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An Application of Vector Space to Sampling Theorem

K. C. Lin

I. Introduction

There are some important concepts in communication theory which are purely of a vector space nature. One of them is the sampling theorem [1], which says that in a certain class of signals a particular signal is completely determined by its values (samples) at an equally spaced set of times extending forever.

II. Subject

Although it is usually not stated explicitly, the set of functions considered as signals forms a vector space over the real numbers. That is, if f(t) and g(t) are signals, then $(f+g)(t) = f(t)+g(t)$ is a signal and $(af)(t) = af(t)$ is also a signal where a is a real number [2]. Usually the vector space of signals is infinite dimensional so that many of the concepts and theorems developed by means of the theory of Fourier integrals [3]. In order to bring the topic within this paper it is assumed that the signals persist for only a finite interval of time and that there is a bound for the highest frequency that will be encountered. If the time interval is of length T, this assumption has the implication that each signal f(t) can be represented as a finite series of the form

$$f(t) = \frac{1}{2} a_o + \sum_{k=1}^{N} a_k \cos \frac{2\pi kt}{T} + \sum_{k=1}^{N} b_k \sin \frac{2\pi kt}{T} \quad (1)$$

Formula (1) is in fact just a precise formulation of the vague statement that the highest frequency to be encountered is bounded. Since the coefficients can be taken to be arbitrary real numbers, the set of signals under consideration forms a real vector space V of dimension 2N+1. We show that f(t) is determined by its values at 2N+1 points equally spaced in time. This statement is known as the finite sampling theorem [3].

The classical infinite sampling theorem [4] from communication theory requires an assumption analogous to the assumption that the highest frequencies are bounded. Only the assumption that the signal persists for a finite interval of time is relaxed. In any practical problem some bound can be placed on the duration of the family of signals under consideration. Thus the restriction on the length of the time interval does not alter the significance or spirit of the theorem in any way.

Consider the function

$$g(t) = \frac{1}{2N+1}\left(1 + 2\sum_{k=1}^{N} \cos\frac{2\pi k}{T} t\right) \in V \qquad (2)$$

$$g(t) = \frac{\sin\frac{\pi}{T}t + \sum_{k=1}^{N} 2\cos\frac{2\pi k}{T}t \,\sin\frac{\pi}{T}t}{(2N+1)\sin\frac{\pi}{T}t}$$

$$= \frac{\sin\frac{\pi}{T}t + \sum_{k=1}^{N} \sin\left(\frac{2\pi k}{T}t + \frac{\pi}{T}t\right) - \sin\left(\frac{2\pi k}{T}t - \frac{\pi}{T}t\right)}{(2N+1)\sin\frac{\pi}{T}t}$$

$$= \frac{\sin\frac{2N+1}{T}\pi t}{(2N+1)\sin\frac{\pi}{T}t} \qquad (3)$$

From (2) and (3) it follows that $g(0)=1$ and $g\left(\frac{Tj}{2N+1}\right)=0$ for $0 \leq |j| \leq N$

Consider the functions

$$g_k(t) = g\left(t - \frac{Tk}{2N+1}\right), \quad \text{for } k = -N, -N+1, \ldots\ldots, N. \qquad (4)$$

These 2N+1 functions are all members of V. Furthermore, for $t_j = \frac{Tj}{2N+1}$ it is easily seen that $g_j(t_j) = 1$ while $g_k(t_j) = 0$ for $k \neq j$. Thus the 2N+1 functions obtained are linearly independent. Since V is of dimension 2N+1 it follows that the set

$$\{g_k(t) \mid k = -N, \ldots\ldots, N\}$$

is a basis of V. These functions are called the sampling functions

If f(t) is any element of V it can be written in the form[5].

$$f(t) = \sum_{k=-N}^{N} d_k g_k(t) \qquad (5)$$

However,

$$f(t_j) = \sum_{k=-N}^{N} d_k g_k(t_j) = d_j \tag{6}$$

consequently it follows that

$$f(t) = \sum_{k=-N}^{N} f(t_k) g_k(t) \tag{7}$$

Thus the coordinates of f(t) with respect to the basis $\{ g_k(t) \}$ are $(f(t_{-N}), \ldots\ldots, f(t_N))$ and these samples are sufficient to determine f(t).

It is of some interest to express the elements of the basis $\{ \frac{1}{2}, \cos\frac{2\pi}{T}t, \ldots\ldots\ldots, \sin\frac{2\pi N}{T}t \}$ in terms of the basis $\{ g_k(t) \}$.

$$\begin{aligned} \frac{1}{2} &= \sum_{k=-N}^{N} \frac{1}{2} g_k(t) \\ \cos\frac{2\pi j}{T}t &= \sum_{k=-N}^{N} \cos\frac{2\pi k}{T} t_j g_k(t) \\ \sin\frac{2\pi j}{T}t &= \sum_{k=-N}^{N} \sin\frac{2\pi k}{T} t_j g_k(t) \end{aligned} \tag{8}$$

Expressing the elements of the basis $\{ g_k(t) \}$ in terms of the basis $\{ \frac{1}{2}, \cos\frac{2\pi}{T}t, \ldots\ldots, \sin\frac{2\pi N}{T}t \}$ is but a matter of the definition of the $g_k(t)$;

$$\begin{aligned} g_k(t) &= g\left(t - \frac{Tk}{2N+1}\right) \\ &= \frac{1}{2N+1}\left[1 + 2\sum_{j=1}^{N} \cos\frac{2\pi j}{T}\left(t - \frac{Tk}{2N+1}\right)\right] \\ &= \frac{1}{2N+1}\left(1 + 2\sum_{j=1}^{N} \cos\frac{2\pi jk}{2N+1}\cos\frac{2\pi j}{T}t + 2\sum_{j=1}^{N} \sin\frac{2\pi jk}{2N+1}\sin\frac{2\pi j}{T}t\right) \end{aligned} \tag{9}$$

With this interpretation, formula (1) is a representation of f(t) in one coordinate system and (7) is a representation of f(t) in another. To express the coefficients in (1) in terms of the coefficients in (7) is but a change of coordinates.

Thus one obtains

$$a_j = \frac{2}{2N+1}\sum_{k=-N}^{N} f(t_k)\cos\frac{2\pi jk}{2N+1} = \frac{2}{2N+1}\sum_{k=-N}^{N} f(t_k)\cos\frac{2\pi j}{T}t_k$$

$$b_j = \frac{2}{2N+1}\sum_{k=-N}^{N} f(t_k)\sin\frac{2\pi jk}{2N+1} = \frac{2}{2N+1}\sum_{k=-N}^{N} f(t_k)\sin\frac{2\pi j}{T}t_k \qquad (10)$$

III. Discussion

There are several ways to look at formula(10). Those familiar with the theory of Fourier series[6] will see the a_j and b_j as Fourier coefficients with formula (10) using finite sums instead of integrals. Those familiar with probability theory[7] will see the a_j as covariance coefficients between the samples of f(t) and the samples $\cos\frac{2\pi j}{T}t$ at times t_k: And we have just viewed them as formulas for a change of coordinates.

The vector space is of dimension aN+1 and we need 2N+1 samples spread equally over an interval of length t, or (2N+1)/T samples per unit time. Since N/T = W is the highest frequency present in the series (1), it is obvious that for large intervals of time approximately 2W samples per unit time are required to determine the signal. The infinite sampling theorem, referred to at the beginning of this paper, says that if W is the highest frequency present, then 2W samples per second suffice to determine the signal. The finite sampling theorem has the practical advantage of providing effective formulas for determining the function f(t) and the Fourier coefficients from the samples.

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