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NULLATOR'S STABILITY CRITERION FOR LINEAR, TIME-INVARIANT PASSIVE RLC NETWORKS

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1. INTRODUCTION

In this paper a new method of determining stability of the linear, time-invariant, passive, portless RLC network is presented. The theory of this method is based on Lasalle theorem, an extension of Liapunov's stability theorem. In order to simplify a complicated network, and to apply the stability theorem, an ideal element, called nullator, is introduced. The nullator is a one-port network element defined by $v=0$ and $i=0$, this means that it is simultaneously an open circuit and a short circuit. (In system theory, nullator may be defined as an ideal component with zero input and zero output for all time).[1]

Some theorems that can be used for testing stability of the network are developed, and they are in terms of network graphs. With these network graph theorems, some of the network can be easily checked by inspection.

If the network is more complicated, the simplification rules developed by nullator can be applied, and then it will not be difficult to find all eigenvalues of the state equation written from the simplified network. The step-by-step testing procedures will be described in section 4.

2. THEOREMS

Theorem 1[2]**:** Given a connected graph G and let $A\subset G$, $B\subset G$, there exist a tree T, such that $A\subset T$ and $B\subset -T$, if and only if

(i) $A\cap B=\phi$

(ii) there is no loop in A.

(iii) there is no cutset in B.

Proof: Sufficient condition,

Let α be a nonempty set of elements of a connected graph G, and contain circuit elements C and noncircuit elements N. Let N be deleted from α but retained in G.

Consider the removal of the maximum circuit element B (containing edges $e_1, e_2, \ldots\ldots e_m$) from C without reducing the rank of G. G_m, the remainder of G after removing B, is connected and contains all vertices of G, so there is no cutset in

B (given).

$\because B \cup G_m = G$ and $B \cap G_m = \phi \longrightarrow G_m = -B$

$A \cap B = \phi$ (given) $\longrightarrow A \subset -B$

$A \subset G_m$ and there is no circuit in A (given)

Any circuit C' of G_m contains at least one edge not in A. Removal of such an edge destroys C' without removing a vertex. Repeated application of this procedure of removing edges from circuit results in a connected set T, since $T \subset G_m \subset G$

$T \subset G$ and contains all vertices of G

$\therefore T$ is a tree.

Since $A \subset G$ and A is not changed while G_m becomes T

$\therefore A \subset T$

Since $B \cap G_m = \phi$ and $T \subset G_m$

$\longrightarrow B \cap T = \phi$

$\longrightarrow B \subset -T$

Necessary condition,

Let G be a connected graph and let A and B be subgraph of G, and let T be any tree of G.

Since $A \subset T$ and $B \subset -T$

$T \cap (-T) = \phi \longrightarrow A \cap B = \phi$

if $A \subset T \longrightarrow$ no loop in A, by definition of tree.

if $B \subset -T \longrightarrow$ no cutset in B, since every link contains no cutset. Q. E. D.

Theorem 2[3]**:** Given an $\{RLC\}$ network, if it has L-loop o. C-cutset, then the network is not asymptotically stable.

Proof: If there exists L-loop, let $\{L_1, L_2, L_3 ...... L_n\}$ be the inductors in the loop, which are orient in the same loop orientation. Let $i_{L_1} = i_{L_2} = ... i_{L_n} = 1$, and all the other currents and voltages be zeros. We will claim that this set of voltages and currents satisfy the network equations.

$$v_R = Ri_R$$

$$v_L = L\frac{di_L}{dt}$$

$$i_c = C\frac{dv_c}{dt}$$

$$Av = 0$$

$$Bi = 0$$

Therefore, we know that, if a network has L-loop, there exists a solution such that $i_L \neq 0$ and $i_c = i_R = v_c = v_L = 0$, for all t, then the network is not asymptotically stable.

Because of duality, the same proof holds for C-cutset. Q. E. D.

Theorem 3: Given an $\{RLC\}$ network, one can replace each R-element by a nullator and use nullator's rules to eliminate L-element and C-element. If neither L-element nor C-element is in the simplified network, then the network is asymptotically stable.

Proof: Let N be an $\{RLC\}$ network, $N(0)$ be the network obtained from N by replacing resistors with nullators.

Let $V(v_c, i_L)=\frac{1}{2}<v_c, Cv_c>+\frac{1}{2}<i_L, Li_L> >0$, for $(v_c, i_L)\neq 0$ (2-1)

$$\dot{V}(v_c, i_L)=\frac{1}{2}<\dot{v}_c, Cv_c>+\frac{1}{2}<v_c, C\dot{v}_c>+\frac{1}{2}<\dot{i}_L, Li_L>+\frac{1}{2}<i_L, L\dot{i}_L>$$

Since L and C are diagonal matrices and

$$C\dot{v}_c=i_c$$
$$L\dot{i}_L=v_L$$
$$\dot{V}(v_c, i_L)=<i_c, v_c>+<i_L, v_L> \tag{2-2}$$

By Tellegen's theorem

$$\dot{V}(v_c, i_L)=-<i_R, Ri_R> \leq 0 \quad \text{for all } v_c, i_L \tag{2-3}$$

With Eq. (2-1) and Eq. (2-3), by LaSalle theorem[4], there exists a set $N(0)$ such that

$$N(0)=\{(v_c, i_L)\mid i_R=0,\ V_R=0\} \tag{2-4}$$

Let M be the largest invariant set of $N(0)$, Then all solutions (v_c, i_L) of N tend to M as $t\to\infty$.

If there is neither L-element nor C-element in $N(0)$, it implies $v_c=0$, and $i_L=0$, that is $M=\{0\}$, so the solutions of network N approach zero as $t\to\infty$, Hence N is asymptotically stable.

Theorem 4: Given an $\{RLC\}$ network, there exists a nonzero D.C. branch current (or D.C. branch voltage) if and only if there is an L-loop (or C-cutset).

Proof: Sufficient condition was proved in theorem 2.

Necessary condition.

Let $\dot{X}=AX$ be the zero-input state equation for an $\{RLC\}$ network where X represents the capacitor-voltage and inductor-current as state vector.

Let there exist a nonzero D.C. current I_0, it implies that there exists a zero eigenvalue of $\dot{X}=AX$, $X(0)=X_0$, that is $\lambda=0$, then there exists X_0, such that $X_0=I_0$ and $X(t)=I_0e^{\lambda t}$

Let branch voltage and branch current be

$$V_i(t)=V_i$$
$$i_i(t)=I_i \quad \text{for all } t,\ i=1, 2, \ldots\ldots b.$$

From element's equations

$$V_L=L\frac{di_L}{dt} \longrightarrow i_L=I_L\ v_L=0$$

$$i_c=C\frac{dv_c}{dt} \longrightarrow v_c=V_c\ i_c=0$$

$$v_R=Ri_R \longrightarrow v_R=V_R\ i_R=I_R.$$

With Tellegen's theorem

$$<i, v>=<i_R, v_R>+<i_L, v_L>+<i_c, v_c>=0$$

since $v_L = i_o = 0$

$$<i, v> = <i_R, v_R> = <i_R, Ri_R> = 0$$

since all elements of R are positive

$$\therefore i_R = 0$$

If there does not exist L-loop, by theorem 1, then there exists a tree, such that the inductors L are in the tree-branch. Since i_L can be represented by linear combination of i_R and i_σ which are link-branch.

$$\because i_R = i_\sigma = 0$$
$$\therefore i_L = 0$$

This contradicts with the assumption $I_0 \neq 0$, Hence we have the conclusion that if there exists a nonzero D.C. branch current, then there is L-loop in the network. Similarly, because of duality, we can prove that if there exists a nonzer D.C. branch voltage, then there is C-cutset in the network.

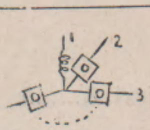
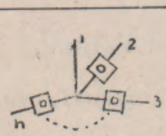
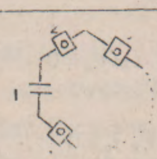
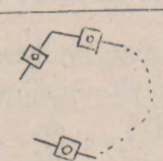
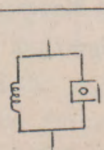
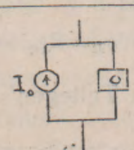
Theorem 5[5]**:** Given a network N, let $N(R=0)$ be the network obtained from N with resistors replaced by short circuits, and similarly $N(R=\infty)$ by open circuits, Let S denote collection of positive limiting sets of solutions of N, and $S(0)$ and $S(\infty)$ for $N(R=0)$ and $N(R=\infty)$, then

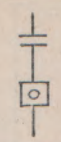
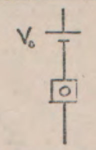
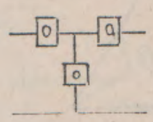
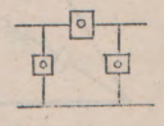
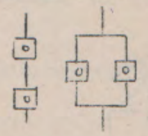
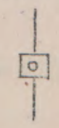
$$S = S(0) \cap S(\infty)$$

The proof is given in Reference [5].

Corallary: If $N(R=0)$ and $N(R=\infty)$ have at least one pair of eigenvalues in common. then the network N is not asymptotically stable. The proof of this corollary is easy to see. Since, from theorem 5, the intersection of $S(0)$ and $S(\infty)$ consists of at least one pair of nonzero solutions of N.

3. SIMPLIFICATION RULES

Rule	Original	Result
1. Replace only inductor with short circuit in nullator cutset.		
2. Replace only capacitor with open circuit in nullator loop.		
3. Replace a parallel inductor-nullator with a D.C. branch current.		

4. Replace a series capacitor-nullator with a D.C. branch voltage.		
5. Nullator T-network can be replaced by nullator π-network and vice vera.		
6. Series or parallel nullators can be replaced by a nullator.		

4. ALGORITHM

We use the theorems described in Section 5.2, and simplification rules to establish the algorithm as follows.

Step 1. Examine the tested network, if there is an L-loop or C-cutset, then the network is not asymptotically stable.

Step 2. If neither L-loop nor C-cutset is in the tested network, use the nullator simplified method to replace each R-element by a nullator. And then by the simplification rules, D. C. branch current I_0 and D. C. branch voltage V_0 may be obtained. With theorem 4 we can replace I_0 with open circuit and V_0 with short circuit. Repeat the simplification until neither L-element nor C-element is in the network, then it is asymptotically stable.

Step 3. If the test fails in step 2, write two state equations, one with all nullators replaced by open circuits, the other with all nullators replaced by short circuits. Find all eigenvalues of the state equations by solving the characteristic equations as

$$g_1(\lambda)=\det(A_1-I)=0 \tag{4-1}$$

$$g_2(\lambda)=\det(A_2-I)=0 \tag{4-2}$$

where A_1 is the system matrix of $N(R=\infty)$, A_2 is that of $N(R=0)$.

If there exists at least one pair of common roots in $g_1(\lambda)=0$ and $g_2(\lambda)=0$, then it is not asymptotically stable, otherwise it is asymptotically stable.

Step 4. If the dimension of system matrix A is too large, put $g(\lambda)=0$ into computer, it is easy to get all roots and then test the stability as in step 3.

5. EXAMPLES

Ex. 1 The network is shown in Fig. 5.1, it is not asymptotically stable, since there is C-cutset in it.

Ex. 2 The network shown in Fig. 5.2 is asymptotically stable, that is easy to see with algorithm check. In Fig. 5.3(a), since there is neither L-loop nor C-cutset in it,

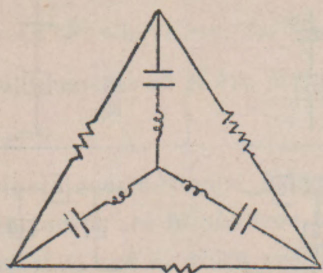


Fig. 5.1 Network with C-cutset elements.

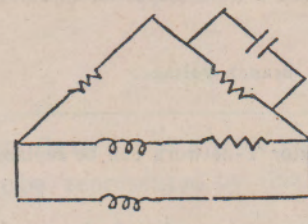


Fig. 5.2 Network of Ex. 2.

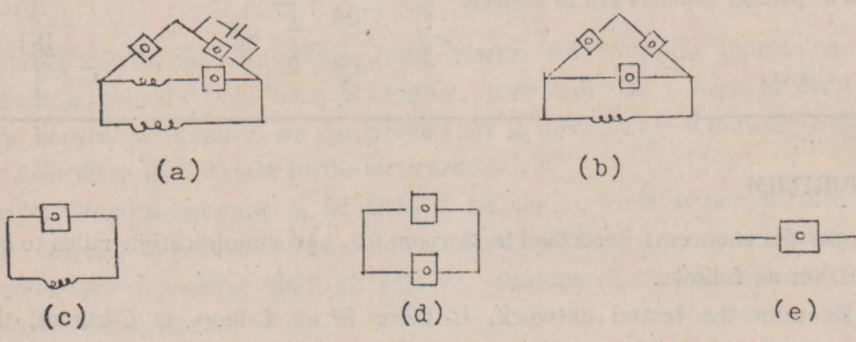


Fig. 5.3 Simplifying a network with nullator's method.

replace all R-element with nullators. Fig. 5.3(b), (c), (d) and (e), show how the network is simplified with the simplification rules.

Ex. 3 The tested network N is shown in Fig. 5.4(a), by algorithm.

Step 1. Neither L-loop nor C-cutset.

Step 2. Replace R-element with nullator, I_0 appears, as shown in Fig. 5.4(b), Replace I_0 with open circuit. There are still L and C elements in the simplified network as shown in Fig. 5.4(c).

Step 3. $N(R=0)$ shown in Fig. 5.4(d) has the state equation

$$\begin{pmatrix} \dot{v}_{c_1} \\ i_L \end{pmatrix} = \begin{pmatrix} 0, & -\frac{1}{C_1+C_2} \\ \frac{1}{L}, & 0 \end{pmatrix} \begin{pmatrix} v_{c_1} \\ i_L \end{pmatrix}$$

$$\lambda = \pm j \frac{1}{\sqrt{L(C_1+C_2)}},$$

$N(R=\infty)$ shown in Fig. 5.4(e) has the state equation

$$\begin{pmatrix} \dot{v}_{c_1} \\ \dot{v}_{c_2} \\ i_L \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{C} \\ 0 & \frac{1}{L} & 0 \end{pmatrix} \begin{pmatrix} v_{c_1} \\ v_{c_2} \\ i_L \end{pmatrix}$$

$$\lambda = 0, \ \lambda = \pm j \frac{1}{\sqrt{LC_2}},$$

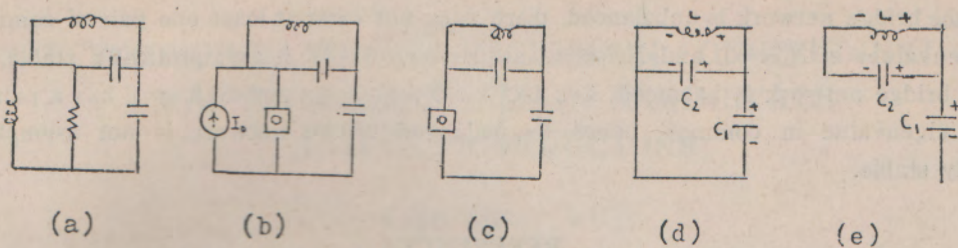


Fig. 5.4 Network N tested by algorithm.

Since there does not exist a pair of common eigenvalues in $N(R=0)$ and $N(R=\infty)$, the network N is asymptotically stable.

Ex. 4 The bridge network N shown in Fig. 5.5(a) is tested by algorithm as follows.

Step 1. Neither L-loop nor C-cutset.

Step 2. Replace R-element with nullator as shown in Fig. 5.5(b), but we can not apply the simplification rule on it.

Step 3. $N(R=0)$, shown in Fig. 5.5(c), has the state equation

$$\begin{pmatrix} \dot{v}_{c_1} \\ \dot{i}_{L_1} \\ \dot{i}_{L_2} \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{C_1+C_2} & \frac{1}{C_1+C_2} \\ \frac{1}{L_1} & 0 & 0 \\ \frac{-1}{L_2} & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{c_1} \\ i_{L_1} \\ i_{L_2} \end{pmatrix}$$

$$\lambda_1=0,\ \lambda_{2,3}=\pm j\frac{L_1+L_2}{\sqrt{L_1L_2(C_1+C_2)}},$$

$N(R=\infty)$, shown in Fig. 5.5(d), has the state equation

$$\begin{pmatrix} \dot{v}_{c_1} \\ \dot{v}_{c_2} \\ \dot{i}_{L_1} \\ \dot{i}_{L_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{C_1} \\ 0 & 0 & \frac{1}{C_2} & 0 \\ 0 & \frac{-1}{L_1} & 0 & 0 \\ \frac{-1}{L_2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_{c_1} \\ v_{c_2} \\ i_{L_1} \\ i_{L_2} \end{pmatrix}$$

$$\lambda_{1,2}=\pm j\frac{1}{\sqrt{L_1C_2}},\quad \lambda_{3,4}=\pm j\frac{1}{\sqrt{L_2C_1}},$$

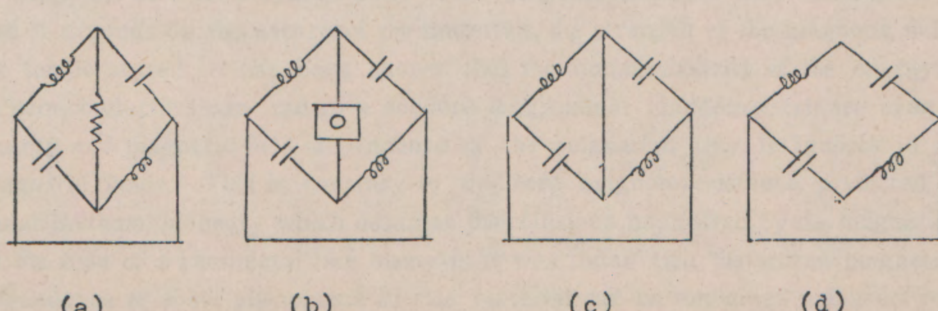


Fig. 5.5 Bridge network tested by algorithm.

If the bridge network is unbalanced, there does not exist at least one pair of common eigenvalues in $N(R=0)$ and $N(R=\infty)$, so the network N is asymptotically stable. If the bridge network is balanced, *i.e.*, $L_1C_2=L_2C_1$, $N(R=0)$ and $N(R=\infty)$, has a pair of the eigenvalue in common, hence the balanced bridge network is not asymptotically stable.

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