

## HANKEL MATRIX, REALIZATION AND IDENTIFICATION

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**Abstract**—This paper studies the realization and the identification problems of linear time-invariant lumped systems. It is assumed that the systems to be identified can be reset to the zero states, and that the measured data are free of noise. The introduced method is based on Hankel matrices. The properties of Hankel matrices are first introduced. A new realization algorithm, which is believed to be the simplest possible, is presented. This result is then used in the identification. Both discrete-time and continuous-time systems are studied.

### I. INTRODUCTION

This paper studies the realization and identification problems of linear time-invariant lumped systems. Identification is a problem of determining the internal structure of a system from the externally measured data<sup>1</sup>; realization is a problem of determining a dynamical equation, an internal description of a system, from a given transfer function matrix, an external description. Though the realization problem can be considered as a special case of the identification problem, it has its own importance in engineering, and has been received considerable attention<sup>2</sup>. It is useful, among others, in analog computer simulations, syntheses of digital filters, and active circuit syntheses<sup>3</sup>.

The identification problem can be roughly divided into two categories: one with noise, the other without noise<sup>1,4</sup>. This paper studies only the latter case, that is, systems and measured data are assumed not contaminated with noise. It is also assumed that the systems to be identified are known to be linear, time invariant, lumped (having finite number of state variables), and can be reset or can return to the zero state after each experiment.

The identification method to be introduced is based on Hankel matrices, and reduces essentially to the irreducible realization problem. The use of Hankel matrices in realizations was initiated, independently, by Ho and Kalman<sup>5</sup>, Silverman<sup>6</sup>, and

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Tissi<sup>8</sup>. Among them, the algorithm developed in<sup>5</sup> seems simplest, and is often referred to as Ho's algorithm<sup>9</sup>. Since then, several modifications have been developed<sup>10-12</sup>. In this paper, a new realization algorithm will be presented.

The realization problem is essentially an identification problem with the application of special inputs: delta function in the continuous-time case, impulse sequence  $\{1000\cdots\}$  in the discrete-time case. Though impulse sequences can be easily generated in practice, delta functions cannot. Therefore it is of interests to be able to identify a system from responses due to inputs other than delta functions. There are some attempts<sup>13,14</sup> of this problem. However, their computations are very involved. We present in this paper a new method which reduces directly to the realization problem. The amount of computation needed is much smaller than those in<sup>13,14</sup>.

This paper is organized as follows. To be self-contained, properties of Hankel matrices, which scattered in<sup>12,15,16</sup>, are developed and refined in Section II. An irreducible realization algorithm, which is believed to be the simplest possible, together with its proof and a numerical example are presented in Sections III to V. The identification of discrete-time systems is presented in Section VI. The continuous-time case is then discussed in Section VII. The reader, who wishes to learn only the realization and identification methods, may skip Sections II and IV.

We introduce some terminology. A transfer function  $\hat{G}(s)$ , or a sampled transfer function  $\hat{G}(z)$  is said to be strictly proper, if the denominator of every entry of  $\hat{G}(s)$  or  $\hat{G}(z)$  has a degree larger than that of the numerator. A rational function is said to be irreducible if the numerator and the denominator have no nontrivial common factor.

## II. PROPERTIES OF HANKEL MATRICES<sup>12,15,16</sup>

Let  $\hat{g}_i(s)$  be a strictly proper rational function expanded as

$$\hat{g}_i(s) = a_1s^{-1} + a_2s^{-2} + a_3s^{-3} + \cdots \tag{1}$$

The constants  $a_1, a_2, \dots$  will be called the Hankel coefficients of  $\hat{g}_i(s)$ . Define the Hankel matrix of order  $k \times l$  of  $\hat{g}_i(s)$  as

$$\tilde{H}_i(k, l) = \begin{bmatrix} a_1 & a_2 \cdots a_l \\ a_2 & a_3 \cdots a_{l+1} \\ \vdots & \vdots \\ a_k & a_{k+1} \cdots a_{l+k-1} \end{bmatrix} \tag{2}$$

**THEOREM 1:** If  $\hat{g}_i(s)$  is irreducible and if its denominator has degree  $n$ , then

$$\rho \tilde{H}_i(n, n) = \rho \tilde{H}_i(k, l) = n \tag{3}$$

for any integers  $k \geq n, l \geq n$ , where  $\rho$  denotes the rank of a matrix.

**PROOF:** Let

$$\hat{g}_i(s) = \frac{\beta_1s^{n-1} + \beta_2s^{n-2} + \cdots + \beta_n}{s^n + \alpha_1s^{n-1} + \cdots + \alpha_n} \tag{4}$$

Since  $\hat{g}_i(s)$  is strictly proper, the degree of the numerator of  $\hat{g}_i(s)$  is at most  $n-1$ . Combining (1) and (4), we obtain

$$\beta_1 s^{n-1} + \beta_2 s^{n-2} + \dots + \beta_n = (s^n + \alpha_1 s^{n-1} + \dots + \alpha_n)(a_1 s^{-1} + a_2 s^{-2} + \dots) \quad (5)$$

Equating the coefficients of  $s^j$  of both sides of (5) yields

$$\begin{aligned} a_1 &= \beta_1 \\ a_2 + \alpha_1 a_1 &= \beta_2 \\ &\vdots \\ a_n + \alpha_1 a_{n-1} + \dots + \alpha_{n-1} a_1 &= \beta_n \\ a_{n+j} + \alpha_1 a_{n+j-1} + \alpha_2 a_{n+j-2} + \dots + \alpha_n a_j &= 0, \quad j=1, 2, \dots \end{aligned} \quad (6)$$

The last equation of (6) can be written as

$$a_{n+j} = -\alpha_1 a_{n+j-1} - \alpha_2 a_{n+j-2} - \dots - \alpha_n a_j, \quad j=1, 2, \dots \quad (7)$$

Because of the structure of Hankel matrices, Eq. (7) implies that the  $(n+1)$ th row of  $\underline{H}_i(k, l)$  depends linearly on the first  $n$  rows of  $\underline{H}_i(k, l)$ . The  $(n+2)$ th row of  $\underline{H}_i(k, l)$  depends linearly on its preceding  $n$  rows, and consequently on the first  $n$  rows. By induction, we can show that every rows of  $\underline{H}_i(k, l)$  depend linearly on the first  $n$  rows of  $\underline{H}_i(k, l)$ . Similarly, every columns of  $\underline{H}_i(k, l)$  depends linearly on the first  $n$  columns of  $\underline{H}_i(k, l)$ . Hence we conclude that

$$\rho \underline{H}_i(n, n) = \rho \underline{H}_i(k, l), \text{ for all integers } k, l, \geq n$$

Next we show that  $\rho \underline{H}_i(n, n) = n$ . Suppose  $\rho \underline{H}_i(n, n) = \underline{H}_i(n, \infty) = m < n$ . Then because of the structure of the Hankel matrix, there exist  $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_m$  such that

$$a_{m+j} + \bar{\alpha}_1 a_{m+j-1} + \dots + \bar{\alpha}_m a_j = 0, \text{ for } j=1, 2, \dots$$

Using (6) with  $n$  replaced by  $m$ , it is possible to write  $\hat{g}_i(s)$  as a rational function with a denominator of degree  $m$ . This, however, contradicts with the assumption that  $\hat{g}_i(s)$  is irreducible and has a denominator of degree  $n$ . Hence we have  $\rho \underline{H}_i(n, n) = n$ . Q. E. D.

In order to extend theorem 1 to the vector case, we need some preliminary results.

LEMMA 1. Let  $f_0(s), f_1(s), \dots, f_p(s)$  be polynomials in  $s$ . Then

$$\text{g. c. d. } \{f_0(s), f_1(s), \dots, f_p(s)\} = 1$$

if and only if there exist some real constants  $c_1, c_2, \dots, c_p$  such that

$$\text{g. c. d. } \{f_0(s), \bar{f}(s)\} = 1$$

where  $\bar{f}(s) \triangleq c_1 f_1(s) + c_2 f_2(s) + \dots + c_p f_p(s)$

and g. c. d.  $\{ \} = 1$  means that the greatest common divisor of the elements in  $\{ \}$  is at most a constant.

PROOF We prove the sufficiency by contradiction. If  $\{f_0(s), f_1(s), \dots, f_p(s)\}$  has a common divisor, say  $s + \lambda$ , then  $\{f_0(s), \bar{f}(s)\}$  has the same common divisor. Hence g. c. d.  $\{f_0(s), \bar{f}(s)\} = 1$  implies g. c. d.  $\{f_0(s), f_1(s), \dots, f_p(s)\} = 1$ .

We show now the necessity. Let  $\lambda_j, j=1, 2, \dots, m$ , be the distinct roots of  $f_0(s)$ .

Let  $\text{Re}[\ ]$  stand for the real part of  $[\ ]$ . If  $\text{Re } f_i(\lambda_j)=0$  for all  $i=1,2,\dots,p$ , we define  $1 \times p$  real matrix  $\underline{M}(\lambda_j)$  as

$$\underline{M}(\lambda_j) = \frac{1}{\sqrt{-1}} [f_1(\lambda_j) f_2(\lambda_j) \dots f_p(\lambda_j)] \tag{8a}$$

Otherwise, we define  $\underline{M}(\lambda_j)$  as

$$\underline{M}(\lambda_j) = [\text{Re } f_1(\lambda_j) \text{Re } f_2(\lambda_j) \dots \text{Re } f_p(\lambda_j)] \tag{8b}$$

Note that  $\underline{M}(\lambda_j)$  can be considered as a linear operator which maps  $p$ -dimensional linear space  $(R^p, R)$  into  $(R, R)$ . Now the assumption  $\text{g. c. d. } \{f_0(s), f_1(s), \dots, f_p(s)\}=1$  implies  $\underline{M}(\lambda_j) \neq 0$  for  $j=1,2,\dots,m$ . Hence the null space of  $\underline{M}(\lambda_i)$ , denoted as  $H_i$ , has dimension at most  $p-1$ . Consider the set  $T = R^p - \bigcup_{i=1}^m H_i$ . Since  $H_i$  has dimension at most  $p-1$ , the union of finite number of  $H_i$  can never fill out the space  $R^p$ . Hence the set  $T$  is non-empty. Now for any vector  $\underline{c}$  in  $T$ , we have  $\underline{M}(\lambda_i)\underline{c} \neq 0$ , or equivalently,  $\bar{f}(\lambda_j) = c_1 f_1(\lambda_j) + c_2 f_2(\lambda_j) + \dots + c_p f_p(\lambda_j) \neq 0$ , for  $j=1,2,\dots,m$ . Hence we conclude that  $\text{g. c. d. } \{f_0(s), \bar{f}(s)\}=1$ . Q. E. D.

LEMMA 2 Let  $\hat{g}_i(s) = \frac{N_i(s)}{D_i(s)}$ ,  $i=1,2,\dots,p$ , be irreducible rational functions, and let  $D(s)$  be the least common denominator of  $\hat{g}_i(s)$ . Then

$$\text{g. c. d. } \{D(s), N_1(s)\bar{N}_1(s), \dots, N_p(s)\bar{N}_p(s)\}=1$$

where  $\bar{N}_i(s) \triangleq D(s)/D_i(s)$ .

PROOF This lemma will be proved by induction. First, we prove that the lemma holds for  $p=2$ . For  $p=2$ , we have  $D(s) = D_1(s)D_2(s)/\text{g. c. d. } \{D_1(s), D_2(s)\}$ , hence

$$\bar{N}_1(s) = D_2(s)/\text{g. c. d. } \{D_1(s), D_2(s)\} \tag{9}$$

and

$$\bar{N}_2(s) = D_1(s)/\text{g. c. d. } \{D_1(s), D_2(s)\} \tag{10}$$

Now we have

$$\begin{aligned} \text{g. c. d. } \{D(s), N_1(s)\bar{N}_1(s), N_2(s)\bar{N}_2(s)\} &= \text{g. c. d. } \{D_1(s)\bar{N}_1(s), \\ N_1(s)\bar{N}_1(s), N_2(s)\bar{N}_2(s)\} &= \text{g. c. d. } \{\bar{N}_1(s), N_2(s)\bar{N}_2(s)\} \\ &= \text{g. c. d. } \{\bar{N}_1(s), \bar{N}_2(s)\} \end{aligned}$$

The last equality follows from the facts that  $\bar{N}_1(s)$  is a subset of  $D_2(s)$ , and  $\text{g. c. d. } \{N_2(s), D_2(s)\}=1$ . From (9) and (10), it is easy to see that  $\text{g. c. d. } \{\bar{N}_1(s), \bar{N}_2(s)\}=1$ . This proves the lemma for  $p=2$ .

Now suppose the lemma holds for  $p-1$ , that is

$$\text{g. c. d. } \{\hat{D}(s), N_1(s)\hat{N}_1(s), \dots, N_{p-1}(s)\hat{N}_{p-1}(s)\}=1$$

where  $\hat{D}(s)$  is the least common denominator of  $\hat{g}_i(s)$ ,  $i=1,2,\dots,p-1$ , and  $\hat{N}_i(s) = \hat{D}(s)/D_i(s)$ . Now we show that the lemma holds for  $p$ . It is clear that

$$\begin{aligned} & \text{g. c. d. } \{D(s), N_1(s)\hat{N}_1(s)\hat{N}_1(s), \dots, N_{p-1}(s)\hat{N}_{p-1}(s)\hat{N}_1(s), N_p(s)\bar{N}_p(s)\} \\ & = \text{g. c. d. } \{\hat{N}_1(s), N_p(s)\bar{N}_p(s)\} \end{aligned}$$

where  $D(s)$  is the least common multiplier of  $\hat{D}(s)$  and  $D_p(s)$ , and

$$\begin{aligned} \hat{N}_1(s) &= \frac{D(s)}{\hat{D}(s)} = \frac{1}{\hat{D}(s)} \cdot \frac{\hat{D}(s)D_p(s)}{\text{g. c. d. } \{\hat{D}(s), D_p(s)\}} = \frac{\hat{D}_p(s)}{\text{g. c. d. } \{\hat{D}(s), D_p(s)\}} \\ \bar{N}_p(s) &= \frac{\hat{D}(s)}{\text{g. c. d. } \{\hat{D}(s), D_p(s)\}} \end{aligned}$$

Using these equations, and the arguments employed for  $p=2$ , we conclude that  $\text{g. c. d. } \{\hat{N}_1(s), N_p(s)\bar{N}_p(s)\} = 1$ . Q. E. D.

LEMMA 3: Let  $\hat{g}_i(s)$ ,  $i=1, 2, \dots, p$ , be strictly proper irreducible rational functions. Let  $D(s)$  be the least common denominator of  $\hat{g}_i(s)$ ,  $i=1, 2, \dots, p$ . Then there exist real constants  $c_i$ , for  $i=1, 2, \dots, p$ , such that the rational function

$$\frac{N(s)}{D(s)} \triangleq \sum_{i=1}^p c_i \hat{g}_i(s)$$

is irreducible, that is,  $N(s)$  and  $D(s)$  have no nontrivial common factor.

PROOF: Let

$$\hat{g}_i(s) = \frac{N_i(s)}{D_i(s)} = \frac{N_i(s)\bar{N}_i(s)}{D(s)}$$

where

$$\bar{N}_i(s) = \frac{D(s)}{D_i(s)}$$

It is shown in Lemma 2 that the irreducibility assumption implies that the greatest common divisor of  $\{D(s), N_1(s)\bar{N}_1(s), N_2(s)\bar{N}_2(s), \dots, N_p(s)\bar{N}_p(s)\}$  is 1. By Lemma 1, there exist real numbers  $c_i$ ,  $i=1, 2, \dots, p$  such that  $D(s)$  and  $N(s) = \sum_{i=1}^p c_i N_i(s)\bar{N}_i(s)$  have no common factor. Q. E. D.

With these lemmas, we can now extend Theorem 1 to the vector case.

THEOREM 2: Let  $\hat{g}_i(s)$ ,  $i=1, 2, \dots, p$ , be strictly proper irreducible rational function of degree  $n_i$ . If the degree of the least common denominator of  $\hat{g}_1(s), \hat{g}_2(s), \dots, \hat{g}_p(s)$  is  $n$ , then

$$\begin{aligned} & \rho[\underline{H}_1(n, n_1)\underline{H}_2(n, n_2)\dots\underline{H}_p(n, n_p)] \\ & = \rho[\underline{H}_1(k, l_1)\underline{H}_2(k, l_2)\dots\underline{H}_p(k, l_p)] = n \end{aligned} \tag{11}$$

and

$$\rho \begin{bmatrix} \underline{H}_1(n_1, n) \\ \underline{H}_2(n_2, n) \\ \vdots \\ \underline{H}_p(n_p, n) \end{bmatrix} = \rho \begin{bmatrix} \underline{H}_1(k_1, l) \\ \underline{H}_2(k_2, l) \\ \vdots \\ \underline{H}_p(k_p, l) \end{bmatrix} = n \tag{12}$$

for any  $k \geq n$ ,  $l \geq n$ ,  $k_i \geq n_i$ , and  $l_i \geq n_i$ ,  $i=1, 2, \dots, p$ ; where  $\underline{H}_i(k, l)$  is the Hankel matrix of  $\hat{g}_i(s)$  of order  $k \times l$ .

PROOF: First, we show that  $\rho[\underline{H}_1(k, l_1)\underline{H}_2(k, l_2)\dots\underline{H}_p(k, l_p)] \leq n$ , for  $k \geq n$ ,  $l_i \geq n_i$ . Let  $D(s)$  be the least common denominator of  $\hat{g}_1(s), \hat{g}_2(s), \dots, \hat{g}_p(s)$ , and let  $H(k, l)$  be

the Hankel matrix of  $1/D(s)$ . Then Theorem 1 and the assumption that the degree of  $D(s)$  is  $n$  imply  $\rho \underline{H}(n, n) = \rho \underline{H}(k, l) = n$  for any  $k \geq n, l \geq n$ . Now, we show that, for each  $i=1, 2, \dots, n$ , the columns of  $\underline{H}_i(k, l_i)$  are linear combinations of the columns of  $\underline{H}(k, n)$ . Indeed, if we write

$$\hat{g}_i(s) = \frac{N_i(s)}{D_i(s)} = \frac{N_i(s)\bar{N}_i(s)}{D(s)} \stackrel{\text{s.a.v.}}{=} \frac{\alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \dots + \alpha_n}{D(s)}$$

then, because of the Hankel coefficients of  $1/D(s)$  starting from  $s^{-n}$ , the Hankel coefficients of  $\hat{g}_i(s)$  are just the multiplication of  $(\alpha_1 s^{n-1} + \dots + \alpha_n)$  with the Hankel coefficients of  $1/D(s)$ . Hence the  $j$ th column of  $\underline{H}_i(k, l_i)$  is equal to  $\underline{H}(k, j-n+j-1)\underline{\alpha}$ , where  $\underline{\alpha} = [\alpha_n \alpha_{n-1} \dots \alpha_1]'$  (the prime denotes the transpose), and  $\underline{H}(k, j-n+j-1)$  is the submatrix of  $\underline{H}(k, n+j-1)$  by deleting the first  $(j-1)$  columns. Since all columns of  $\underline{H}(k, l)$  with  $l \geq n$  are linear combinations of the columns of  $\underline{H}(k, n)$ , we conclude that all columns of  $\underline{H}_i(k, l_i)$  are linear combinations of the columns of  $\underline{H}(k, n)$ . The rank of  $\underline{H}(k, n)$  is  $n$  by Theorem 1, hence the rank of  $[\underline{H}_1(k, l_1)\underline{H}_2(k, l_2)\dots\underline{H}_p(k, l_p)]$  is at most  $n$ .

Next, we show that  $\rho[\underline{H}_1(n, n)\underline{H}_2(n, n)\dots\underline{H}_p(n, n)] = n$ . It follows from Lemma 3 that there exist real constants  $c_i$  such that the rational function  $\hat{g}(s) \triangleq \sum_{i=1}^p c_i \hat{g}_i(s)$  is irreducible and its denominator has degree  $n$ . Let  $\underline{K}(n, n)$  be the Hankel matrix of  $\hat{g}(s)$ . Then Theorem 1 implies  $\rho \underline{K}(n, n) = n$ . Note that  $\underline{K}(n, n)$  can be expressed in terms of the Hankel matrices of  $\hat{g}_i(s)$ , for  $i=1, 2, \dots, p$ , as

$$\underline{K}(n, n) = c_1 \underline{H}_1(n, n) + c_2 \underline{H}_2(n, n) + \dots + c_p \underline{H}_p(n, n) \tag{13}$$

Now it is claimed that  $\rho[\underline{H}_1(n, n)\underline{H}_2(n, n)\dots\underline{H}_p(n, n)] = n$ . Indeed, if  $\rho[\underline{H}_1(n, n)\underline{H}_2(n, n)\dots\underline{H}_p(n, n)] < n$ , then there exists a  $1 \times n$  nonzero real vector  $\underline{\beta}$  such that  $\underline{\beta} \underline{H}_i(n, n) = 0$  for  $i=1, 2, \dots, p$ . This implies, by using (13),  $\underline{\beta} \underline{K}(n, n) = 0$ , which contradicts  $\rho \underline{K}(n, n) = n$ . Hence, we conclude that  $\rho[\underline{H}_1(n, n)\underline{H}_2(n, n)\dots\underline{H}_p(n, n)] = n$ . Using again Theorem 1, we conclude  $\rho[\underline{H}_1(n, n_1)\dots\underline{H}_p(n, n_p)] = \rho[\underline{H}_1(n, n)\dots\underline{H}_p(n, n)] = n$ .

Now, every  $[\underline{H}_1(k, l_1)\underline{H}_2(k, l_2)\dots\underline{H}_p(k, l_p)]$  for  $k \geq n, l_i \geq n_i$  contains  $[\underline{H}_1(n, n_1)\underline{H}_2(n, n_2)\dots\underline{H}_p(n, n_p)]$  as a submatrix. The former has rank at most  $n$ ; the latter has rank  $n$ . Hence (11) is proved. The proof of (12) is exactly the same as (11) and is omitted. Q. E. D.

### III. REALIZATION ALGORITHM

In this section, an irreducible realization procedure will be presented. We first state formally the problem: Given a strictly proper transfer function matrix  $\hat{G}(s)$ , or sampled transfer function matrix  $\hat{G}(z)$ , find a continuous-time dynamical equation

$$\begin{aligned} \dot{\underline{x}}(t) &= \underline{A} \underline{x}(t) + \underline{B} u(t) \\ \underline{y}(t) &= \underline{C} \underline{x}(t) \end{aligned}$$

or a discrete-time dynamical equation

$$\begin{aligned} \underline{x}(n+1) &= \underline{A} \underline{x}(n) + \underline{B} u(n) \\ \underline{y}(n) &= \underline{C} \underline{x}(n) \end{aligned}$$

so that  $\hat{G}(s) = \underline{C}(s\underline{I} - \underline{A})^{-1}\underline{B}$ , or  $\hat{G}(z) = \underline{C}(z\underline{I} - \underline{A})^{-1}\underline{B}$ . Furthermore, the dynamical equations are required to be irreducible, or equivalently, controllable and observable [2]. The realization procedure to be introduced is applicable without any modification to either case.

Consider a  $q \times p$  strictly proper rational matrix  $\hat{G}(s) = (\hat{g}_{ij}(s))$ . Let  $\hat{g}_{ij}(s)$  be expanded as

$$\hat{g}_{ij}(s) = h_{ij}(1)s^{-1} + h_{ij}(2)s^{-2} + h_{ij}(3)s^{-3} + \dots \tag{14}$$

Note that these Hankel coefficients  $h_{ij}(k)$  can also be obtained by measurements if the use of pure differentiators is permitted. We apply a delta function at the input, the  $k$ th derivative of the output at  $t=0$  yields directly  $h_{ij}(k)$ . In the case of discrete-time case, if we apply  $u(0)=1$  and  $u(i)=0$ , for  $i=1, 2, \dots$ , then the measured output at  $n=1, 2, 3, \dots$  yields the Hankel coefficients.

We now form the Hankel matrix of  $\hat{g}_{ij}(s)$  of order  $\alpha \times \beta$  as

$$\underline{H}_{ij}(\alpha, \beta) = \begin{bmatrix} h_{ij}(1) & h_{ij}(2) & \dots & h_{ij}(\beta) \\ h_{ij}(2) & h_{ij}(3) & \dots & h_{ij}(\beta+1) \\ \vdots & \vdots & \ddots & \vdots \\ h_{ij}(\alpha) & h_{ij}(\alpha+1) & \dots & h_{ij}(\alpha+\beta-1) \end{bmatrix} \tag{15}$$

Let the degree of the least common denominator of the  $i$ th row of  $\hat{G}(s)$  be  $\alpha_i$ , and that of the  $j$ th column of  $\hat{G}(s)$  be  $\beta_j$ . Form the generalized Hankel matrix of  $\hat{G}(s)$  as

$$\underline{H} \triangleq \begin{bmatrix} \underline{H}_{11}(\alpha_1+1, \beta_1) & \underline{H}_{12}(\alpha_1+1, \beta_2) & \dots & \underline{H}_{1p}(\alpha_1+1, \beta_p) \\ \underline{H}_{21}(\alpha_2+1, \beta_1) & \underline{H}_{22}(\alpha_2+1, \beta_2) & \dots & \underline{H}_{2p}(\alpha_2+1, \beta_p) \\ \vdots & \vdots & \ddots & \vdots \\ \underline{H}_{q1}(\alpha_q+1, \beta_1) & \underline{H}_{q2}(\alpha_q+1, \beta_2) & \dots & \underline{H}_{qp}(\alpha_q+1, \beta_p) \end{bmatrix} \triangleq \begin{bmatrix} \underline{H}_1 \\ \underline{H}_2 \\ \vdots \\ \underline{H}_q \end{bmatrix} \tag{16}$$

where  $\underline{H}_i$  is an  $(\alpha_i+1) \times \beta$  matrix with  $\beta \Delta \sum_{i=1}^p \beta_i$  and is called the  $i$ th block of  $\underline{H}$ . Before proceeding, we discuss some properties of  $\underline{H}$ .

Let the rank of  $\underline{H}$  be  $n$ , and let the  $n$  linearly independent rows of  $\underline{H}$  be chosen in its natural ordering (that is, beginning from 1st row, 2nd row and so forth). We use  $\Sigma$  to denote this set of linearly independent rows. Then, because of the of the properties of Hankel matrices, for each  $\underline{H}_i$ , there exists an integer  $\sigma_i$  with  $0 \leq \sigma_i \leq \alpha_i$  so that the first  $\sigma_i$  rows of  $\underline{H}_i$  belong to  $\Sigma$ . Note that  $\sigma_1 = \alpha_1$ , but not necessarily for  $\sigma_i = \alpha_i$  with  $i > 1$ . The assumption that  $\underline{H}$  has rank  $n$  implies

$$\sigma_1 + \sigma_2 + \dots + \sigma_q = n \tag{17}$$

In the following, a transformation will be introduced to yield such set of linearly independent rows. Let matrix  $\underline{U}$  be of the same dimension as  $\underline{H}$  and be of the form

$$\underline{U}' = \underbrace{[u'_{11} \dots u'_{1\sigma_1} \ 0']}_{(\alpha_1+1) \text{ columns}} \underbrace{[u'_{21} \dots u'_{2\sigma_2} \ 0' \dots 0']}_{(\alpha_2+1) \text{ columns}} \dots \underbrace{[u'_{q1} \dots u'_{q\sigma_q} \ 0' \dots 0']}_{(\alpha_q+1) \text{ column}} \tag{18}$$

where the prime stands for the transpose. Note that  $\underline{U}$  has  $n$  nonzero rows (or  $\underline{U}'$  has  $n$  nonzero columns) whose positions correspond to those of  $\Sigma$  in  $\underline{H}$ . The transformation of  $\underline{H}$  into  $\underline{U}$  can be achieved by elementary transformations as follows. Let

$$\underline{U} = \underline{K}_l \underline{K}_{l-1} \cdots \underline{K}_2 \underline{K}_1 \underline{H} \triangleq \underline{K} \underline{H} \tag{19}$$

where the  $\underline{K}_i$  are lower triangular matrices with all diagonal elements unity, and are obtained in the following manner. Let  $h_1(j)$  be the first nonzero element from the left in the first row of  $\underline{H}$ . Then  $\underline{K}_1$  is chosen so that all, except the first, elements of the  $j$ th column of  $\underline{K}_1 \underline{H}$  are zero. Let  $h_2(k)$  be the first nonzero element from the left of the second row of  $\underline{K}_1 \underline{H}$ . Then  $\underline{K}_2$  is chosen so that all, except the first two, elements of the  $k$ th column of  $\underline{K}_2 \underline{K}_1 \underline{H}$  are zero. Proceeding in this manner, the matrix  $\underline{H}$  can be transformed into  $\underline{U}$ . Note that in this process, if one row is identically zero, then proceed to the next row. We see that this process is similar to the Gaussian elimination except that the order of the rows is not changed, and that the elimination is carried out not necessarily in the natural ordering (that is, in the order of 1st column, 2nd column, ...). By multiplying these  $\underline{K}_i$ , we obtain  $\underline{K} = \underline{K}_l \cdots \underline{K}_2 \underline{K}_1$ . Note that  $l$ , the number of  $\underline{K}_i$ , is equal to or smaller than the rank of  $\underline{H}$ . In the matrix  $\underline{K}$ , only the  $(\sigma_1+1)$ th,  $(\alpha_1+1+\sigma_2+1)$ th,  $\cdots$ ,  $(\sum_{i=1}^{q-1} (\alpha_i+1) + \sigma_q + 1)$ th row are needed in the realization to be proposed. Let

$$m_j \triangleq \sum_{i=1}^{j-1} (\alpha_i + 1) + \sigma_j + 1, \quad j = 1, 2, \dots, q$$

Then the  $m_j$ th row, denoted as  $k_{mj}$ , of  $\underline{K}$  can be written as

$$\begin{aligned} k_{m_1} &= [\overbrace{a_{11}(1) a_{11}(2) \cdots a_{11}(\sigma_1)}^{\alpha_1+1} \mid 0 \quad \cdots \quad 0 \quad \cdots \quad 0 \mid \cdots \mid 0 \quad \cdots \quad 0 \quad \cdots \quad 0] \\ k_{m_2} &= [a_{21}(1) a_{21}(2) \cdots a_{21}(\sigma_1) \mid 0 \quad \overbrace{a_{22}(1) \cdots a_{22}(\sigma_2)}^{\alpha_2+1} \mid \cdots \mid 0 \quad \cdots \quad 0 \quad \cdots \quad 0] \\ k_{m_q} &= [a_{q1}(1) a_{q1}(2) \cdots a_{q1}(\sigma_1) \mid a_{q2}(1) \cdots a_{q2}(\sigma_2) \mid \cdots \mid a_{qq}(1) \cdots a_{qq}(\sigma_q)] \end{aligned} \tag{20}$$

Note that the vector  $k_{mj}$  has, excluding element 1, only  $(\sigma_1 + \sigma_2 + \cdots + \sigma_j)$  possible nonzero elements. These elements are the coefficients of the linear combination of the  $m_j$ th row of  $\underline{H}$  in terms of the preceding rows in  $\Sigma$ . With this preliminary, we can now state the proposed algorithm.

ALGORITHM: Given a  $q \times p$  strictly proper rational matrix  $\hat{G}(s)$ .

- (i) Expand each element of  $\hat{G}(s)$  into the form as shown in (14).
- (ii) Form the generalized Hankel matrix  $\underline{H}$  as shown in (16).
- (iii) Transform the matrix  $\underline{H}$  into the form of  $\underline{U}$  shown in (18) by using a sequence of elementary transformations discussed. Write  $\underline{K} \underline{H} = \underline{U}$ .
- (iv) Then by using appropriate rows of  $\underline{K}$ , as in (20), an irreducible realization of  $\hat{G}(s)$  can be obtained as

$$\dot{\underline{x}} = \underline{A} \underline{x} + \underline{B} u \tag{21a}$$

$$\underline{y} = \underline{C} \underline{x} \tag{21b}$$

with

$$\underline{A} = \begin{bmatrix} \underline{A}_{11} & 0 & \cdots & 0 \\ \underline{A}_{21} & \underline{A}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \underline{A}_{q1} & \underline{A}_{q2} & \cdots & \underline{A}_{qq} \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} \underline{B}_1 \\ \underline{B}_2 \\ \vdots \\ \underline{B}_q \end{bmatrix} \tag{22}$$



where, for  $i=1, 2, \dots, q$ ,

$$\tilde{A}_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_{ii}(1) & -a_{ii}(2) & -a_{ii}(3) & \dots & -a_{ii}(\sigma_i) \end{bmatrix} \quad (\sigma_i \times \sigma_i) \text{ matrix}$$

for  $i \neq j$ ,

$$\tilde{A}_{ij} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ -a_{ij}(1) & -a_{ij}(2) & \dots & -a_{ij}(\sigma_j) \end{bmatrix} \quad (\sigma_i \times \sigma_j) \text{ matrix}$$

and, for  $i=1, 2, \dots, q$ ,

$$\tilde{B}_i = \begin{bmatrix} h_{i1}(1) & h_{i2}(1) & \dots & h_{ip}(1) \\ h_{i1}(2) & h_{i2}(2) & \dots & h_{ip}(2) \\ \vdots & \vdots & \dots & \vdots \\ h_{i1}(\sigma_i) & h_{i2}(\sigma_i) & \dots & h_{ip}(\sigma_i) \end{bmatrix} \quad (\sigma_i \times p) \text{ matrix}$$

The matrix  $\tilde{C}$  is given by, if  $\sigma_i \neq 0$ , for  $i=1, 2, \dots, q$ ,

$$\tilde{C} = \left[ \underbrace{\begin{bmatrix} 1 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ 0 & 0 \dots 0 \end{bmatrix}}_{\sigma_1} \quad \underbrace{\begin{bmatrix} 0 & 0 \dots 0 \\ 1 & 0 \dots 0 \\ 0 & 0 \dots 0 \end{bmatrix}}_{\sigma_2} \quad \dots \quad \underbrace{\begin{bmatrix} 0 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ 1 & 0 \dots 0 \end{bmatrix}}_{\sigma_q} \right] \quad (24a)$$

If  $\sigma_i = 0$  for some  $i$ , say  $i=3$  for convenience of illustration, then

$$\tilde{C} = \left[ \underbrace{\begin{bmatrix} 1 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ -a_{31}(1) & \dots & -a_{31}(\sigma_1) \\ 0 & 0 \dots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \dots 0 \end{bmatrix}}_{\sigma_1} \quad \underbrace{\begin{bmatrix} 0 & 0 \dots 0 \\ 1 & 0 \dots 0 \\ -a_{32}(1) & \dots & -a_{32}(\sigma_2) \\ 0 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ 0 & 0 \dots 0 \end{bmatrix}}_{\sigma_2} \quad \underbrace{\begin{bmatrix} 0 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ 1 & 0 \dots 0 \\ 0 & 0 \dots 0 \end{bmatrix}}_{\sigma_4} \quad \dots \quad \underbrace{\begin{bmatrix} 0 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ 1 & 0 \dots 0 \end{bmatrix}}_{\sigma_q} \right] \quad (24b)$$

The dimension of this realization is  $n = \sigma_1 + \sigma_2 + \dots + \sigma_q$ .

We note that the form of the resulting dynamical equation (21) is identical to the one in [11]. The result in [11] is obtained by repeatedly solving algebraic equations. The algorithm presented here is believed to be new, and is simpler than that in [11]. The algorithm is inspired by [10].

#### IV. PROOF OF THE ALGORITHM

In this section, we shall show that the dynamical equation (21) is indeed a realization of  $\hat{G}(s)$ , and is irreducible. We use the fact that (21) is a realization of  $\hat{G}(s)$  if and only if

$$h_{ij}(n) = \tilde{c}_i \tilde{A}^{n-1} \tilde{b}_j \quad \text{for } n=1, 2, 3, \dots \quad (25)$$

where  $\underline{c}_i$  is the  $i$ th row of  $\underline{C}$ , and  $\underline{b}_j$  is the  $j$ th column of  $\underline{B}$ . Using (20),  $h_{ij}(\sigma_i+1)$  can be written in terms  $h_{kj}(n)$ ,  $k=1, 2, \dots, j$ , as

$$h_{ij}(\sigma_i+1) = - \sum_{k=1}^j \sum_{m=1}^{\sigma_k} a_{ik}(m) h_{kj}(m)$$

Because of the structure of the Hankel matrix  $\underline{H}$ , this can be generalized to

$$h_{ij}(\sigma_i+l) = - \sum_{k=1}^j \sum_{m=1}^{\sigma_k} a_{ik}(m) h_{kj}(m+l-1), \text{ for } l=1, 2, 3, \dots \tag{26}$$

Using this equation, (25) can be directly verified.

To show that the realization is irreducible, we show that  $\{A, B\}$  is controllable and  $\{A, C\}$  is observable. Indeed, it can be directly verified that

$$[\underline{b}_1 \underline{A} \underline{b}_1 \dots \underline{A}^{\beta_1-1} \underline{b}_1 \underline{b}_2 \underline{A} \underline{b}_2 \dots \underline{A}^{\beta_2-1} \underline{b}_2 \dots \underline{b}_p \dots \underline{A}^{\beta_p-1} \underline{b}_p] = \underline{H}_x \tag{27}$$

where  $\underline{H}_x$  consists of only the linearly independent rows of  $\underline{H}$ . Since the rank of  $\underline{H}_x$  is  $n$ , hence  $\{\underline{A}, \underline{B}\}$  is controllable. Similarly, it can be easily verified that the matrix

$$\begin{bmatrix} \underline{c}_1 \\ \underline{c}_1 \underline{A} \\ \vdots \\ \underline{c}_1 \underline{A}^{\sigma_1-1} \\ \vdots \\ \underline{c}_q \\ \vdots \\ \underline{c}_q \underline{A}^{\sigma_q-1} \end{bmatrix} \tag{28}$$

is an identity matrix of order  $n$ , hence  $\{\underline{A}, \underline{C}\}$  is observable. Note that in the formation of (28), if  $\sigma_i=0$ , then the corresponding  $\underline{c}_i$  is omitted. This proves that the realization (21) is irreducible.

**V. NUMERICAL EXAMPLE**

Consider the sampled transfer function matrix

$$\hat{\underline{G}}(z) = \begin{bmatrix} \frac{1}{(z+2)^2} & \frac{1}{z+1} \\ \frac{z+3}{(z+1)(z+2)} & \frac{z}{(z+1)^2} \end{bmatrix}$$

It is clear that  $\alpha_1=3$ ,  $\alpha_2=3$ ,  $\beta_1=3$ , and  $\beta_2=2$ . We form

$$\underline{H} \triangleq \begin{bmatrix} \underline{H}_{11}(4, 3) & \underline{H}_{12}(4, 2) \\ \underline{H}_{21}(4, 3) & \underline{H}_{22}(4, 2) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 & | & 1 & -1 \\ 1 & -4 & 12 & | & -1 & 1 \\ -4 & 12 & -32 & | & 1 & -1 \\ 12 & -32 & 80 & | & -1 & 1 \\ \hline 1 & 0 & -2 & | & 1 & -2 \\ 0 & -2 & 6 & | & -2 & 3 \\ -2 & 6 & -14 & | & 3 & -4 \\ 6 & -14 & 30 & | & -4 & 5 \end{bmatrix}$$

The elementary transformations  $K_1, K_2, \dots$  defined in (19) can be easily found as

$$\tilde{K}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -12 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 32 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -6 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 14 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \tilde{K}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -12 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -6 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\tilde{K}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \tilde{K}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The applications of these elementary transformations to  $\underline{H}$  results in

$$U = \begin{bmatrix} 0 & 1 & -4 & 1 & -1 \\ 1 & 0 & -4 & 3 & -3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow \sigma_1 = 3 \\ \leftarrow \sigma_2 = 1 \end{matrix}$$

Hence the dimension of the irreducible realization is  $\sigma_1 + \sigma_2 = 4$ . The  $(\sigma_1 + 1)$ th and the  $[(\sigma_1 + 1) + (\sigma_2 + 1)]$ th rows of  $\tilde{K}_4 \tilde{K}_3 \tilde{K}_2 \tilde{K}_1$  can be easily computed as

$$k_{m1} = [4 \ 8 \ 5 \ 1 : 0 \ 0 \ 0 \ 0]$$

$$k_{m2} = [6 \ 7 \ 2 \ 0 : 1 \ 1 \ 0 \ 0]$$

Hence an irreducible realization of  $\hat{G}(z)$  is given by

$$\tilde{x}(n+1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & -8 & -5 & 0 \\ \hline -6 & -7 & -2 & -1 \end{bmatrix} \tilde{x}(n) + \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -4 & 1 \\ \hline 1 & 1 \end{bmatrix} u(n)$$

$$\tilde{y}(n) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{bmatrix} \tilde{x}(n)$$

## VI. IDENTIFICATION OF DISCRETE-TIME SYSTEMS

For convenience, we study in this section only discrete-time systems. The realization problem introduced in the previous sections starts from sampled transfer function matrices, or equivalently, from responses,  $\underline{y}(n)$ , due to the applications of impulse sequences. In this section, the inputs to be applied will be extended to other sequences.

Let the system to be identified is a  $p$ -input  $q$ -output system, and is described by

$$\begin{aligned}\underline{x}(n+1) &= \underline{A}\underline{x}(n) + \underline{B}\underline{u}(n) \\ \underline{y}(n) &= \underline{C}\underline{x}(n)\end{aligned}\quad (28)$$

or its equivalent dynamical equation, where  $\underline{A}, \underline{B}, \underline{C}$  are still not known. In order to apply the result obtained in the previous sections, the inputs are required to be generated by a known  $p$ -input  $p$ -output dynamical equation, say,

$$\begin{aligned}\underline{x}_1(n+1) &= \underline{A}_1\underline{x}_1(n) + \underline{B}_1\underline{u}_1(n) \\ \underline{u}(n) &= \underline{C}_1\underline{x}_1(n)\end{aligned}\quad (29)$$

with input  $u_{1j}(0)=1$ ,  $u_{1j}(i)=0$ ,  $i=1, 2, \dots$ , where  $u_{1j}$  is the  $j$ th component of  $\underline{u}_1$ . Then the output  $\underline{y}(n)$  due to the application of  $\underline{u}(n)$  in (28) gives the Hankel coefficients of the tandem connection of (29) followed by (28). Hence, from these Hankel coefficients, an irreducible dynamical equation, say,

$$\begin{aligned}\underline{z}(n+1) &= \underline{F}\underline{z}(n) + \underline{G}\underline{u}_1(n) \\ \underline{y}(n) &= \underline{H}\underline{z}(n)\end{aligned}\quad (30)$$

can be obtained. If equation (30) is equivalent to the tandem connection of (29) followed by (28), that is

$$\begin{aligned}\begin{bmatrix} \underline{x}_1(n+1) \\ \underline{x}(n+1) \end{bmatrix} &= \begin{bmatrix} \underline{A}_1 & 0 \\ \underline{B}\underline{C}_1 & \underline{A} \end{bmatrix} \begin{bmatrix} \underline{x}_1(n) \\ \underline{x}(n) \end{bmatrix} + \begin{bmatrix} \underline{B}_1 \\ 0 \end{bmatrix} \underline{u}_1(n) \\ \underline{y}(n) &= [0 \quad \underline{C}] \begin{bmatrix} \underline{x}_1(n) \\ \underline{x}(n) \end{bmatrix}\end{aligned}\quad (31)$$

then (28) or its equivalent equation can be solved from (29) and (30). Unfortunately, equation (30) may not always equivalent to (31). For example, if

$$\begin{aligned}\underline{A} &= \begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \underline{C} = [1 \quad 1] \\ \underline{A}_1 &= \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}, \quad \underline{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \underline{C}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\end{aligned}$$

then the equation

$$\underline{F} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -6 & -5 \end{bmatrix}, \quad \underline{G} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ -2 & -7 \end{bmatrix}, \quad \underline{H} = [1 \quad 0 \quad 0]$$

which is obtained by using the algorithm in Section III, is not equivalent to (30). This is due to the fact that the tandem connection of (29) followed by (28) may

not always be controllable and observable [2]. In this case, the determination of (28) from (29) and (30) will not be a simple task. Hence, the identification by using exclusively dynamical equations seems not very promising.

Let the sampled transfer function matrices of (28) and (29) be, respectively,  $\hat{G}(z)$  and  $\hat{G}_1(z)$ . If we apply to (28) an input generated by (29), then the measured data at  $n=1, 2, \dots$  are the Hankel coefficients of  $\hat{G}(z)$   $\hat{G}_1(z)$ , that is

$$\hat{G}(z)\hat{G}_1(z) = \left( \sum_{k=1}^{\infty} h_{ij}(k)z^{-k} \right) \quad (32)$$

Now if  $G_1(z)$  is nonsingular, then (32) implies

$$\hat{G}(z) = \begin{bmatrix} \sum_{k=1}^{\infty} h_{11}(k)z^{-k} & \sum h_{12}(k)z^{-k} \dots \sum h_{1p}(k)z^{-k} \\ \sum h_{21}(k)z^{-k} & \sum h_{22}(k)z^{-k} \dots \sum h_{2p}(k)z^{-k} \\ \vdots & \vdots \\ \sum h_{q1}(k)z^{-k} & \sum h_{q2}(k)z^{-k} \dots \sum h_{qp}(k)z^{-k} \end{bmatrix} G_1^{-1}(z) \quad (33)$$

From this equation, the Hankel coefficients of  $\hat{G}(z)$  can be easily obtained. Using these Hankel coefficients, an irreducible dynamical equation can then be found.

In using (32) to identify a system, not all the Hankel coefficients are needed. If an upper bound, say  $n$ , of the dimension of the system is known, then at most the first  $2n$  Hankel coefficients of each entry of  $\hat{G}(z)$  are needed. That is, we form  $\hat{H}_{ij}(n+1, n)$  in (15), and then proceed. If we do not know the upper bound of the dimension of a system, we may proceed as follows: we first find the least integer  $\alpha_1$  so that the increase of  $\alpha_1$  will not increase the rank of the matrix

$$\begin{bmatrix} \hat{H}_{11}(\alpha_1+1, \alpha_1) & \hat{H}_{12}(\alpha_1+1, \alpha_1) \dots \hat{H}_{1p}(\alpha_1+1, \alpha_1) \\ \hat{H}_{21}(\alpha_1+1, \alpha_1) & \hat{H}_{22}(\alpha_1+1, \alpha_1) \dots \hat{H}_{2p}(\alpha_1+1, \alpha_1) \\ \vdots & \vdots \\ \hat{H}_{q1}(\alpha_1+1, \alpha_1) & \hat{H}_{q2}(\alpha_1+1, \alpha_1) \dots \hat{H}_{qp}(\alpha_1+1, \alpha_1) \end{bmatrix}$$

Next find the least integer  $\alpha_2$  so that the increase of  $\alpha_2$  will not increase the rank of the matrix

$$\begin{bmatrix} \hat{H}_{11}(\alpha_1+1, \alpha_2) & \hat{H}_{12}(\alpha_1+1, \alpha_2) \dots \hat{H}_{1p}(\alpha_1+1, \alpha_2) \\ \hat{H}_{21}(\alpha_2+1, \alpha_2) & \hat{H}_{22}(\alpha_2+1, \alpha_2) \dots \hat{H}_{2p}(\alpha_2+1, \alpha_2) \\ \vdots & \vdots \\ \hat{H}_{q1}(\alpha_2+1, \alpha_2) & \hat{H}_{q2}(\alpha_2+1, \alpha_2) \dots \hat{H}_{qp}(\alpha_2+1, \alpha_2) \end{bmatrix}$$

Proceed in this manner until the least integer  $\alpha_q$  is found so that the increase of  $\alpha_q$  will not increase the rank of the matrix

$$\begin{bmatrix} \hat{H}_{11}(\alpha_1+1, \alpha_q) & \hat{H}_{12}(\alpha_1+1, \alpha_q) \dots \hat{H}_{1p}(\alpha_1+1, \alpha_q) \\ \hat{H}_{21}(\alpha_2+1, \alpha_q) & \hat{H}_{22}(\alpha_2+1, \alpha_q) \dots \hat{H}_{2p}(\alpha_2+1, \alpha_q) \\ \vdots & \vdots \\ \hat{H}_{q1}(\alpha_q+1, \alpha_q) & \hat{H}_{q2}(\alpha_q+1, \alpha_q) \dots \hat{H}_{qp}(\alpha_q+1, \alpha_q) \end{bmatrix}$$

In this process, the algorithm developed in (19) can be used. Once the integer  $\alpha_q$  is found, the dynamical equation description of the system can be readily obtained.

Although the idea of the use of (32) to identify a system is extremely simple, it seems never been proposed. Compared with the identification procedures in [13, 14], the proposed method is conceptually and computationally simpler. The method also seems simpler than the conventional impulse response identification [1].

We give a simple example to conclude this section. Let the system to be identified is given by [14]

$$\tilde{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tilde{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

We apply an input sequence (1, 2, 4, 8, ...) which is generated by  $(z-2)^{-1}$ . Let the measured output  $y(n)$  at  $n=1, 2, \dots$ , be

$$\left\{ \begin{array}{l} 0, 0, 0, 1, 5, 16, 42, \dots \\ 0, 1, 4, 11, 26, 57, 120, \dots \end{array} \right\}$$

Then the Hankel coefficients of the system to be identified can be easily computed as, by using (32),

$$\left\{ \begin{array}{l} 0, 0, 1, 3, 6, 10, 15, \dots \\ 1, 2, 3, 4, 5, 6, 7, \dots \end{array} \right\}$$

If the upper bound of the dimension of the system is given as 4, Then, using the algorithm in Section III, and using (24b), we obtain

$$\tilde{\tilde{A}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}, \tilde{\tilde{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tilde{\tilde{C}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

which can be shown to be equivalent to  $\{\tilde{A}, \tilde{B}, \tilde{C}\}$ .

### VII. IDENTIFICATION OF CONTINUOUS-TIME SYSTEMS

The identification procedure developed in the discrete-time case can be applied without any modification to the continuous-time case, if pure differentiators can be employed. In the continuous-time case, (33) reads as

$$\hat{\tilde{G}}(s) = \begin{bmatrix} \sum_{k=1}^{\infty} h_{11}(k)s^{-k} & \sum h_{12}(k)s^{-k} & \dots & \sum h_{1p}(k)s^{-k} \\ \sum h_{21}(k)s^{-k} & \sum h_{22}(k)s^{-k} & \dots & \sum h_{2p}(k)s^{-k} \\ \vdots & \vdots & \ddots & \vdots \\ \sum h_{q1}(k)s^{-k} & \sum h_{q2}(k)s^{-k} & \dots & \sum h_{qp}(k)s^{-k} \end{bmatrix} \hat{\tilde{G}}_1^{-1}(s) \tag{34}$$

The  $h_{ij}(k)$  can be obtained by measuring the outputs and its derivatives at  $t=0$ . The tranfer function matrix  $\hat{\tilde{G}}_1(s)$  is used to generate the inputs. Once the Hankel coefficients of  $\hat{\tilde{G}}(s)$  is computed from (34), a dynamical equation description of the system to be identified can be obtained.

The use of differentiators is however not recommended in practice, because it may introduce large errors in measurements. We introduce in the following an identification scheme which does not require differentiations. Consider a system described by

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \tag{35a}$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) \tag{35b}$$

This system is to be identified. If we apply a piecewise constant input, that is

$$\tilde{u}(t) = \tilde{u}(n) \text{ for } nT \leq t < (n+1)T; \quad n=0, 1, 2, \dots \tag{36}$$

where  $T$  is a positive constant, then the responses of (35) at  $t=0, T, 2T, \dots$  can be described by the discrete-time equation [2]

$$\tilde{x}(n+1) = \tilde{A}\tilde{x}(n) + \tilde{B}u(n) \tag{37a}$$

$$\tilde{y}(n) = \tilde{C}\tilde{x}(n) \tag{37b}$$

with  $\tilde{A} = e^{AT}$  (38a)

$$\tilde{B} = \left( \int_0^T e^{A^T t} dt \right) B \triangleq \tilde{M}B \tag{38b}$$

$$\tilde{C} = C \tag{38c}$$

Note that (36) can be generated by a sampler and a zero-order hold [17] and that if (35) is controllable and observable, then for almost every  $T$ , (37) is also controllable and observable [2]. Now the discrete-time dynamical equation (37) can be identified by using the algorithm developed in the previous section and the measured data  $\tilde{y}(t)$  at  $t=0, T, 2T, \dots$ . Once  $\tilde{A}$  is known,  $\tilde{A}$  can be computed from (38a) [18]. If  $\tilde{A}$  is known,  $\tilde{M}$  in (38b) can be computed, and can be shown to be nonsingular [2]. Hence  $\tilde{B}$  can be uniquely determined from (38b). Consequently, a continuous-time system can be identified indirectly from a discrete-time equation. By so doing, the use of differentiators can be avoided.

There are two other existing identification schemes which do not use differentiators [19, 20]. The method in [19] needs integration over  $[0, \infty)$ ; the one in [20] needs multiple integrations. Therefore the method in this section may be preferable. However, the determination of  $\tilde{A}$ ,  $\tilde{B}$  from  $\tilde{A}$ ,  $\tilde{B}$  many require considerable amount of computations.

### VIII. CONCLUSIONS

We have presented in this paper an irreducible realization algorithm and a method of identifying linear time-invariant lumped systems. The introduced realization algorithm computes essentially only the rank of  $\tilde{H}$ . The other existing methods need the rank of  $\tilde{H}$  as well as some additional multiplications or inversions of matrices, hence the amount of computations needed in this paper is the smallest. Since the rank of  $\tilde{H}$  is essential in any irreducible realization, and since the proposed algorithm computes only the rank, hence it is believed that the proposed algorithm cannot be further simplified. Furthermore, the method can be easily programmed on a digital computer.

The identification method is carried out partly in the frequency domain and partly in the time domain. By so doing, the difficulty of controllability and observability, which may occur in using exclusively dynamical equations, is eliminated. The identification scheme is easy to apply, and compares favorable with the conventional impulse response identification.

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