.P. Boas, Jr.[2], but here we prove that $Df(x)$ and $Dg(x)$ are finite and equal on $[a,b]$ except at a countable number of points.

5. Concluding Remarks

We may apply the above results to prove some fundamental theorems of the calculus. For example,

Suppose f is a continuous mapping of $[a,b]$ into R such that, at every $x \in [a,b]$, f has a derivative with respect to $[a,b]$, and $m \leq f'(x) \leq M$. Then

$$m(b-a) \leq f(b) - f(a) \leq M(b-a).$$

Moreover, we can infer them to discuss the convergent or uniformly convergent sequence $\{f_n(x)\}$. Further research on a vector-valued mapping is continued.

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