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A GENERALIZED FUNCTION-THEORETIC TECHNIQUE FOR THE SOLUTION OF SOME DIFFRACTION PROBLEMS*

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Abstract—*A straightforward approach to the solution of a general class of Wiener-Hopf equations is developed. The advantage of the present procedure is to apply the topological concept so that a rather difficult step of performing factorization in the usual procedure can be avoided. A new theorem regarding additive decomposition is deduced and verified in some detail; a theorem of approximating a function in a topological space is presented; in addition, an alternative view of Laplace transformation is given. The solutions of two typical diffraction problems are used to demonstrate the method; moreover application of the concept of a generalized operator to simplity problem formulation is well illustrated.*

1. INTRODUCTION

It has been shown in recent years that by an application of the Wiener-Hopf technique it is possible to obtain exact solutions of some problems in diffraction theory[1]. Radlow[2,3] has generalized the Wiener-Hopf technique from one to two complex variables, and so obtained the closed-form solutions for both right-angled dielectric wedge and quarter plane problems. His method in dealing with the transform equations is less systematic than the one presented here and moreover, the solutions are too complicated to be inverse transformable. To the auther's knowledge his solutions are wrong. In fact, most diffraction problems cannot be solved exactly. Therefore, we are seeking an approximate technique which will provide the physical insight of a problem.

Abandonning the hope of securring an exact solution, we develop a method without the necessity of performing factorization, which has been acknowledged as a crucial step in the Wiener-Hopf procedure. This work presents a systematic procedure for obtaining an inverse transformable solution which is a linear combination of dense sets of functions in the Banach spaces. These dense set can be conveniently combined into the discrete and continuous modes which are characterized by the physical geometry of the problem considered. The present procedure enable us to tackle any transform equation of one, two, or n complex variables, and will give a solution which provides the best physical insight.

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The relevent notations and theory required are rigorously summarized in the next two sections. In Section 4 and 5, two physical illustrations are given.

2. LAPLACE TRANSFORMATIONS

This section is primarily concerned with those aspects of this vast subject which are relevant to the material in the present technique.

A. One Dimension

To understand Laplace transformation thoroughly we start the discussion with a spectral representation of the operator $T=\frac{d^2}{dx^2}$ whose domain is the set of all twice differentiable functions $u(x)$, $-\infty<x<\infty$, such that

$$\int_{-\infty}^{\infty}|u(x)|^2\,dx<\infty \tag{2.1}$$

Note that the set of all square-integrable functions will be denoted by $L_2(-\infty,\infty)$. We can readily show that all improper eigenfunctions $\exp(-kx)$, $-\infty<k<\infty$ of T form a complete set in the space $L_2(-\infty,\infty)$. If we normalize the eigenfunctions $u(x,s)$ in accordance with the sense

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\exp(-sx)\exp(s'x)dx=\delta(s-s') \tag{2.3}$$

then we immediately obtain the completeness relation by replacing s by x, s' by x', and x by s:

$$\delta(x-x')=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\exp[-s(x-x')]\,ds. \tag{2.4}$$

Now every function $f(x)$ in $L_2(-\infty,\infty)$ can be expressed in terms of these improper eigenfunctions as

$$f(x)=\int_{-\infty}^{\infty}f(\xi)\delta(x-\xi)d\xi=\frac{1}{2\pi i}\int_{-\infty}^{\infty}F(s)\exp(sx)ds. \tag{2.5}$$

where $$F(s)=\int_{-\infty}^{\infty}f(\xi)\exp(-s\xi)d\xi \tag{2.6}$$

We call Eq. (2.6) as Laplace transformation and Eq. (2.5) as inverse Laplace transformation. Therefore we see that Laplace transformation is nothing but an improper eigenfunction expansion.

In this paper we usually write Laplace transform and its inverse symbolically as

$$F(s)=\mathfrak{L}_x f(x) \quad \text{and} \quad f(x)=\mathfrak{L}_x^{-1}F(s) \tag{2.7}$$

where $s=u+iv$. It is understood that these Laplace integrals are written in the Lebesgue sense. Any ordinary function $f(x)$ can be decomposed as

$$f(x)=f_+(x)l(x)+f_-(x)l(-x) \tag{2.8}$$

where $f_+(x)=\begin{cases}f(x), & x>0\\ 0, & x<0\end{cases}$; $f_-(x)=\begin{cases}0, & x>0\\ f(x), & x<0\end{cases}$;

$l(x)$ denote the unit step function. This follows the expressions

$$F(s)=F_+(s)+F_-(s)\,;\qquad F_+(s)=\mathfrak{L}_x f_+(x)\,;\qquad F_-(s)=\mathfrak{L}_x f_-(x)\,. \tag{2.9}$$

To fit for our purpose, a basic theorem concerning the analyticity of $F(s)$ is written in a particular form:

Theorem 2.1 If

$$\int_0^\infty |f_+(x)|^2\,dx<\infty$$

and $\quad |f_+(x)|<c_1\exp(u_+x)\quad$ as $\quad x\to\infty,$

then in the right half-plane $u>u_+$, $F_+(s)$ is analytic and square integrable in the following sense

$$\int_{-\infty}^\infty |F_+(u+iv)|^2\,dv<\infty\,. \tag{2.10}$$

Similarly, if

$$\int_{-\infty}^0 |f_-(x)|^2\,dx<\infty$$

and $\quad |f_-(x)|<c_2\exp(u_-x)\quad$ as $\quad x\to-\infty,$

then $F_-(s)$ is analytic and square integrable for all $u<u_-$. This follows that $F(s)$ is analytic and square integrable in the strip $u_+<u<u_-$.

Proof. It is sufficient to prove the first part of the theorem. By Schwarz's inequality, we have

$$\begin{aligned}&\int_0^\infty |f_+(x)\exp(-sx)|\,dx\\ &\qquad=\int_0^M |f_+(x)\exp(-ux)|\,dx+\int_M^\infty |f_+(x)\exp(-ux)|\,dx\\ &\qquad\le\Big\{\int_0^M |f_+(x)|^2\,dx\int_0^M\exp(-2ux)\,dx\Big\}^{1/2}+\int_M^\infty\exp[-(u-u_+)x]\,dx\end{aligned}$$

The first term on the right converges, while the second term also converges if $u>u_+$. This follows immediately that the Laplace integral $\mathfrak{L}_x f_+(x)$ converges uniformly and so is analytic all $u>u_+$. By Parseval's formula, we have

$$\int_{-\infty}^\infty |F_+(u+iv)|^2\,dv=2\pi\int_0^\infty |f_+(x)\exp(-ux)|^2\,dx \tag{2.11}$$

Separating the interval $(0,\infty)$ into $(0,M)$ and (M,∞) we may readily prove that the integral on the right converges, so does the integral on the left. This completes the proof of the theorem.

Hereafter we shall use H_+ to denote the right half-plane, defined by $u>u_+$, while use H_- to denote the left half-plane $u<u_-$. This set of all functions analytic and square integrable in H_+ forms a Banach space with the norm specified by

$$\|F_+(s)\|=\Big[\int_{-\infty}^\infty |F_+(iv)|^2\,dv\Big]^{1/2} \tag{2.12}$$

This Banach space will be denoted by $B(H_+)$. Similarly we let $B(H_-)$ denote the Banach space of all functions analytic and square integrable in H_-, the corresponding norm is given by

$$\|F_-(s)\| = \left[\int_{-\infty}^{\infty} |F_-(iv)|^2 dv\right]^{1/2}. \tag{2.13}$$

The proof for the completeness has been given in[5].

It is known that: If $F(s)$ has a bounded norm, then there exists a unique inverse $f(x)$.

B. Two Dimensions

The boundary-value problems occurring in mathematical physics usually involve functions which are square integrable, and therefore we shall restrict our discussion to such functions. Here we just write a theorem for a function of two independent variables; however, the theorem may be generalized for a function of n independent variables.

We first introduce the double Laplace transform defined by

$$F(\underline{s}) = \mathfrak{L}_{\underline{x}} f(\underline{x}) = \int_X f(\underline{x}) \exp(-\underline{s}\cdot\underline{x})\, d^2x$$

where $\underline{x} = (x_1, x_2)$, $\underline{s} = (s_1, s_2)$, $s_j = u_j + iv_j$, $(j=1,2)$, $\underline{s}\cdot\underline{x} = s_1x_1 + s_2x_2$, and X denote the entire x_1x_2-plane.

If we introduce the functions, $(n=1,2,3,4)$,

$$f_n(x) = \begin{cases} f(\underline{x}), & \underline{x} \in Q_n \\ 0, & \underline{x} \in (X-Q_n) \end{cases},$$

where Q_n denotes the nth quadrant of X and $X-Q_n$ denotes the domain outside Q_n, then

$$F_n(\underline{s}) = \mathfrak{L}_{\underline{x}} f_x(x),$$

and

$$F(s) = \sum_{n=1}^{4} F_n(s). \tag{2.14}$$

A basic theorem concerning the existence of $F_n(s)$ is contained in the following:

Theorem 2.2 If the function $f(\underline{x}) \exp(-\underline{s}\cdot\underline{x})$ is absolutely integrable over Q_1 in the Lebesque sense for a pair of real values $(u_1, u_2) = (c_1, c_2)$, then the integral of $f(x) \exp(-s\cdot x)$ converges uniformly for $u_1 \geq c_1$, $u_2 \geq c_2$ and so analytic in the corresponding domain. Similarly, we may state the existence theorem for the Laplace integrals $\mathfrak{L}_{\underline{x}} f_n(\underline{x})$, $(n=2,3,4)$, accordingly. Therefore, the analytic domains $T(D,n)$ of $F_n(s)$ are defined by $-\infty < v_j < \infty$, $(j=1,2)$ and

$$(D,1): \ u_1 > u_1^+, \ u_2 > u_2^+;$$
$$(D,2): \ u_1 < u_1^-, \ u_2 > u_2^+;$$
$$(D,3): \ u_1 < u_1^-, \ u_2 < u_2^-;$$
$$(D,4): \ u_1 > u_1^+, \ u_2 < u_2^-. \tag{2.15}$$

Speaking of the existence theorem, there is a difference between the one-dimensional Laplace integral and the two-dimensional Laplace integral. Since this difference does not affect our special application, we shall not study it here. Now we rewrite Theorem 2.2 in a setting which will be fit for our work.

Corollary If $f_n(\underline{x})$, $(n=1,2,3,4)$, is square integrable over X, then $F_n(\underline{s})$ is analytic in $T(D,n)$. Moreover, $F_n(\underline{s})$ has a bounded norm

$$\|F_n(\underline{s})\|=\left[\int |F_n(\underline{u}+i\underline{v})|^2\, d^2v\right]^{1/2}$$

for all $\underline{u}$ in (D,n). If $f_n(x)$ is square integrable, then $f_n(\underline{x})\exp(-\underline{s}\cdot\underline{x})$ is absolutely integrable for some $\underline{u}$. For the second part of the corollary, we may prove it by using Parseval's formula.

3. ANALYTIC FUNCTION THEORY

We shall develop the theory and representations of the functions which are analytic and of bounded norm in associated half-planes.

Theorem 3.1 Consider the functions defined by the integrals of the type

$$F(s)=\int_C g(z)\,h(s,z)\,dz \tag{3.1}$$

Suppose (1) $h(s,z)$ is a function of complex variables s and z where s lies inside a region R and z lies on a rectifiable contour C; (2) $h(s,z)$ is analytic function of s for every z on C; (3) $g(z)$ has only a finite number of finite discontinuities on C; (4) $g(z)$ is bounded except a finite number of points. If z_0 is such a point, then

$$\int_C g(z)\,h(s,z)\,dz=\lim_{\delta\to 0}\int_{C-\delta} g(z)\,h(s,z)\,dz$$

exists. Then,

(1) $F(s)$ defined by (3.1) is analytic in R;

(2) For every s_0 in R, we have

$$\lim_{s\to s_0} F(s)=\int_C g(z)\lim_{s\to s_0} h(s,z)\,dz; \tag{3.2}$$

(3)
$$\frac{d^n F(s)}{ds^n}=\int_C g(z)\frac{\partial^n}{\partial s^n}h(s,z)\,dz. \tag{3.3}$$

This theorem may be readily proved by using the uniform convergence of the integral (3.1). Refer to[6].

Theorem 3.2 Let $F(s)$ be an analytic function of one complex variable s, analytic in the strip $u_+<u<u_-$, such that $F(s)=Q(|s|^{-p})$, $p>2$, for $|s|\to\infty$, uniformly for all arguments of s in the strip. Then, for $u_+<c_+<u<c_-<u_-$,

$$F(s)=F_+(s)+F_-(s) \tag{3.4}$$

where $\quad F_+(s)=-\frac{1}{2\pi i}\int_{c_+ +i\infty}^{c_+ +i\infty}\frac{F(z)\,dz}{z-s},\qquad F_-(s)=\frac{1}{2\pi i}\int_{c_- -i\infty}^{c_- +i\infty}\frac{F(z)}{z-s}\,dz,$

$F_+(s)$ is analytic for $u>c^+$ and $F_-(s)$ is analytic for $u<c_-$; and moreover,

$$\lim_{s\to\infty} s\,F_+(s) = \lim_{s\to-\infty} (-s)\,F_-(s), \tag{3.5}$$

$$\lim_{s\to\infty} (-s^2)\frac{d}{ds}[sF^+(s)] = \lim_{s\to-\infty} s^2\frac{d}{ds}[sF_-(s)]\,. \tag{3.6}$$

Proof. Integrating $\frac{F(z)}{z-s}$ over the rectangle with the vertices $c_+\pm ia$ and $c_-\pm ia$, by Canchy's theorem we obtain

$$2\pi i F(s) = \left\{\int_{c_++ia}^{c_+-ia} + \int_{c_+-ia}^{c_--ia} + \int_{c_--ia}^{c_-+ia} + \int_{c_-+ia}^{c_++ia}\right\} \frac{F(z)}{z-s}dz\,.$$

From our assumption as regards the behavior of $F(z)$ as $|z|\to\infty$ in the strip, the second and fourth integrals vanish as $a\to\infty$, therefore we are left with the required equation

$$F(z) = -\frac{1}{2\pi i}\int_{c_+-i\infty}^{c_++i\infty} \frac{F(z)\,dz}{z-s} + \frac{1}{2\pi i}\int_{c_--i\infty}^{c_-+i\infty} \frac{F(z)}{z-s}dz$$

As the first integral on the right converges uniformly for all $u>c_+$, we have

$$\frac{d^n}{ds^n}\left[-\frac{1}{2\pi i}\int_{c_+-i\infty}^{c_++i\infty} \frac{F(z)}{z-s}dz\right] = (-1)^{n+1}\frac{n!}{2\pi i}\int_{c_+-i\infty}^{c_++i\infty} \frac{F(z)}{(z-s)^{n+1}}dz$$

Thus, the first integral is analytic in the right half-plane H_+; similarly the second integral is anlaytic in H_-. This completes the proof of Eq. (3.4).

To prove Eq. (3.5), without loss of generality we assume that $F(z)$ has poles in addition to the branch points, one located on the left of the strip and the other on the right. By Theorem 3.1 deform the contour of the integral $F_+(s)$ to the left half-plane so that the limiting operation and the integrating operator are interchangeable. As shown in Fig. 1, this deformation can be accomplished by connecting the contour $z=c_+$ and the contour l_+ around the branch cut with the infinite arcs.

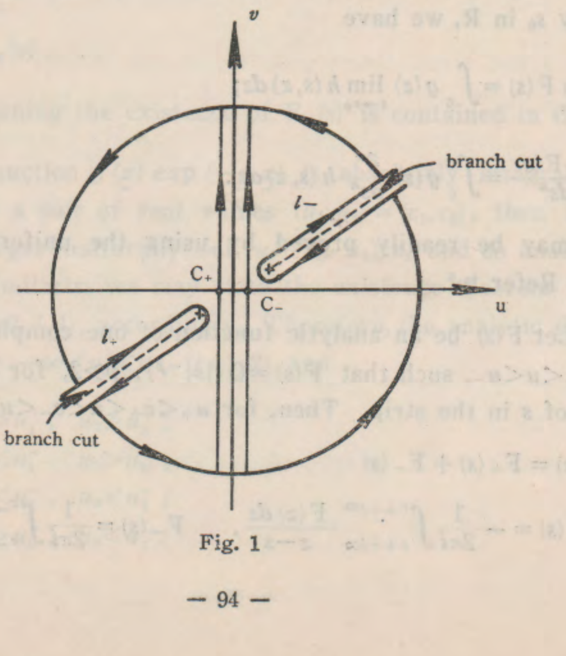

Fig. 1

As the integral over the infinite arc vanishes, we obtain

$$\lim_{s\to\infty} sF_+ = \frac{1}{2\pi i}\int_{c+} F(z)\,dz - \frac{1}{2\pi i}\int_{l+} F(z)\,dz$$
$$= R[F; H_-] - \frac{1}{2\pi i}\int_{l+} F(z)\,dz \qquad (3.7)$$

where the closed contour C_+ consists of the line $z=c_+$, the contour l_+ and the infinite arc. $R[F; H_+]$ denote the sum of the residues of poles of $F(z)$ in H_-. With the arguments similar to those above, we have

$$\lim_{s\to-\infty}(-s)F_-(s) = -R[F; H_+] + \frac{1}{2\pi i}\int_{l_-} F(z)\,dz. \qquad (3.8)$$

where $R[F; H_+]$ denotes the sum of residues of poles of $F(z)$ in H_+. Now we perform the integration of $F(z)$ over the infinite circle by deforming the contour around the branch cuts. With the residue theorem we find

$$\frac{1}{2\pi i}\left\{\int_{l+} + \int_{l_-}\right\} F(z)\,dz = R[F; H_+] + R[F; H_-]. \qquad (3.9)$$

With Eqs. (3.7), (3.8) and (3.9) we may readily prove that

$$\lim_{s\to\infty} sF_+(s) = \lim_{s\to-\infty}(-s)F_-(s)$$

The other asymtotic relation (3.6) may be verified by similar means together with the residue relation,

$$\frac{1}{2\pi i}\int_{l+} zF(z)\,dz + \frac{1}{2\pi i}\int_{l_-} zF(z)\,dz = R[zF; H_-] + R[zF; H_+]. \qquad (3.10)$$

This completes the proof of the theorem.

It remains to investigate the structure of $B(H_+)$ and $B(H_-)$. We have shown that the set of functions, $(a>0)$.

$$\frac{(s-a)^m}{(s+a)^{m+1}}, \quad m=0, 1, 2, \ldots.$$

is complete in the space $B(H_+)$, while the set

$$\frac{(s+a)^m}{(s-a)^{m+1}}, \quad m=0, 1, 2, \ldots.$$

is complete in $B(H_-)$. However, these sets are impractical, since these will give very slowly convergent serieses[7]. For practical use, we shall present the dense set in the following.

Theorem 3.3 The set of functions

$$\frac{1}{c_+ + im\Delta v - s}, \quad m=0, \pm1, \pm2, \ldots. \qquad (3.11)$$

is dense in $B(H_+)$, while the set

$$\frac{1}{c_- + im\Delta v + s}, \quad m=0, \pm1, \pm2, \ldots. \qquad (3.13)$$

is dense in $B(H_-)$.

Proof. It is sufficient to prove the first part of the theorem. Every function $F_+(s)$ in $B(H_+)$ can be expressed as an integral of the type

$$F_+(s) = -\frac{1}{2\pi i}\int_{c_+-i\infty}^{c_++i\infty}\frac{F_+(z)}{z-s}\,dz$$

Now for every $\varepsilon>0$, there exist an interval Δv small enough such that

$$\left|F_+(s) - \sum_{m=-\infty}^{\infty}\frac{-1}{2\pi i}\,\frac{F_+(c_++im\Delta v)\,\Delta v}{c_++im\Delta v-s}\right| < \varepsilon.$$

This implies that $F_+(s)$ can be expressed approximately as a linear combination of the set (3.11). This complete the proof.

B. Two Variables

In this paragraph we shall confine our discussion on the class of functions which are analytic and of the bounded norm,

$$\|F(u+iv)\| = \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(u+iv|^2 d^2 v\right]^{1/2} < \infty, \tag{3.13}$$

over a tube $T(D)$ defined by

$$T(D):\ u_j^+ < u_j < u_j^-,\quad -\infty < v_j < \infty,\quad (j=1,2)\,. \tag{3.14}$$

Let $B(D)$ denote the Banach space of such functions as described bove.

Theorem 3.4[5] If $F(\underline{s})$ belongs to $B(D)$, then (1) for

$$u_j^+ < c_j^+ < u_j < c_j^- < u_j^-,\quad (j=1,2),$$

$$F(\underline{s}) = \sum_{n=1}^{4} F_n(\underline{s}), \tag{3.15}$$

$$F_n(s) = \frac{1}{(2\pi i)^2}\int_{Z_1^{\delta_n}}\int_{Z_2^{\tau_n}}\frac{F(z_1,z_2)\,dz_1\,dz_2}{(\delta_n 1)\,(s_1-z_1)\,(\tau_n 1)\,(s_2-z_2)} \tag{3.16}$$

where Z_1^+, Z_1^-, Z_2^+, Z_2^- denote the vertical contours defined by the lines $z_1=c_1^+$, $z_1=c_1^-$, $z_2=c_2^+$, $z_2=c_2^-$ respectively, and δ_n, τ_n denote the sign symbols defined by

$$\delta_n = \begin{cases} +, & n=1,4 \\ -, & n=2,3 \end{cases},\quad \tau_n = \begin{cases} +, & n=1,2 \\ -, & n=2,3 \end{cases} \tag{3.17}$$

This decomposition is unique up to an additive constant. However, such additive constant has no physical meaning; (2) For $n=1,2,3,4$, $F_n(\underline{s})$ is analytic and of bounded norm for all $\underline{s}$ in $T(D,n)$ with the basis (D,n) defined by (2.15), and thus belongs to $B(D,n)$; (3) $|F_n(s)| \to |c|s_1|^{-p}|s_2|^{-q}$, $p>0$, $q>0$, as $|s_j|\to\infty$, $(j=1,2)$, uniformly with respect to all arguments of s in $T(D,n)$.

Theorem 3.5 For $n=1,2,3,4$, any function $F_n(s)$ in $B(D,n)$ has an approximate representation,

$$F_n(\underline{s}) = \sum_{l,m=-\infty}^{\infty}\frac{a_{lm}^n}{(c_1^{\delta n}+il\Delta v_1-\delta_n s_1)\,(c_2^{\tau n}+im\Delta v_2-\tau_n s_2)} \tag{3.18}$$

Δv_1 and Δv_2 may be chosen as small as possible that the error due to this approximation will be within our expectation. The proof of the theorem is omitted here, as it is the straightforward generalization from Theorem 3.3. In fact, the practical details of numerical computation for the two-variable technique are much involved.

4. SURFACE WAVE DIFFRACTION BY A STEP

A. Formulation of the Problem

The discontinuity studied in this section is a reactive step of height a as shown in Fig. 2.

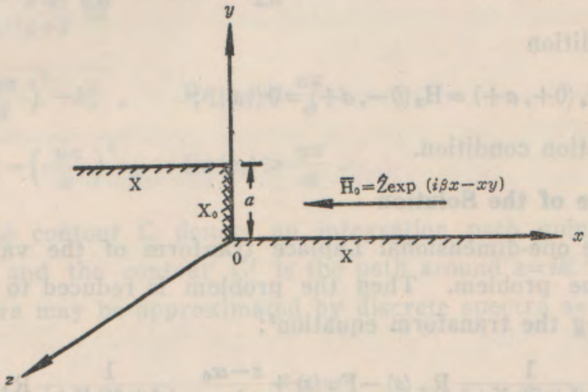

Fig. 2

Two semi-infinite planes ($x>0$, $y=0$ and $x<0$, $y=a$) with a normalized reactance X are joined together by the step ($x=0$, $0<y<a$) with a normalized reactance X_0. The incident field is a TM surface wave travelling in the negative x-direction. The step produces the reflected and transmitted surface waves and a radiated field.

To formulate the boundary-value problem, divide the entire space into three regions: (1) $x>0$, $0<y<a$; (2) $x>0$, $y>a$; (3) $x<0$, $y>a$. This is a two-dimensional problem, as the incident as well as scattered waves contain the z-component only and are functions of x and y. With reference to the time factor $\exp(i\omega t)$, denote the magnetic intensity of the incident wave as

$$H_0(x,y)=\exp(i\beta x-\alpha y) \tag{4.1}$$

where $\beta=k_0\sqrt{1+X^2}$, $\alpha=k_0X$ and $k_0=\omega\sqrt{\mu_0\varepsilon_0}$.

The branch of k_0 is taken that $I_m k_0<0$. Define $H_n(x,y)$, $n=1,2$ as the magnetic intensity of a scattered field in the regions $x>0$ and $H_3(x,y)$ as that of a scattered field minus the incident field in the region $x<0$. Then, $H_n(x,y)$, $(n=1,2,3)$ satisfy the Helmholtz's wave equations:

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_0^2\right)H_n(x,y)=0\,,\quad \text{for } (x,y) \text{ in Region } (n)\,. \tag{4.2}$$

These equations are to be solved subject to the boundary conditions:

$$\frac{\partial H_1}{\partial y}(x,0+)=-\alpha H_1(x,0+)\,,\qquad (x>0)\,;$$

$$\frac{\partial H_3}{\partial y}(x,a+)=-\alpha H_3(x,a+)\,,\qquad (x<0)\,;$$

$$\frac{\partial H_1}{\partial x}(0+,y)=-\alpha_0 H_1(0+,y)-(\alpha_0+i\beta)e^{-\alpha y}, \qquad (0<y<a), \tag{4.3}$$

where $\alpha_0=k_0X_0$;

the continuity conditions

$$H_1(x,a-)=H_2(x,a+), \qquad \frac{\partial H_1}{\partial y}(x,a-)=\frac{\partial H_2}{\partial y}(x,a+), \qquad (x>0),$$

$$H_2(0+,y)=H_3(0-,y), \qquad \frac{\partial H_2}{\partial x}(0+,y)=\frac{\partial H_3}{\partial x}(0-,y), \qquad (y>a); \tag{4.4}$$

the edge condition

$$H_2(0+,a+)=H_3(0-,a+)=0(|x|); \tag{4.5}$$

and the radiation condition.

B. Procedure of the Solution

Apply the one-dimensional Laplace tranform of the variable x to the above boundary-value problem. Then the problem is reduced to an equivalent problem, that of solving the transform equation[8]:

$$\frac{1}{(s^2+\beta^2)K(s)}R_+(s)-F_-(s)+\frac{s-\alpha_0}{s+\alpha_0}\,\frac{1}{(s^2+\beta^2)K(s)}R_+(-s)$$
$$-\frac{s-\alpha_0}{s+\alpha_0}F_-(-s)=-2(\alpha_0+i\beta)e^{-\alpha a}\frac{s}{s+\alpha_0}\,\frac{1}{s^2+\beta^2} \tag{4.6}$$

where $R_+(s)=(\alpha-iW)\,\mathfrak{L}_x\{H_2(x,a+)+H_3(x,a+)\}$,

$F_-(s)=\mathfrak{L}_x H_3(x,a+), \qquad W=\sqrt{s^2+k_0^2}$,

$K(s)=\exp(-iWa)\sin Wa/W$.

If $R_+(s)$ were known, then the solution can be obtained through the inverse transformation

$$H_2(x,y)\,l(x)+H_3(x,y)\,l(-x)$$
$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{R_+(s)}{\alpha-iW}\exp[-iW(y-a)+sx]\,ds, \qquad y>a \tag{4.7}$$

It reamins to solve Eq. (4.6). By Theorem 3.2, we may additively decompose the 1st, 3rd and 5th terms of (4.6). This separates the transform equation into two independent integral equations:

$$\left[\frac{R_+(s)}{(s^2+\beta^2)K(s)}\right]_+ + \left[\frac{s-\alpha_0}{s+\alpha_0}\,\frac{R_+(s)}{(s^2+\beta^2)K(s)}\right]_+ - \frac{s-\alpha_0}{s+\alpha_0}F_-(-s)$$
$$=e^{-\alpha a}\left\{\frac{2\alpha_0}{\alpha_0-i\beta}\,\frac{1}{s+\alpha_0}-\frac{\alpha_0+i\beta}{\alpha_0-i\beta}\,\frac{1}{s+i\beta}\right\};$$

$$\left[\frac{R_+(s)}{(s^2+\beta^2)K(s)}\right]_- + \left[\frac{s-\alpha_0}{s+\alpha_0}\,\frac{R_+(s)}{(s^2+\beta^2)K(s)}\right]_- - F_-(s)$$
$$=-e^{-\alpha a}\frac{1}{s-i\beta}. \tag{4.8}$$

Here we employ the brackets with '+' and '−' subscripts to denote the decomposed functions which are analytic in H_+ and H_-, respectively.

Now the functions $R_+(s)$ and $F_-(s)$ may be represented in terms of modal expansions. These modal expansions are characterized by the geometry of the problem. Generally, in open region the field is characterized by continuous eigenvalue while in closed region the field is characterized by discrete eigenvalues. For the present problem we have both continuous and discrete spectra as

$$R_+(s)=\frac{d_1}{s+i\beta}+\frac{d_2}{s+\alpha_0}+\sum_{n=3}^{\infty}\frac{d_n}{s+c_{n-2}}+\int_C\frac{u(z)}{z-s}dz;$$

$$F_-(s)=\int_{C'}\frac{v(z)}{z+s}dz$$

where $$c_n=\begin{cases}\sqrt{\left(\frac{n\pi}{a}\right)^2-k_0^2}, & Re(k_0)<\frac{n\pi}{a};\\ i\sqrt{k_0^2-\left(\frac{n\pi}{a}\right)^2}, & Re(k_0)>\frac{n\pi}{a}.\end{cases}$$

As shown in Fig. 3, the contour C denote an integration path going around the branch point $z=-ik$, and the contour C′ is the path around $z=ik$. Furthermore, these continuous spectra may be approximated by discrete spectra as follows:

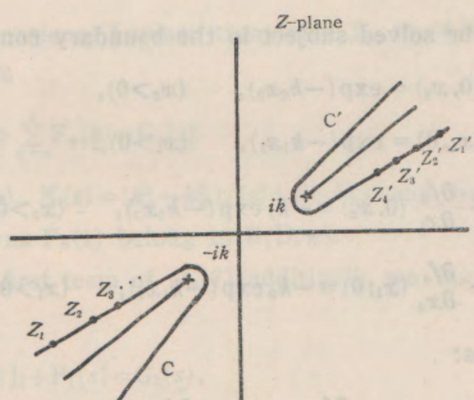

Fig. 3

$$\int_C\frac{u(z)}{z-s}dz=\sum_{n=1}^{\infty}\frac{c_n}{z_n-s},$$

$$\int_{C'}\frac{v(z)}{z+s}dz=\sum_{n=1}^{\infty}\frac{c'_n}{z'_n+s} \tag{4.10}$$

Inserting (4.9) into (4.8), we shall obtain two matrix equations for the unknown constants d_n, c_n, c'_n, $(n=1, 2, \ldots.)$. The practical details of the analysis together with the numerical result will be presented in a subsequent paper.

5. A QUARTER-ANGLED DIELECTRIC WEDGE

A. Formulation of the Problem

The problem considered is that of determining the scattered fields inside and outside a right-angled dielectric wedge which is specified by $x_1\geq0$, $x_2\geq0$, $-\infty<x_3$

$<\infty$, where x_1, x_2, x_3, form a right-handed Cartesian coordinate system. The permittivity and permeability of the dielectric will be denoted by the ε_a, μ_0, and the corresponding constants of the medium exterior to the wedge by ε_b, μ_0 in which ε_a and ε_b are complex. The propagation constants are given by

$$k_a^2=\omega^2\mu_0\varepsilon_a \quad \text{and} \quad k_b^2=\omega^2\mu_0'\varepsilon_b\,.$$

With reference to the time factor $\exp(i\omega t)$, the branches of k must be chosen such that $I_m(k)<0$. The primary field is a linearly E-polarized plane wave whose electric intensity is specified by $\hat{x}_3\exp(-k_1x_1-k_2x_2)$ where $k_1=ik_b\cos\theta_0$, $k_2=ik_b\sin\theta_0$ and $0<\theta_0<\pi/2$.

To formulate the boundary-value problem, let $f_n(\underline{x})=f_n(x_1,x_2)$ denote the electric intensity of the secondary waves in the nth quadrant Q_n of the x_1x_2-plane. Now $f_n(\underline{x})$, $(n=1,2,3,4)$, satisfy the Helmholtz' wave equations

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+k_a^2\right)f_1(\underline{x})=0, \qquad (\underline{x}\text{ in }Q_1);$$

$$\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+k_b^2\right)f_n(\underline{x})=0, \qquad (\underline{x}\text{ in }Q_n), \qquad (n=2,3,4) \tag{4.11}$$

These equations are to be solved subject to the boundary conditions:

$$f_1(0,x_2)-f_2(0,x_2)=\exp(-k_2x_2), \qquad (x_2>0),$$

$$f_1(x_1,0)-f_4(x_1,0)=\exp(-k_1x_1), \qquad (x_1>0),$$

$$\frac{\partial f_1}{\partial x_1}(0,x_2)-\frac{\partial f_2}{\partial x_1}(0,x_2)=-k_1\exp(-k_2x_2), \qquad (x_2>0),$$

$$\frac{\partial f_1}{\partial x_2}(x_1,0)-\frac{\partial f_4}{\partial x_2}(x_1,0)=-k_2\exp(-k_1x_1), \qquad (x_1>0); \tag{4.12}$$

the continuity conditions:

$$f_2(x_1,0)=f_3(x_1,0), \qquad \frac{\partial f_2}{\partial x_2}(x_1,0)=\frac{\partial f_3}{\partial x_2}(x_1,0), \qquad (x_1<0),$$

$$f_3(0,x_2)=f_4(0,x_2), \qquad \frac{\partial f_3}{\partial x_1}(0,x_2)=\frac{\partial f_4}{\partial x_1}(0,x_2), \qquad (x_2<0) \tag{4.13}$$

the integrability condition:

$$\int_{Q_n}|f_n(\underline{x})|^2d^2x<\infty, \qquad (n=1,2,3,4); \tag{4.14}$$

and the radiation condition. A detailed physical interpretation of these conditions is given in[5].

B. Procedure of the Solution

Note that the derivatives in (4.11) through (4.13) are performed in the ordinary sense. If we apply the concept of a generalized operator, i. e., to perform differentiation in the distributional sense, we may reduce the boundary-value problem to a compact form. Let D_{x_1}, D_{x_2} denote the generalized differential operators. If

$$f(\underline{x})=\sum_{n=1}^{4} f_n(\underline{x})\, l(\delta_n x_1)\, l(\tau_n x_2)$$

then
$$D_{x_1} f(\underline{x})=\frac{\partial}{\partial x_1} f(\underline{x})+\exp(-k_2 x_2)\,\delta(x_1)\, l(x_2),$$

$$D^2_{x_1}(\underline{x})=\frac{\partial^2}{\partial x_1^2} f(\underline{x})+\exp(-k_2 x_2)\,\delta'(x_1)\, l(x_2)$$
$$-k_1\exp(-k_2 x_2)\,\delta(x_1)\, l(x_2),$$

$$D^2_{x_2} f(\underline{x})=\frac{\partial^2}{\partial x_2^2} f(\underline{x})+\exp(-k_1 x_1)\,\delta'(x_2)\, l(x_1)$$
$$-k_2\exp(-k_1 x_1)\,\delta(x_2)\, l(x_1) \qquad (4.15)$$

Note that the conditions (4.12) and (4.13) has been used in the above manipulations. With the last two expressions we may combine (4.11) through (4.13) into a single equation

$$[D^2_{x_1}+D^2_{x_2}+k_b^2+(k_a^2-k_b^2)\, l(x_1)\, l(x_2)]\, f(x)$$
$$=[\delta'(x_1)-k_2\delta(x_1)]\, l(x_2)\exp(-k_2 x_2)$$
$$+[\delta'(x_2)-k_2\delta(x_2)]\, l(x_1)\exp(-k_1 x_1) \qquad (4.16)$$

Applying the two-dimensional Laplace transform $\mathfrak{L}_x$ to (4.16) and simplifying yield the transform equation

$$K(\underline{s})F_1(\underline{s})+\sum_{n=1}^{4} F_n(\underline{s})=G_1(\underline{s}) \qquad (4.17)$$

where $F_n(\underline{s})=\mathfrak{L}_{\underline{x}} f_n(\underline{x})$, $K(\underline{s})=(k_a^2-k_b^2)/(s_1^2+s_2^2+k_b^2)$ and $G_1(\underline{s})=1/[(s_1+k_1)(s_2+k_2)]$. Note that the unknowns $F_n(s)$ belong to $B(D,n)$.

Decomposing the first term of (4.17) additively, we may obtain a set of four equations:

$$[K(\underline{s})F_1(\underline{s})]_1+F_1(\underline{s})=G_1(\underline{s}),$$

$$[K(\underline{s})F_1(\underline{s})]_n+F_n(\underline{s})=0, \quad (n=2,3,4) \qquad (4.17)$$

where $[KF_1]_n$, $(n=1,2,3,4)$, denote the decomposed functions of KF_1, which belong to $B(D,n)$.

Inserting the approximate expressions (3.18) into (4.17) we may reduce the integral equations to the matrix equations. The prectical details of manipulation and verification together with the numerical result will be given in a subsequent paper.

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