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A STUDY ON ORBITS OF ARTIFICIAL SATELLITES

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I. INTRODUCTION

Technological developments in the fields of rocket propulsion and guidance have made possible the launching of artificial satellites into one of the many possible space orbits. Once a body has been put into an orbit beyond the influence of the atmosphere, its total energy remains constant and equal to that at rocket burnout. The total energy establishes the class of orbit, which may vary from an ellipse to a hyperbola, the circle and the parabola being limiting cases. However, an infinite number of orbits of the same class are possible for a given total energy at a specified point in space, and the heading angle of the velocity vector at rocket burnout must specify the particular orbit from the many possible orbits passing through this point.

II. GENERAL MOTION UNDER AN INVERSE SQARE CENTRAL FORCE

We will assume that the satellite is launched to a sufficient height beyond the influence of the atmosphere. The equations of motion are then[1,2].

Radial force

$$\ddot{r}-r\dot{\theta}^2=-\frac{\mu}{r^2} \tag{1}$$

Force normal to radius

$$r\ddot{\theta}+2\dot{r}\dot{\theta}=\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})=0 \tag{2}$$

Wherein r is distance from centre of Earth, θ is angle or from reference r_1, $\mu=GM=g_0R^2$. As a consequence of equation (2) we arrive at the result:

$$r^2\dot{\theta}=\lambda \tag{3}$$

where λ is a constant equal to twice the area swept out by the radial line per unit of time.

By the substitution of $r=\frac{1}{u}$, the independent variable t is changed to the independent variable θ, and equation (1) becomes:

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{\lambda^2} \tag{4}$$

with the general solution,

$$u=\frac{\mu}{\lambda^2}+C\cos(\theta-\theta_1) \qquad (5)$$

We can next consider the energy relationship per unit mass. The kinetic energy per unit mass is

$$T=\frac{1}{2}\nu^2=\frac{1}{2}[\dot{r}+(r\dot{\theta})^2]=\frac{\lambda^2}{2}\left[\left(\frac{du}{d\theta}\right)^2+u^2\right] \qquad (6)$$

Where ν is velocity, with no dissipation, the sum of the kinetic and potential energies per unit mass is a constant equal to the total energy per unit mass at burnout.

$$E=T+V \qquad (7)$$

Where T, V is kinetic and potential energy per unit mass respectively. Equation (6) can now be written as

$$\left(\frac{du}{d\theta}\right)^2+u^2=\frac{2(E-V)}{\lambda^2} \qquad (8)$$

which is the energy equation accompanying equation (4).

For the launching of satellites, the total energy with respect to the Earth's surface is of interest, so that we will take the potential energy to be zero at the Earth's surface.

$$V=\int_R^r \frac{\mu}{r^2}dr=\frac{\mu}{R}\left(1-\frac{R}{r}\right)=\frac{\mu}{R}(1-Ru) \qquad (9)$$

Where R is radius of Earth. Substituting equations (5) and (9) into equation (8), the constant C can be evaluated[3] as

$$C^2=\left(\frac{\mu}{\lambda^2}\right)^2\left[1+\frac{2E\lambda^2}{\mu^2}-\frac{2\lambda^2}{\mu R}\right] \qquad (10)$$

and by rotating the base line of $\theta=0$ to perigee (point of minimum distance from the origin of r), equation (5) can be put in the final form,

$$u=\frac{1}{r}=\frac{\mu}{\lambda^2}(1+e\cos\theta) \qquad (11)$$

where eccentricity e is

$$e=\sqrt{1+\frac{2E\lambda^2}{\mu^2}-\frac{2\lambda^2}{\mu R}} \qquad (12)$$

This is the most general solution for the inverse square central force problem, and the class of orbit is established by the value of e as follows:

Hyperbola if $e>1$
Parbola if $e=1$
Ellipse if $e<1$, with perigee corresponding to $\theta=0$
Circle if $e=0$
Ellipse if $e<0$, with apogee corresponding to $\theta=0$

The value of e is determined by the initial conditions at rocket burnout, indicated by the subscript 0, and we will examine the problem without placing restrictions on e.

Letting r_1 be the perigee distance, equation (11) becomes

$$u=\frac{1+e\,\mathrm{coe}\,\theta}{r_1(1+e)} \tag{13}$$

$$\frac{\lambda^2}{\mu}=r_1(1+e) \tag{14}$$

From the kinetic energy and total energy, two additional equations can be obtained and expressed in non-dimensional form:

$$\left(\frac{r\nu^2}{\mu}\right)=2-\frac{r}{r_1}(1-e) \tag{15}$$

$$\left(\frac{\mathrm{ER}}{\mu}\right)=1-\frac{\mathrm{R}}{2r}\left[2-\left(\frac{r\nu^2}{\mu}\right)\right] \tag{16}$$

III. INITIAL CONDITIONS

With these equations we will specify the initial values as burnout, which are shown in Fig. 1.

$$r=r_0 \qquad \beta=\beta_0$$

$$\nu=\nu_0 \qquad \theta=\theta_0$$

Fig. 1. Initial conditions.

The components of the initial velocity[4] are:

$$\nu_0\cos\beta_0=r_0\dot{\theta}_0=\frac{\lambda}{r_0}$$

$$\nu_0\sin\beta_0=\dot{r}_0=-\lambda\left(\frac{du}{d\theta}\right)_{\theta=\theta_0}=\frac{\mu e\sin\theta_0}{\nu_0 r_0\cos\beta_0}$$

From equation (11) we have at launch:

$$\frac{1}{r_0}=\frac{\mu(1+e\cos\theta_0)}{\nu_0^2 r_0^2\cos\beta_0}$$

Solving for $e\sin\theta_0$ and $e\cos\theta_0$ and dividing, the angular position from perigee is found:

$$\tan\theta_0=\frac{\left(\frac{r_0\nu_0^2}{\mu}\right)\sin\beta_0\cos\beta_0}{\left(\frac{r_0\nu_0^2}{\mu}\right)\cos^2\beta_0-1} \tag{17}$$

Since the value of E is constant after burnout, equation (16) can now be written as:

$$\left(\frac{ER}{\mu}\right)=1-\frac{R}{2r_0}\left[2-\left(\frac{r_0\nu_0^2}{\mu}\right)\right] \tag{18}$$

From equations (12), (18), and $\lambda=r_0\nu_0\cos\beta_0$, the quantity e is found:

$$e^2=\left[\left(\frac{r_0\nu_0^2}{\mu}\right)-1\right]^2\cos^2\beta_0+\sin^2\beta_0 \tag{19}$$

Equations (17) and (19) are sufficient to completely establish the orbit for any initial condition, and equation (18) indicates the required energy.

IV. INTERPRETATION OF RESULTS

1. Special Case $\beta_0=0$

We will first examine the special case where the satellite is launched with a zero heading angle, $\beta_0=0$. From equation (17) it is evident that $\theta_0=0$, so that $r_0=r_1$. Equation (15) now becomes:

$$e=\left(\frac{r_0\nu_0^2}{\mu}\right)-1 \tag{15*}$$

where the (*) is used for the special case $\beta_0=0$, and (**) for the further restricted case of circular orbit. The quantity e and the total energy from equation (18) are plotted in Fig. 2 and 3, which clearly classify the type of orbit.

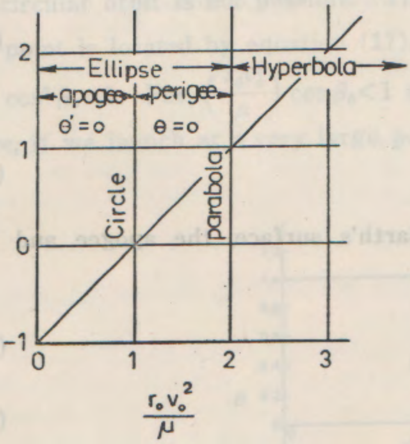

Fig. 2.

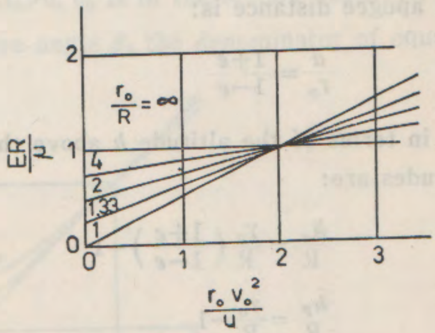

Fig. 3.

If we are able to launch a satellite with $\beta_0=0$ at a given value of r_0/R, as shown in Fig. 4, we would get a circular orbit ($e=0$) only when $r_0\nu_0^2/\mu=1$. The corsesponding total energy is then:

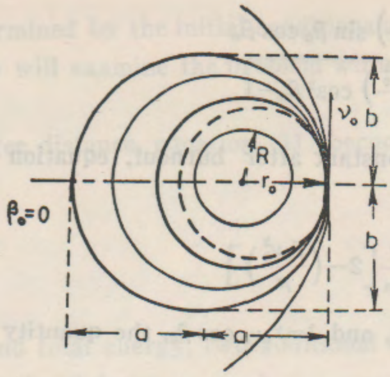

Fig. 4.

$$\left(\frac{ER}{\mu}\right)=1-\frac{R}{2r_0} \tag{18**}$$

If ν_0 is then increased so that $\frac{r_0\nu_0^2}{\mu}>1$, the orbit will be an ellipse with e and ER/μ as determined from equations (15*) and (18) or from Fig. 2 and 3. For values of $\frac{r_0\nu_0^2}{\mu}>2$ the orbit will become a hyperbola, and the satellite will escape from the Earth, never to return. Thus $\frac{r_0\nu_0^2}{\mu}=2$ corresponds to the velocity of escape at height $r_0=R+h$

$$\nu_e=\sqrt{\frac{2\mu}{r_0}}=R\sqrt{\frac{2g_0}{r_0}} \tag{20}$$

Considering the geometry of the elliptic orbit, the semimajor and semi=minor axes are:

$$\text{semi-major } \frac{a}{r_0}=\frac{1}{1-e} \tag{21}$$

$$\text{semi-minor } \frac{b}{r_0}=\sqrt{\frac{1+e}{1-e}} \tag{22}$$

The apogee distance is:

$$\frac{a}{r_0}=\frac{1+e}{1-e} \tag{23}$$

and in terms of the altitude h above the Earth's surface, the apogee and perigee altitudes are:

$$\frac{h_a}{R}=\frac{r_0}{R}\left(\frac{1+e}{1-e}\right)-1 \tag{24}$$

$$\frac{h_p}{R}=\frac{r_0}{R}-1 \tag{25}$$

Numerical values for a few values of e are given in Table to illustrate that the elliptic orbits are nearly circular. The ratio of apogee to perigee altitudes, however, can be quite large. For an eccentricity of 0.20, this ratio is 6.50 when the launch height is $\frac{r_0}{R}=1.10$, or approximately 400 miles above the Earth's surface, It is interesting to note that the ratio of the semi-major to semi-minor axes is only 1.02, and that these results are attained by an increase of launching speed of only 9.6% above that of the circular orbit speed.

Table for Calculation of Launching Altitude $\frac{r_0}{R}=1.10$

e	$\frac{1+e}{1-e}$	$\frac{h_a}{h_p}$	$\frac{a}{b}$	$\frac{\gamma_0 v_0^2}{\mu}$	$\left(\frac{\text{Elliptic Speed}}{\text{Circular Speed}}\right)$ at launch
0.00	1.00	1.00	1.00	1.00	1.00
0.05	1.105	2.15	1.00+	1.05	1.025
0.10	1.22	3.40	1.007	1.10	1.05
0.20	1.50	6.50	1.02	1.20	1.096

We can next investigate the case $\frac{r_0 v_0^2}{\mu}<1.0$. Examination of equation (11) with negative e shows that we have an ellipse with the starting point corresponding to apogee, an perigee is at $\theta=180°$. The speed is then not sufficient to balance the attractive force of the Earth, and the satelite distance r will diminish from its initial value r. With negative e, the centre of the allipse falls between the origin and the launching point. It is evident from the previous set of numbers that the satellite will fall into a region when atmospheric drag becomes important, even for a small negative e.

2. General Case $\beta_0 \neq 0$

We now ecamine the more general case where the heading angle at launch is not zero. For any value of $\frac{r_0 v_0^2}{\mu}$ and, the quantity e is established from Eq. (19) which is plotted in Fig. 5. It is evident that if $\beta_0 \neq 0$ e can never become zero, so that a circular orbit is not possible. The perigee position θ_0 with respect to the launch point is located by equation (17). The equation indicates that θ_0 is $\frac{\pi}{2}$ when $\left(\frac{r_0 v_0^2}{\mu}\right)\cos^2\beta_0=1$. For $\left(\frac{r_0 v_0^2}{\mu}\right)\cos\beta_0<1$ and $\beta_0>0$, θ_0 is in the second quadrant. For instance, if we launch at a very large positive angle β_0 the denominator of equation

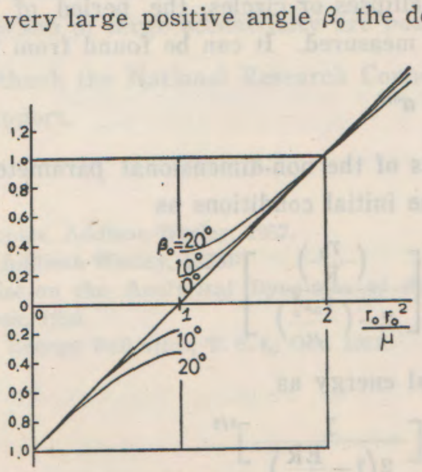

Fig. 5.

(17) is nearly -1, so that $\tan\theta_0$ approaches zero in the second quadrant, or θ_0 is slightly under 180°, and we have a very slender ellipse by equation (19).

To determine the effect of β_0 alone, we might take a case where the speed at launch is equal to that of the circular orbit, $\left(\frac{r_0\nu_0^2}{\mu}\right)=1$, and vary β_0. The total energy $\frac{\text{ER}}{\mu}$ depends only on $\frac{r_0\nu_0^2}{\mu}$ and $\frac{r_0}{\text{R}}$, so that it will be the same for all orbits passing through the launch point with the same speed. For $\beta_0=0$ and $\frac{r_0\nu_0^2}{\mu}=1$, we have a circle and θ_0 has no significance. For very surall angle β_0, and $\frac{r_0\nu_0^2}{\mu}=1$, equation (17) can be written as:

$$\tan\theta_0\cong-\frac{1}{\beta_0} \tag{26}$$

therefore we find θ_0 proceeding from $\frac{\pi}{2}$ to π as β_0 increases. If β_0 is negative, θ_0 proceeds from $\frac{3\pi}{2}$ to 2π.

If we next hole β_0 at some fired positive value and increase $\frac{r_0\nu_0^2}{\mu}$, we would find e increasing and θ_0 increasing from some value near $\frac{\pi}{2}$ in the first quadrant when $\left(\frac{r_0\nu_0^2}{\mu}\right)\cos^2\beta_0>1$, to $\frac{\pi}{2}$ at $\left(\frac{r_0\nu_0^2}{\mu}\right)\cos^2\beta_0=1$, and into the second quadrant for $\left(\frac{r_0\nu_0^2}{\mu}\right)\cos^2\beta_0<1$.

When $r_0\nu_0^2=2$, $e=1$, and we have a parabola regardless of the value of β_0. The total energy is then $\frac{\text{ER}}{\mu}=1$, and the reference angle θ_0 starts at 0 for small β_0 and increases to 90° when $\cos^2\beta_0=\frac{1}{2}$ or $\beta_0=45°$. Further increase in β_0 shifts θ_0 into the second quadrant.

For $\left(\frac{r_0\nu_0^2}{\mu}\right)>2$, similar conclusions hold, except that the orbit is now a hyperbola for all values of β_0, which can only change the reference angle θ_0.

V. PERIOD OF CLOSED ORBITS

For closed orbits, ellipses or circles, the period of the satellite is a quantity which cass be readily measured. It can be found from the areal rate as

$$\tau=\frac{2\pi}{\sqrt{\mu}}a^{3/2} \tag{26}$$

Substituting a in terms of the non-dimensional parameters, the period can be expressed in terms of the initial conditions as

$$\tau=\frac{2\pi}{\sqrt{\frac{g_0}{\text{R}}}}\left[\frac{\left(\frac{r_0}{\text{R}}\right)}{2-\left(\frac{r_0\nu_0^2}{\mu}\right)}\right]^{3/2} \tag{27}$$

or in terms of the total energy as

$$\tau=\frac{2\pi}{\sqrt{\frac{g_0}{\text{R}}}}\left[\frac{1}{2\left(1-\frac{\text{ER}}{\mu}\right)}\right]^{3/2} \tag{28}$$

where g_0 is the acceler-ation of gravity at Earth's surface, $h=0$. Thus we find that the period depends only on the total energy, and all orbits passing through a giving point with the same velocity will have the same period, regardless of the heading angle.

VI. CONCLUSIONS

(1) The orbits of artificial satellites are specified by three non-dimensional paramctors at rocket burnout $\frac{r_0}{R}$, $\frac{r_0\nu_0^2}{\mu}$ and β_0.

(2) A circular orbit is possible only under the condition $\beta_0=0$ and $\frac{r_0\nu_0^2}{\mu}=1$ and the required energy is given by equation (18**).

(3) With $\beta_0=0$ and $\frac{r_0\nu_0^2}{\mu}\neq 1$, we have the following orbits

(a) Ellipse with launch point corresponding to apogee if $0<\left(\frac{r_0\nu_0^2}{\mu}\right)>1$.

(b) Ellipse with launch point corresponding to perigee if $1<\left(\frac{r_0\nu_0^2}{\mu}\right)<2$.

(c) Parabola with launch point corresponding to perigee if $\frac{r_0\nu_0^2}{\mu}=2$.

(d) Hyperbola with launch point corresponding to perigee if $\frac{r_0\nu_0^2}{\mu}>2$.

Corresponding values of e are found from Eq. (15*).

(4) For the general case $\beta_0\neq 0$, the quantity e and the perigee position with respect to the launch point are established from equation (19) and (17) respectively, and the orbits are

(a) Ellipses for $0<\left(\frac{r_0\nu_0^2}{\mu}\right)<2$.

(b) Parabola for $\frac{r_0\nu_0^2}{\mu}=0$.

(c) Hyperbola for $\frac{r_0\nu_0^2}{\mu}>2$.

(5) For the case $\beta_0=0$, the ratio of apogee to perigee heights above Earth surface may deviate greatly from unity, although the orbits are nearly circular.

(6) When $\beta_0\neq 0$ elliptic orbits of large eccentricity are possible for large β_0.

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