

ANALYSIS OF LOW FREQUENCY FLUCTUATIONS IN AN ARC DISCHARGE COLUMN

K. C. HUANG and J. L. CHENG

College of Engineering, National Chiao Tung University

(Received December 8, 1972)

Abstract—The main objectives of this paper are the theoretical and experimental investigations of low frequency fluctuations in a weakly turbulent plasma generated by a hollow cathode arc discharge.¹

The dispersion characteristics of the weakly plasma turbulence caused by the electro-static ion cyclotron instability was derived. The derivation of the analysis is based on the kinetic model. The experimental observations were found to be in good agreement with our theoretical predictions.

THEORY

It is well known that a gaseous plasma is rarely quiescent. Many forms of fluctuations such as random noise and partially coherent oscillation arise spontaneously in the plasma. These fluctuations are intimately related to the instabilities and the transport properties of the plasma. This aspect of plasma nature has been discussed to certain extent in the papers of Chung and Rose², and others³.

An independent derivation based on the kinetic model was performed in the following. The problem to be solved here is the Vlasov equation

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla + \frac{q}{m} (\hat{\underline{E}} + \underline{v} \times \hat{\underline{B}}) \cdot \nabla_v \right) \hat{f} = 0 \quad (1)$$

where $\hat{\underline{E}} = \underline{E}_0 + \tilde{\underline{E}}$, $\hat{\underline{B}} = \underline{B}_0 + \tilde{\underline{B}}$, and $\hat{f} = f + \tilde{f}$, i. e. the total quantity is composed of background and fluctuation quantities. Assuming that $\underline{E}_0 = 0$, $\underline{B}_0 = 0$ and $\tilde{f} \ll f$, linearization leads to background equation

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla + \frac{q}{m} \underline{v} \times \underline{B}_0 \cdot \nabla_v \right) f(\underline{r}, \underline{v}, t) = -\frac{q}{m} \langle \tilde{\underline{E}} \cdot \nabla_v \tilde{f} \rangle \quad (2)$$

The wave fluctuation component of Vlasov equation can be obtained from eq. (1) subtracted by eq. (2)

$$\left(\frac{\partial}{\partial t} + \underline{v} \cdot \nabla + \frac{q}{m} \underline{v} \times \underline{B}_0 \cdot \nabla_v \right) \tilde{f} = -\frac{q}{m} (\tilde{\underline{E}} \cdot \nabla_v f) \quad (3)$$

In order to solve f , one can define the "characteristic trajectory" as follows:

$$\begin{aligned} \underline{r}(t) &\equiv \underline{r}(t; \underline{r}', \underline{v}', t') \\ \frac{d\underline{r}}{dt} &= \underline{v}, \quad \frac{d\underline{v}}{dt} = \frac{q}{m} \underline{v} \times \underline{B}_0, \quad \underline{B}_0 = B_0 \underline{z}_0 \end{aligned} \quad (4)$$

Obviously, eq. (4) will satisfy eq. (2) when the right hand side of eq. (2) is zero. Eventually f can be found as

$$f = f(v_{\perp}^2, v_z^2, \underline{v}_{\perp} + w_c \underline{z}_0 \times \underline{r}_{\perp}) \quad (5)$$

where $w_c = \frac{qB_0}{m}$ is a cyclotron frequency. Let us choose

$$\begin{aligned} f &= n \left(x + \frac{v_y}{w_c} \right) \frac{\exp\left(-\frac{v_{\perp}^2}{a_{\perp}^2}\right)}{\pi a_{\perp}^2} \frac{\exp\left(-\frac{v_z^2}{a^2}\right)}{\sqrt{\pi} a} \\ &\equiv n \left(x + \frac{v_y}{w_c} \right) F(v_{\perp}) g(v_z) \end{aligned} \quad (6)$$

where

$$\begin{aligned} a &= \sqrt{\frac{2R_B T}{m}}; \quad a_{\perp} = \sqrt{\frac{2R_B T_{\perp}}{m}} \\ F(v_{\perp}) &\equiv \frac{\exp\left(-\frac{v_{\perp}^2}{a_{\perp}^2}\right)}{\pi a_{\perp}^2}; \quad \int F(v_{\perp}) dv_{\perp} = 1 \\ g(v_z) &\equiv \frac{\exp\left(-\frac{v_z^2}{a^2}\right)}{\sqrt{\pi} a}; \quad \int g(v_z) dv_z = 1 \end{aligned}$$

The explicit form of trajectory, by solving eq. (4), is

$$\begin{aligned} \underline{r} &= \underline{r}' + \underline{v}'_{\perp} \frac{\sin w_c(t-t')}{w_c} - \underline{z}_0 \times \underline{v}'_{\perp} \frac{(1 - \cos w_c(t-t'))}{w_c} \\ &\quad + v'_z(t-t') \underline{z}_0 \equiv \underline{r}' + \underline{\beta}(\tau) \cdot \underline{v}' \end{aligned} \quad (7)$$

where

$$\tau \equiv t - t'$$

From eq. (3) associated with the defined trajectory, we get

$$\frac{d\tilde{f}}{dt} = -\frac{q}{m} \tilde{\mathbf{E}} \cdot \nabla_{\underline{v}} f \quad (8)$$

Suppose that we create a particle at $(\underline{r}', \underline{v}', t')$, then the Green function is

$$G = \delta(\underline{r} - \underline{r}(t)) \delta(\underline{v} - \underline{v}(t)) \eta(t - t')$$

where

$$\eta(t - t') = \begin{cases} 0 & \text{for } t < t' \\ 1 & \text{for } t > t' \end{cases} \quad (9)$$

Eq. (8) gives

$$\tilde{f}(\underline{r}, \underline{v}, t) = - \int_{-\infty}^{\infty} \frac{q}{m} \tilde{\mathbf{E}}(\underline{r}', t') \cdot \nabla_{\underline{v}'} f \delta(\underline{r} - \underline{r}(t)) \delta(\underline{v} - \underline{v}(t)) \eta(t - t') dt' d\underline{r}' d\underline{v}' \quad (10)$$

Setting $\tau \equiv t - t'$ and using the property of $\tilde{f} = 0$ at $t = -\infty$, eq. (10), by integrator

over the velocity, gives the following result for density fluctuation immediately,

$$\tilde{n}(\underline{r}, t) = -\frac{q}{m} \int_0^\infty d\tau \iint \tilde{\mathbf{E}}(\underline{r}', t-\tau) \cdot \nabla_{v'} f \delta(\underline{r}-\underline{r}(t)) d\underline{r}' d\underline{v}' \quad (11)$$

Substituting eq. (7) for $\underline{r}(t)$ into eq. (11), yields

$$\tilde{n}(\underline{r}, t) = -\frac{q}{m} \int_0^\infty d\tau \int d\underline{v} \tilde{\mathbf{E}}(\underline{r}-\underline{\beta}(\tau) \cdot \underline{v}, t-\tau) \cdot \nabla_v f(v_\perp^2, v_z^2, x + \frac{v_y}{w_c}) \quad (12)$$

Taking Fourier transform on eq. (12) into \underline{k}, w basis, we get

$$\tilde{n}_{\underline{k}w} = -\frac{q}{m} \int_0^\infty d\tau \int d\underline{v} \tilde{\mathbf{E}}_{\underline{k}w} \cdot \nabla_v f e^{i(w\tau - \underline{k} \cdot \underline{\beta}(\tau) \cdot \underline{v})} \quad (13)$$

We seek wave fields such that

$$\left. \begin{aligned} \underline{k} &= k_y \underline{y}_0 + k_z \underline{z}_0 \\ \tilde{\mathbf{E}} &= \tilde{E}_y \underline{y}_0 + \tilde{E}_z \underline{z}_0 \end{aligned} \right\} \quad (14)$$

where

$$\left. \begin{aligned} e^{i(w\tau - \underline{k} \cdot \underline{\beta}(\tau) \cdot \underline{v})} &= e^{i(w - k_z v_z)\tau} e^{-i\mathcal{R} \cdot \underline{v}_\perp} \\ \mathcal{R} \cdot \underline{v}_\perp &\equiv \frac{k_y v_y}{w_c} \sin w_c \tau - \frac{k_z v_x}{w_c} (1 - \cos w_c \tau) \end{aligned} \right\} \quad (15)$$

Substituting eq. (14), (15) into eq. (13), yields

$$\frac{\tilde{n}_{\underline{k}w}}{n_0} = -\frac{q}{m} \int_0^\infty d\tau \int_{-\infty}^{+\infty} d v_z \left[\left(i \frac{k_y \sin w_c \tau}{w_c} + \frac{\mathcal{H}}{w_c} \right) \tilde{E}_y - \frac{2v_z}{a^2} \tilde{E}_z \right] e^{-\alpha(1 - \cos w_c \tau)} g(v_z) e^{i(w - k_z v_z)\tau} \quad (16)$$

where

$$\begin{aligned} \mathcal{H} &\equiv \frac{n'_0}{n_0} = \frac{1}{n_0} \frac{\partial n_0}{\partial x} \\ \alpha &= \frac{k_y^2 a^2}{2w_c^2} \end{aligned}$$

Note the identity

$$e^{\alpha \cos w_c \tau} = \sum_{m=-\infty}^{\infty} I_m(\alpha) e^{im w_c \tau} \quad (17)$$

where I_m is the modified Bessel function of the first kind of order m .

Since for longitudinal fields

$$\tilde{\mathbf{E}} = -ik\tilde{\phi} = -i[k_y \underline{y}_0 + k_z \underline{z}_0] \tilde{\phi} = \tilde{E}_y \underline{y}_0 + \tilde{E}_z \underline{z}_0 \quad (18)$$

Eq. (16) becomes

$$\begin{aligned} \frac{\tilde{n}_{\underline{k}w}}{n_0} &= -\frac{q\tilde{\phi}}{m} \sum_{m=-\infty}^{\infty} e^{-\alpha} I_m(\alpha) \left\{ \frac{k_y^2}{2w_c} \int_{-\infty}^{\infty} \frac{g(v_z) d v_z}{w + (m+1)w_c - k_z v_z} \right. \\ &\quad - \frac{k_y^2}{2w_c} \int_{-\infty}^{\infty} \frac{g(v_z) d v_z}{w + (m-1)w_c - k_z v_z} + \frac{k_y}{w_c} \int_{-\infty}^{\infty} \frac{g(v_z) d v_z}{w + m w_c - k_z v_z} \\ &\quad \left. - \frac{2k_z}{a^2} \int_{-\infty}^{\infty} \frac{v_z g(v_z) d v_z}{w + m w_c - k_z v_z} \right\} \quad (19) \end{aligned}$$

We can simplify eq. (19), by utilizing the following formula:

$$\int_{-\infty}^{+\infty} \frac{g(v_z) dv_z}{i(w + mw_c - k_z v_z)} = \frac{1}{a} Z\left(\frac{w + mw_c}{k_z a}\right) \quad (20)$$

$$\int_{-\infty}^{\infty} \frac{g'(v_z) dv_z}{i(w + mw_c - k_z v_z)} = \frac{1}{a^2} Z'\left(\frac{w + mw_c}{k_z a}\right) \quad (21)$$

$$\frac{I_{m-1}(\alpha) - I_{m+1}(\alpha)}{2} = \frac{m I_m(\alpha)}{\alpha} \quad (22)$$

and

$$I_m(\alpha) = I_{-m}(\alpha) \quad (23)$$

where $Z(\zeta)$ is the plasma dispersion function⁴ and is defined as follows

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-\xi^2} d\xi}{\xi - \zeta} \quad (24)$$

Finally, we get

$$\frac{\tilde{n}_{kw}}{n_0} = -\frac{q\tilde{\phi}}{k_B T} \left\{ \sum_{m=-\infty}^{\infty} \frac{w - \frac{k_y a^2}{2w_c} - mw_c \left(1 - \frac{T_{\perp}}{T}\right)}{k_z a} Z\left(\frac{w - mw_c}{k_z a}\right) + 1 \right\} \cdot e^{-\alpha} I_m(\alpha) \quad (25)$$

Equation (25) is the expression of density fluctuation for both electrons and ions by taking the finite gyroradius effect into consideration. All information can be derived from this expression.

Let us define the diamagnetic frequency w_* as

$$w_* = -k_y \frac{n'_0}{n_0} \frac{k_B T}{e B_0} = \frac{k_y \mathcal{H} a_e^2}{2 \frac{q_e B}{m_e}} = \frac{k_y \mathcal{H} a_e^2}{2 w_{ce}} \quad (26)$$

Thus

$$\frac{k_y \mathcal{H} a_i^2}{2 w_{ci}} = k_y \frac{n'_0}{n_0} \frac{k_B T_i}{e B_0} = -w_* \frac{T_i}{T_e} \quad (27)$$

The Poisson equation states that

$$k^2 \tilde{\phi}_{kw} = \sum_l \frac{\tilde{n}_l w_l q_l}{\epsilon_0} \quad l = \text{electron or ion} \quad (28)$$

Substituting eq. (25) into eq. (28), yields

$$\begin{aligned} & \sum_{m=-\infty}^{\infty} \left[\frac{w - w_* - mw_{ce} \left(1 - \frac{T_e}{T_i}\right)}{k_z a_e} Z\left(\frac{w - mw_{ce}}{k_z a_e}\right) + 1 \right] e^{-\alpha_e} I_m(\alpha_e) \\ & + \frac{T_e}{T_i} \sum_{m=-\infty}^{\infty} \left[\frac{w + w_* \frac{T_i}{T_e} - mw_{ci} \left(1 - \frac{T_i}{T_{\perp i}}\right)}{k_z a_i} Z\left(\frac{w - mw_{ci}}{k_z a_i}\right) + 1 \right] \cdot e^{-\alpha_i} I_m(\alpha_i) \\ & + \frac{k^2}{k_{de}^2} = 0 \end{aligned} \quad (29)$$

where $k_d e^2 = \frac{n_0 e^2}{\epsilon_0 k_B T_e}$ is the square of electron Debye vector. T_i and T_e are the ion and electron temperature respectively.

Equation (29) is a general longitudinal dispersion relation in electron-ion plasma. In the case of a hollow cathode arc plasma, we have $\alpha_e \ll 1$, therefore

$$e^{-\alpha_e} I_0(\alpha_e) \simeq 1, \text{ for } m=0$$

$$e^{-\alpha_e} I_m(\alpha_e) \simeq 0, \text{ for } m \neq 0$$

Since $\frac{w}{k_z a_e} \ll 1$ and $\frac{w}{k_z a_i} \gg 1$, the small argument approximation for electron and the large argument approximation for ion can be applied to the plasma dispersion function $Z(\zeta)$. Therefore, from equation (29), the real frequency w_r can be evaluated from the following equation:

$$\left[\frac{w_r - w_*}{k_z a_e} \left(-\frac{2w_r}{k_z a_e} \right) + 1 \right] \frac{w_*}{w_r} e^{-\alpha_i} I_0(\alpha_i) - \left(\frac{T_e}{T_i} \right) \frac{2(w_{ci}^2 + w_r w_* \frac{T_i}{T_e})}{w_r^2 - w_{ci}^2} \cdot e^{-\alpha_i} I_1(\alpha_i) + \frac{k^2}{k_d^2} \simeq 0 \tag{30}$$

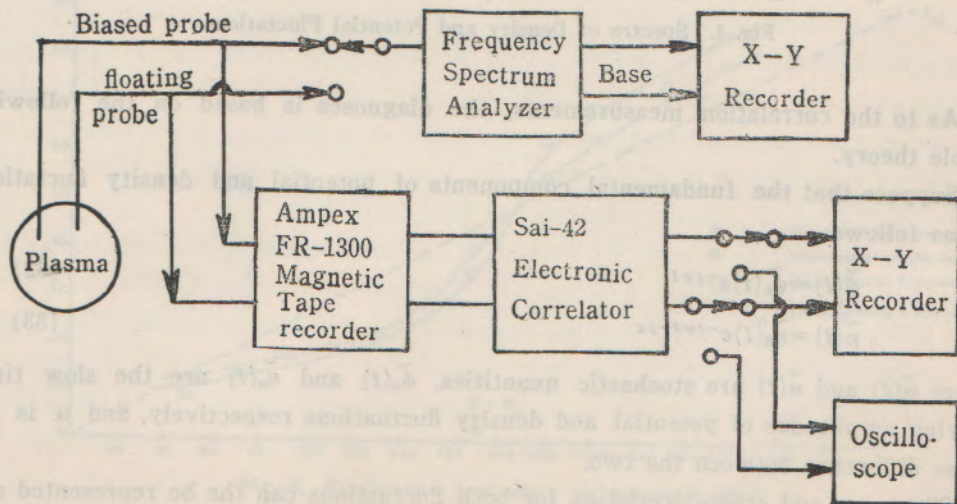
If we assume the imaginary frequency is much smaller than the real frequency, namely $\gamma \ll w_r$, then γ can be estimated from the following expression:

$$\sqrt{\pi} \frac{w_r - w_*}{k_z a_e} - \frac{2(2w_r - w_*)\gamma}{k_z^2 a_e^2} + \frac{w_* \gamma}{w_r^2 + \gamma^2} e^{-\alpha_i} I_0(\alpha_i) + \left(\frac{T_e}{T_i} \right) e^{-\alpha_i} I_1(\alpha_i) \frac{4w_{ci}^2 w_r \gamma + 2\gamma w_* \frac{T_i}{T_e} (w_r^2 + w_{ci}^2 + \gamma^2)}{(w_r^2 - w_{ci}^2 - \gamma^2)^2 + 4w_r^2 \gamma^2} \simeq 0 \tag{31}$$

EXPERIMENT

A highly ionized argon arc plasma with density $10^{12} - 4 \times 10^{13} \text{cm}^{-3}$ and moderate temperatures ($T_i \lesssim 1 \text{ev}$, $T_e \lesssim 10 \text{ev}$) was generated by means of a long hollow cathode discharge facility. The experimental set-up and its plasma parameter were reported elsewhere.¹

The block diagram for our measurements is as follows



The power spectra of the potential fluctuation and density fluctuation in the HCD plasma follow the similar pattern for any cases including the plasma with axial current and the plasma without the axial current. No matter how we changed the punping speed and gas flow rate, power spectra of both fluctuation still keep the similar patterns. Figure (1) verified the statement. This phenomena also confirm a fact that the plasma instability is an electrostatic mode. Taking the electro-static ion cyclotron wave instability as an example, we have derived equation (25). It shows that all quantities in the bracket are known and measurable. Under one operating condition, these quantities have definite values. Therefore, \tilde{n}_{kw}/n_0 and $q\tilde{\phi}_{kw}/k_B T$ vary in a similar fashion.

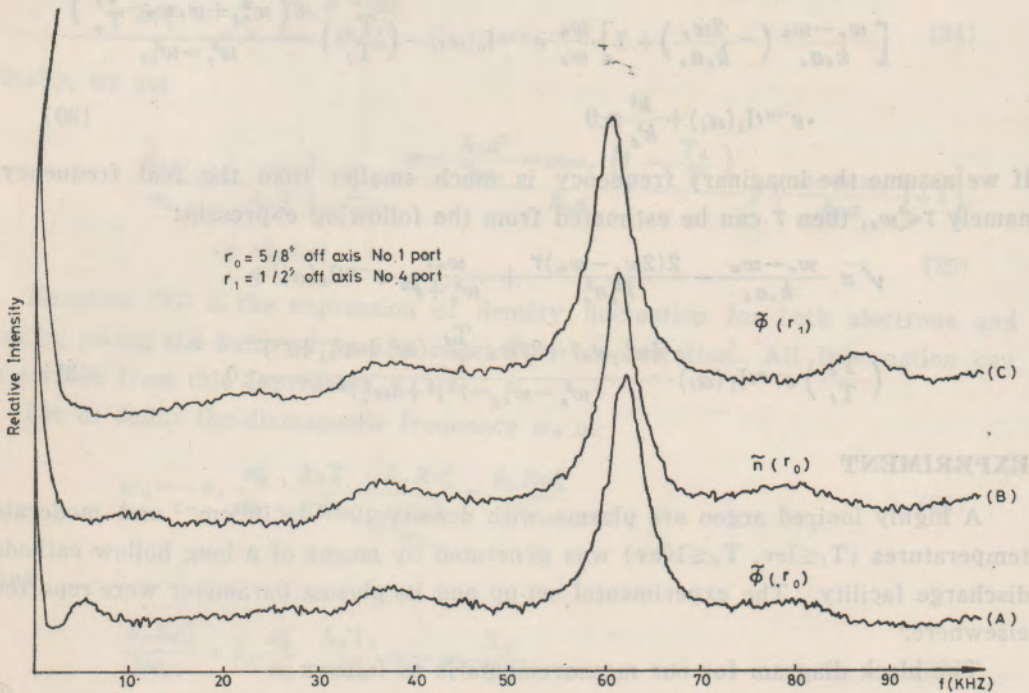


Fig. 1. Spectra of Density and Potential Fluctuations

As to the correlation measurements, the diagnoses is based on the following simple theory.

Suppose that the fundamental components of potential and density fluctuations are as follows

$$\tilde{\phi}(t) = \tilde{\phi}_0(t)e^{-i\omega t} \tag{32}$$

$$\tilde{n}(t) = \tilde{n}_0(t)e^{-i\omega t - i\alpha} \tag{33}$$

where $\tilde{\phi}(t)$ and $\tilde{n}(t)$ are stochastic quantities, $\tilde{\phi}_0(t)$ and $\tilde{n}_0(t)$ are the slow time-varying amplitudes of potential and density fluctuations respectively, and α is the phase difference between the two.

The auto- and cross-correlation for both fluctuations can be represented as

$$\langle \tilde{\phi}(t)\tilde{\phi}^*(t+\tau) \rangle = |\tilde{\phi}_0(t)|^2 e^{i\omega\tau} \quad (34)$$

$$\langle \tilde{n}(t)\tilde{n}^*(t+\tau) \rangle = |\tilde{n}_0(t)|^2 e^{i\omega\tau} \quad (35)$$

$$\langle \tilde{n}(t)\tilde{\phi}^*(t+\tau) \rangle = |\tilde{n}(t)\tilde{\phi}^*(t)| e^{i\omega\tau} e^{i\alpha} \quad (36)$$

Examining eq. (34)–(36), we realized the frequency of the special wave in plasma can be obtained. The auto- and cross-correlograms $\langle \tilde{\phi}\tilde{\phi}^* \rangle$, $\langle \tilde{n}\tilde{n}^* \rangle$ and $\langle \tilde{n}\tilde{\phi}^* \rangle$ for a weakly turbulent arc plasma were shown in Fig. (2).

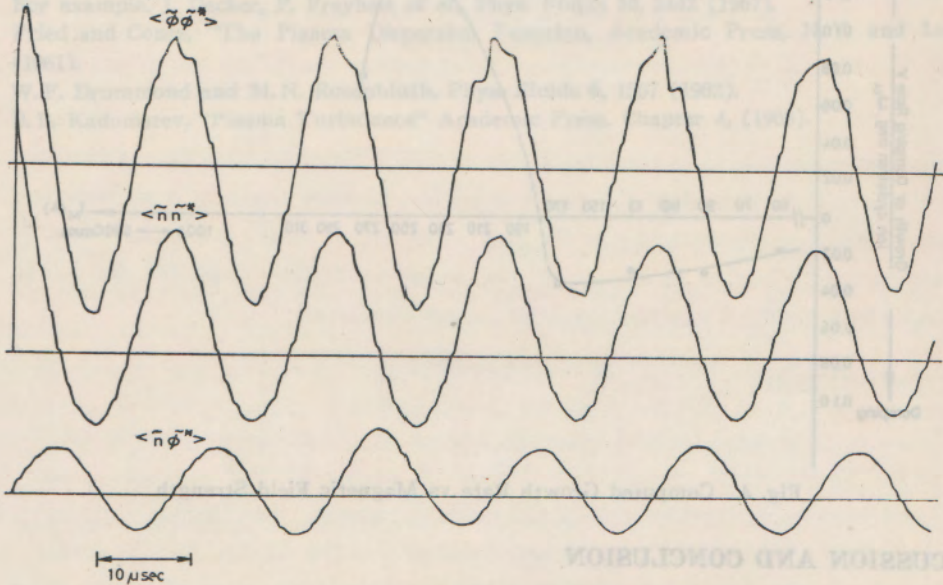


Fig. 2. Auto- and Cross correlograms

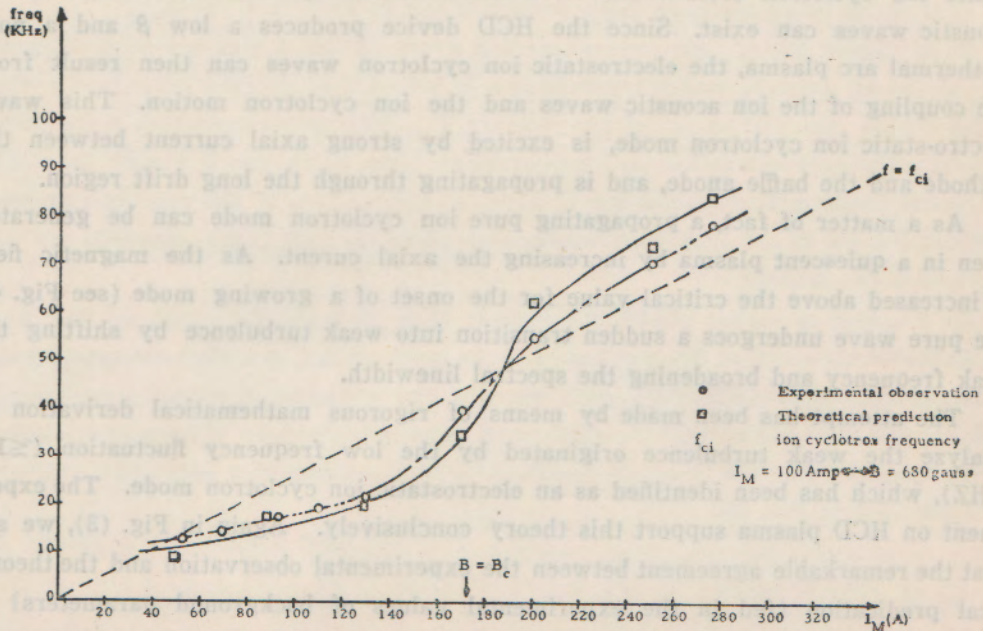


Fig. 3. Fluctuation frequency vs. magnetic field

The measured wave frequency and the calculated value, based on eq. (30) are plotted in Fig. (3) a linear curve for ion cyclotron frequency was taken as a reference.

In Fig. (4) the computed data, based on eq. (31), for growth or damping rate were presented.

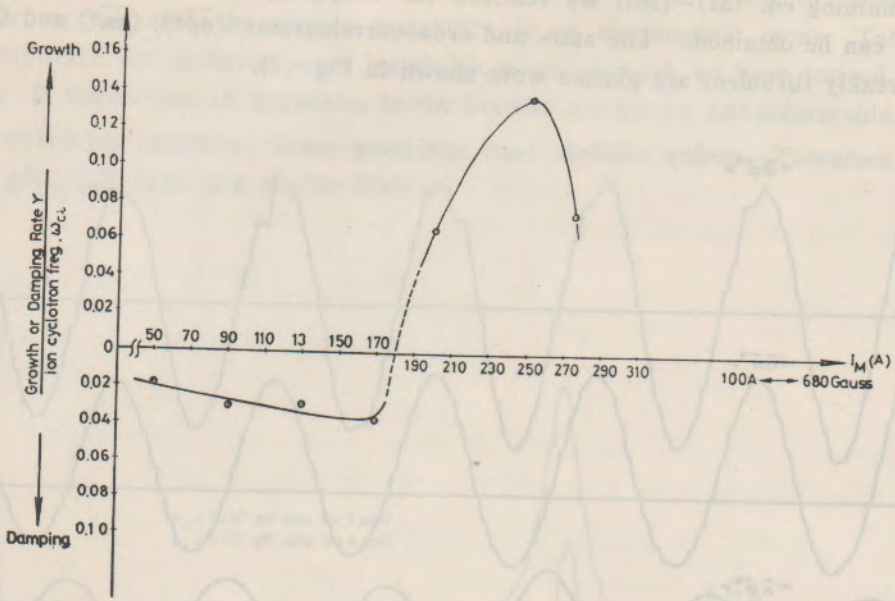


Fig. 4. Computed Growth Rate vs Magnetic Field Strength

DISCUSSION AND CONCLUSION

It is well known that^{5,6} in a low β plasma, the longitudinal electron flux may excite ion cyclotron oscillations. In a non-isothermal plasma where $T_e \gg T_i$, ion acoustic waves can exist. Since the HCD device produces a low β and a non-isothermal arc plasma, the electrostatic ion cyclotron waves can then result from the coupling of the ion acoustic waves and the ion cyclotron motion. This wave, electro-static ion cyclotron mode, is excited by strong axial current between the cathode and the baffle anode, and is propagating through the long drift region.

As a matter of fact, a propagating pure ion cyclotron mode can be generated even in a quiescent plasma by increasing the axial current. As the magnetic field is increased above the critical value for the onset of a growing mode (see Fig. 4), the pure wave undergoes a sudden transition into weak turbulence by shifting the peak frequency and broadening the spectral linewidth.

The attempt has been made by means of rigorous mathematical derivation to analyze the weak turbulence originated by the low frequency fluctuation (≤ 100 KHZ), which has been identified as an electrostatic ion cyclotron mode. The experiment on HCD plasma support this theory conclusively. Again in Fig. (3), we see that the remarkable agreement between the experimental observation and the theoretical predication (fed in the experimental values of background parameters) is achieved.

ACKNOWLEDGEMENT

The experiment was performed at the Polytechnic Institute of Brooklyn.

REFERENCES

1. K. C. Huang, K. Chung and E. Levi "Investigation of Loss Process in an Argon Discharge Column" (to be published, 1972).
2. K. Chung and D. J. Rose "Low Frequency fluctuations in a weakly turbulent Hollow Cathode Discharge Plasma", PIB MRI Proceeding of Symposium on Turbulence of Fluids and Plasma, New York, N. Y. (1968).
3. For example, J. Decker, P. Freyheit *et al.*, Phys. Fluids **10**, 2442 (1967).
4. Fried and Conte, "The Plasma Dispersion Function, Academic Press, N. Y. and London (1961).
5. W. F. Drummond and M. N. Rosenbluth, Phys. Fluids **5**, 1507 (1962).
6. B. B. Kadomstev, "Plasma Turbulence" Academic Press. Chapter 4, (1965).