Science Bulletin of National Chiao Tung University, Vol. VI, No. 1

A HYBRID METHOD FOR SOLVING PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

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(Received November 16, 1972)

Abstract—*The temperature distribution acoss an one-dimensional thermally conductive, uniform slab was found at discrete space points using continuous-time, discrete-space (CTDS) technique on the University of Cincinnate AD-4 hybrid computer facilities. In addition, a pure analog computer solution was also investigated, so that a comparison can be made between these two methods. The agreement between the analog and hybrid methods is satisfactorily good.*

INTRODUCTION

A hybrid computer is in principle a combination of a digital and an analog computer togather with some interface devices in such a way that these two can operate as one unit. Analog computers are particularly well suited for the solution of large set of coupled simultaneous ordinary differential equations. These equations can be solved simultaneously on the analog computer at high speed without computing ting penalty. On the other hand, digital computers are capable of preforming logic decisions both quickly and accurately. By combining analog computer's ability to quickly solve simultaneous differential equations with a digital computer's prodigious memory and ability to rapidly make complex logic decisions, one can obtain a hybrid system that combines the best features of both analog and digital machines.[1,6]

The interface devices of a hybrid computer include analong to digital converters (ADC), digital to analog converter (DAC) and some logic control signals. The analog to digital converters provide for the conversion of analog signals present at the analog console to sixteen digital words. A multiplexer provides for scanning sixteen such analog signals in succession and stores in the digital memory for later playback. The digital to analog converters are really digital registers that control electronic switches attached to resistor networks. The output terminal of the resistor neworks can be connected to the summing junctions of computing amplifiers and thereby provide a very high speed technique for transferring a digital value held in the digital computer to a corresponding analog value which then appears at the output terminal of the computing operational amplifier. Thus in summary, digital or analong information can be accepted by the system and can be exchanged by the system at high speed under the control of digital logic signals or by digital computer program written in the convenient language of Fortran. Of course, the analog or digitaal computr of the hybrid system can work as a separate computer independent of the other if desired. Fig. 1 shows a schematic diagram for a hybrid computer.

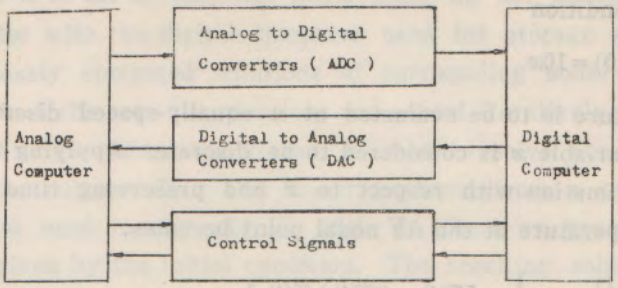

Fig. 1. Hybrid Computer Schematic Diagram

For a number of years it has been suggested that one of the fruitful field of application for hybrid computation might lie in the study of distributed systems. In principle, the combination of analog speed with the memory and logic capabilities of digital computers should make it possible to solve partial differential equations both efficiently and rapidly. One of the most promising technique for hybrid solution of partial differential equations involves in a CTDS (continuous-time discrete-space) analog mechanization of a space nodal point (or a group of nodal points) with the digital computer used for storage and playback of guessed or prcviously computed soluions of the surrounding nodal points. The nodal point simulated on the analog computer is then integrated over the entire region to obtain the solution for the first major iteration. Iterations will continue until solutions converge to a satisfactory degree of accuracy.

In the digital computer solutian of a partial differential equations, both time and space variables are discreatized by applying finite difference method.[4] The number of difference equations equal the number of space nodal points times the number of time increments. In the analog computer solution of partial differential equations, the equations can be appoximated by a set of coupled ordinary differential equations and can be solved simultaneously. However, unless the problem is simple, and the number of nodal points is small, pure analog methods frequently requires large quantities of hardware and pure digital solutions are always extremely time-consuming, thereby justifying the idea of hybrid computation.

ONE-DIMENSIONAL DIFFUSION EQUATIONS

An infinite one-dimensional thermally conductive, uniform slab of length L is initially at a temperature distribution of $10x$. Its left boundary at $x=0$ is held at zero temperature and the right boundary at $x=\mathrm{L}$ is held at 100°C. The temperature temperature distribution along this slab is described by the equation

$$\frac{\partial \mathrm{T}(x,\, t)}{\partial t}=\frac{\partial^2 \mathrm{T}(x,\, t)}{\partial x^2} \qquad (1)$$

with the boundary conditions

$$T(0, t)=0$$

$$T(L, t)=100$$

and the initial condition

$$T(x, 0)=10x$$

The temperature is to be evaluated at n equally-spaced discrete nodal points, i. e. the spatial variable x is considered to be discrete. Applying the central finite difference approximation with respect to x and preserving time as a continuous variable, the temperature at the ith nodal point becomes:

$$\frac{dT_i(t)}{dt}=\frac{1}{(\Delta x)^2}[T_{i-1}^{(t)}-2T_i^{(t)}+T_{i+1}^{(t)}] \qquad (2)$$

$$i=1, 2, \cdots, n$$

where

$T_i(t)$ = Temperature at the ith nodal point.

Δx = The distance between two nodal points.

HYBRID COMUTER SET-UP AND SOLUTION

The analog circuit for solving the resulting n simultaneous differential equations of eqn. (2) all in parallel is shown in Fig. 2. Although the all-parallel analog approach is quite feasible for one-dimensional diffusion equations with a small

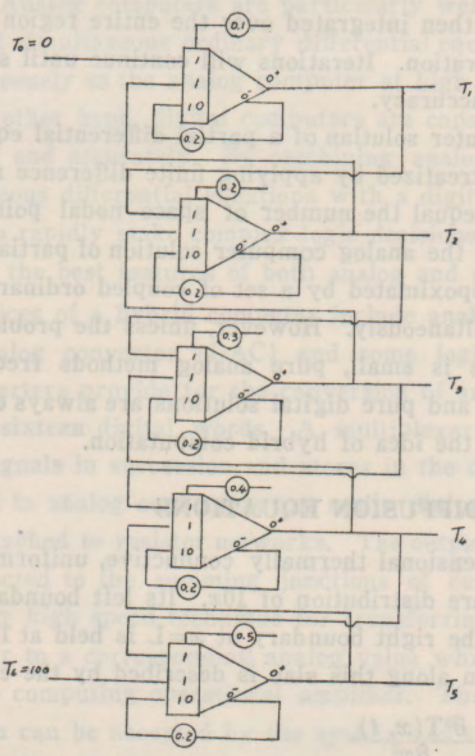

Fig. 2. Analog Circuit For Solving One-dimensional Heat Diffusion Equation with 5 Space Nodal Points.

number of nodal points, it soon becomes impractical due to the analog hardware limitations when a large number of nodal points, say, 50 points, is required.

An alternative is to set up only one nodal point on the analog patchboard of the hybrid computer with the digital computer used for storage and playback of guessed or previously computed solutions of surrounding nodal points. In this CTDS hybrid method, the set of n equations are solved iteratively, one at a time. Thus in Fig. 3 the single integrator circuit is used to solve eqn, (2) at the ith nodal point with the voltages T_{i-1} and T_{i+1} reproduced from a previous solution. This circuit is first used to compute T_1 with T_0 given by prescribed boundary condition and T_2 given by the initial condition. The resulting solution T_1 is then sampled and converted to digital words and stored in digital computer. Next the circuit is used to compute T_2 at the second nodal point with T_1 obtained by playing back the just-computed solution through a DAC (digital to analog converter) and T_3 given by the initial condition. The resulting solution T_2 is then converted to digital form again and stored for playback. The process is repeated, using the circuit shown in Fig. 3 to compute T_3, $T_4, \cdots$, T_n respectively. This completes a major iteration.

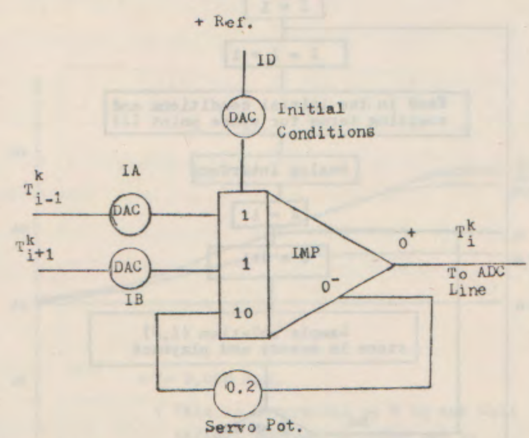

Fig. 3. Hybrid Computer Circuit for Solving One-dimensional Heat Equations.

The second major iteration is begun with the circuit again representing T_1, but this time T_2 was the solution obtained previously in the first major iteration. This single integrator circuit is then iterated sequentially across the one-dimensional slab over and over again until all the T_i's converge to within a small prescribed value between successive iterations. In this arrangement, the temperature at the ith nodal point for the kth major iteration is represented as:[5]

$$\frac{dT_i^k}{dt}=\frac{1}{(\Delta x)^2}(T_{i-1}^k-2T_i^k+T_{i+1}^{k-1}) \tag{3}$$

where

i=Number of nodal points.

k=Number of major iterations.

Is is always wanted to estimate the time needed to achieve the solution. This is a direct function of the number of iterations necessary for convergence. The speed of convergence depends upon the number of nodal points, and the initial guess used to start the iteration procedure.

RESULTS OF HYBRID CALCULATION

A flow chart for hybrid program used for solving the foregoing heat flow problem is shown in fig. 4. The hybrid program written in the convenient language of FORTRAN is listed in appendix. The solutions were calculated on the University of Cincinnati hybrid facilities which is a combination of an AD-4 analog computer and and IBM 1130 digital computer. All the hybrid communication routines and the addresse of the hybrid devices used in the program were written by the auther according to the "HYBRID COMMUNICATION ROUTINES OPERATORS'S MANUAL

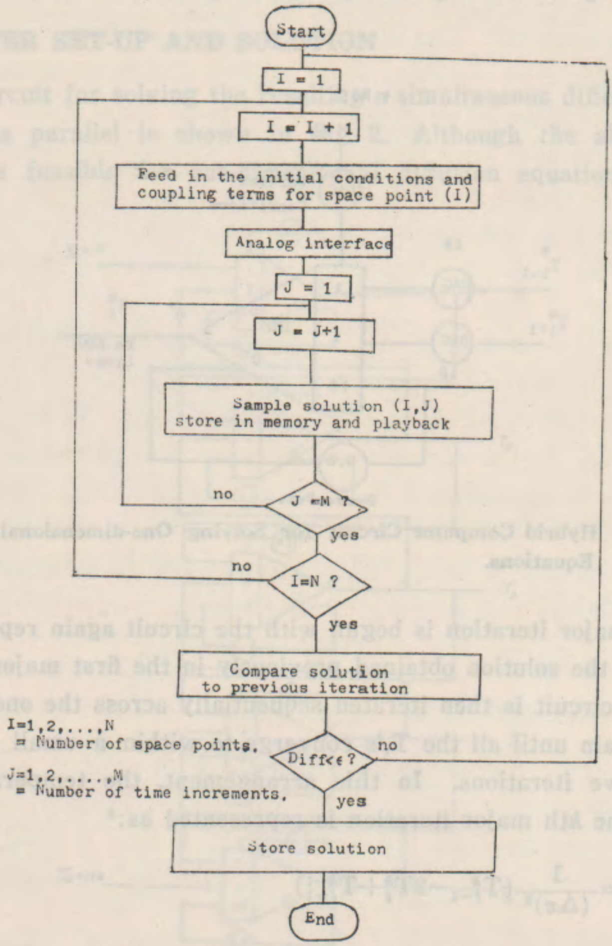

Fig. 4. Hybrid Flow Chart For Solving One-dimensional parabolic Partial Differential Equations.

BY APPLIED DYNAMICS, INC."[7] A detail explanation of the program is also provided in appendix.

Table 1 shows the number of iterations and total computer time needed for solving the heat flow problems for five space nodal points and for three space nodal points. Actually the total computer time includes the time for execution and the print-out time. It is seen both computer time and number of iterations increses with the increse in nodal points.

Table 1. Comparison between 5 station problems and 3 station problems

	Number of Iterations	Total Computer Time
Five Stations	12	5 min. 20 sec.
Three Stations	6	3 min. 10 sec.

Fig. 5 shows the solutions of T_3^k for a five nodal stations probem after various iterations. It can be seen from Fig. 5 that the solutions converge after 12 iterations. The numerical values for each nodal station aftər 12 iterations are shown in table 2.

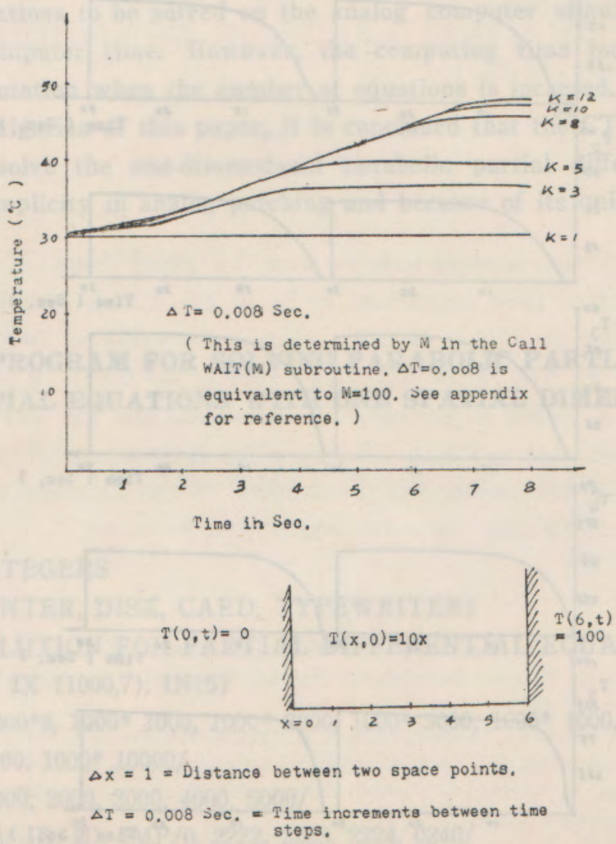

Fig. 5 Iterative solutions for T_3^k by hybrid computation

Table 2. Hybrid results for 5 nodal stations of an one-dimensional heat flow problem

Time (Sec.)	T_1	T_2	T_3	T_4	T_5
0	1000	2000	3000	4000	5000
0.8	1004	2024	3130	4549	6751
1.6	1038	2124	3433	5014	7323
2.4	1106	2311	3730	5480	7606
3.2	1190	2484	3978	5744	7782
4.0	1271	2638	4177	5937	7903
4.8	1343	2769	4338	6085	7992
5.6	1403	2876	4467	6199	8060
6.4	1453	2961	4569	6290	8113
7.2	1490	3030	4650	6362	8156
8.0	1502	3096	4703	6414	8186

Note: The exact value = The printout value $\times 10^{-2}$. for example, 1000 is equivalent to 10.

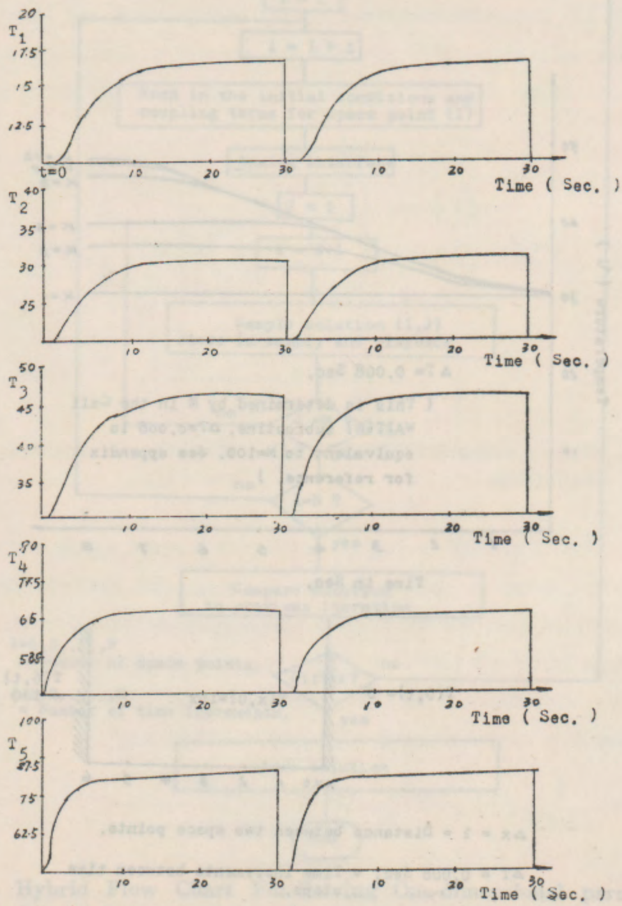

Fig. 6. Analong Results for 5 Nodal Stations Heat Flow Problem

The results obtained from pure analog method for 5 stations problem also are shown in Fig. 6. Observation of the results in table 2 and Fig. 6 suggests that the agreement between hybrid method and pure analog method is satisfactorily good.

CONCLUSIONS

Although the problem solved is one-dimensional, extension to two-dimensional problems is quite simple by using the method of lines, i. e. solving the two-dimensional partial differential equations in parallel along one row of nodes and use previously-obtained solutions for the row above and row below.[3] In this arrangement the iterative equation obtained by applying the central finite-difference approximation to the spatial variable x and y in the two-dimensional slab heat flow equations are:

$$\frac{dT^k_{i,j}}{dt}=\frac{1}{h^2}[T^k_{i+1,j}+T^k_{i-1,j}+T^k_{i,j-1}+T^{k-1}_{i,j+1}-4T^k_{i,j}]$$

$$h=\Delta x=\Delta y$$

k=Number of iterations

where i corresponds to x and j corresponds to y.

In this case, the advantages of hybrid computation stand out very clearly. It allows more equations to be solved on the analog computer simutaneously without the increse of computer time. However, the computing time increses linearly in pure digital comutation when the number of equations is incresed.

By the investigation of this paper, it is concluded that the CTDS hybrid method is adequate to solve the one-dimensional parabolic partial differential equations because of its simplicity in analog patching and because of its quickness in convergence.

APPENDIX

THE HYBRID PROGRAM FOR SOLVING PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH ONE SPATIAL DIMENSION

```
//JOB
//FOR
*LIST ALL
*ONE WORD INTEGERS
*IOCS (1132 PRINTER, DISK, CARD, TYPEWRITER)
C  HYBRID SOLUTION FOR PARTIAL DIFFERENTIAL EQUATIONS
   DIMENSION IX (1000,7), IN(5)
   DATA IX/1000*0, 1000* 1000, 1000* 2000, 1000* 3000, 1000* 4000,
   1   1000* 5000, 1000* 10000/
   DATA IN/1000, 2000, 3000, 4000, 5000/
   DATA N, IA, IB, ID, IMP/0, 2222, 2223, 2224, 0240/
   M=100
```

```
    CALL INITA (E, N)
    CALL HYTST (3, 0)
    CALL RUN (E)
    DO 60 K=1, 15
    DO 20 I=1, 5
    CALL IC (E)
    CALL WAIT (300)
    DO 20 J=1, 1000
    CALL STIND (E, ID, IN(I))
    CALL SIND (E, IA, IX (J, I))
    CALL STIND (E, IB, IX (J, I+2))
    CALL OP (E)
    CALL WAIT (M)
    CALL HOLD (E)
    CALL READ (E, IMP, IX (J, I+1))
20  CONTINUE
    WRITE (3, 30) K
30  FORMAT (2X, 'MAJOR ITERATION', I3, //, 6X 'TI', 9X, 'T2',
    1    9X, 'T3', 9X, 'T4', 9X, 'T5')
    DO 40 L=1, 1000, 100
40  WRITE (3, 50) IX (L, 2), IX (L, 3) IX (L, 4), IX (L, 5), IX (L, 6)
50  FORMAT (3X, 5 (16, 5X))
60  CONTINUE
    CALL STP (E)
    CALL EXIT
    END
```

The process of incrementing time is done by going from IC MODE into OP MODE for a length of time determined by M in the CALL WAIT (M) statement and then proceeding to HOLD MODE, wherein the value of T*i* is picked off and stored in memory by ADC lines using the subroutine CALL READ. The initial conditions and the values of surrounding nodal stations are fed in by means of DAC'S using the subroutine CALL STIND. The subroutines CALL INITIA, CALL HYTST. are special subroutins used to initialize all the interface functions to zero state and provide to indicate the error signals for the users.

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