A STUDY ON MEASURES ON NONMEASURABLE SUBSETS

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1. INTRODUCTION

Let $(X, \mathcal{A}, \mu)$ be a measure space and E is a μ-measurable subset (that is $E \in \mathcal{A}$). Let $\mathcal{A}_E=\{F \in \mathcal{A} | F \subset E\}$, and let μ_E be the restriction of μ to $\mathcal{A}_E$, then $(E, \mathcal{A}_E, \mu_E)$ is a measure space. If Y is a non-μ-measurable subset of X (that is $Y \notin \mathcal{A}$) such that if $B \subset Y'$ and $B \in \mathcal{A}$, then $\mu(B)=0$. Let $\mathcal{A}^+=\{M^+=Y \cap M | M \in \mathcal{A}\}$, define $\mu^+ : \mathcal{A}^+ \to R^\#$ by $\mu^+(M^+)=\mu(M)$ if $M^+=Y \cap M$. It was proved that $(Y, \mathcal{A}^+, \mu^+)$ is a measure space. In this paper, the existence of such a subset Y when μ is the Lebesgue outer measure λ on R is proved in detail by the concepts of cardinal numbers and ordinal numbers.

2. PRELIMINARY

a. Cardinal numbers and ordinal numbers

If A is a set, we let $\overline{\overline{A}}$ denote the cardinal number of A, thus $\overline{\overline{\phi}}=0$, $\overline{\overline{\{1, 2, \ldots, n\}}}=n$, $\overline{\overline{N}}=N_o$ (N is the set of all positive integers), $\overline{\overline{R}}=C$ (R is the set of all real numbers) etc......

(1) If set B is one to one correspondence (or equivalent) to A $(B \sim A)$, then $\overline{\overline{B}}=\overline{\overline{A}}$.

(2) Let A, B be sets, and $\overline{\overline{A}}=a$, $\overline{\overline{B}}=b$. We write $a \leqq b$ if A is equivalent to some subset of B $(a<b)$ if $a \leqq b$ and $a \neq b$).

(3) Any set of cardinal numbers is a linearly ordered set.

(4) Continuum hypothesis: there is no cardinal number strictly between N_o and C.

If A is a well-ordered set. We use ord$A$ to denote the ordinal number of A, thus ord $\phi=0$, ord $\{1, 2, \ldots, n\}=n$, ord $N=\omega$, etc.

(5) If A, B are two well-ordered sets, and ord$A=\alpha$, ord$B=\beta$, then $\alpha=\beta$; if there is an order isomorphism from A onto B $(A \approx B)$.

(6) $\alpha<\beta$ if $\exists\ x \in B$ such that $A \approx$ the initial segment P_x of B determined by x.

(7) Any set of ordinal numbers forms a linearly ordered set.

Lemma 1. Let α be an ordinal number, $\alpha>0$, and let P_α be the set of all ordinal numbers $<\alpha$. Then P_α is a well-ordered set and ord$P_\alpha=\alpha$.

Lemma 2. Let a be a cardinal number, Then there exists an ordinal number α such that $\overline{\overline{P_\alpha}}=a$.

Lemma 3. There is a smallest ordinal number Ω such that P_Ω is uncountable,

b. Outer measures and measures

Definition 1; Let X be a set. A set function μ defined on $\mathcal{P}(X)$ into $R^{\#}$ is called an outer measure if the following relations hold;

(i) $0 \leqq \mu(A) \leqq \infty$ for all $A \subset X$:

(ii) $\mu(\phi)=0$:

(iii) $\mu(A) \leqq \mu(B)$ if $A \subset B \subset X$;

(iv) $\mu(\bigcup_{n=1}^{\infty} A_n) \leqq \sum_{n=1}^{\infty} \mu(A_n)$ for all sequences $(A_n)_{n=1}^{\infty}$ of subsets of X.

Definition 2: Let X be a set and $\mathcal{A} \subset \mathcal{P}(X)$ be a σ-algebra. A set function μ: $\mathcal{A} \to R^{\#}$ is called a measure if

(i) $0 \leqq \mu(A) \leqq \infty$ for all $A \varepsilon \mathcal{A}$;

(ii) $\mu(\phi)=0$;

(iii) $\mu(\bigcup_{n=1}^{\infty} A_n)=\sum_{n=1}^{\infty} \mu(A_n)$ for all pairwise disjoint sequences $(A_n)_{n=1}^{\infty}$ of $\mathcal{A}$.

If μ is a measure on $\mathcal{A}$, then the triple $(X, \mathcal{A}, \mu)$ is called a measure space.

Definition 3. Let X be a set and μ an outer measure on $\mathcal{P}(X)$. A subset A of X is said to be μ-measurable if $\mu(T)=\mu(T \cap A)+\mu(T \cap A')$ for all $T \subset X$.

Lemma 4. Let X be a set and μ an outer measure on $\mathcal{P}(X)$ Then the set N_μ of all μ-measurable subsets forms a σ-algebra, and (X, M_μ, μ) forms a measure space.

Lemma 5. For the Lebesgue outer measure λ on $\mathcal{P}(R)$, if A is a λ-meassurable subset of R such that $A \subset \bigcup_{n=1}^{\infty} B_n$ for some sequences $(B_n)_{n=1}^{\infty}$ of sets such thal $\lambda(B_n)<\infty$ for all n, then $\lambda(A)=\sup\{\lambda(F) \mid F \text{ is compact}, F \subset A\}$.

3. MAIN THEOREM REFERENCES

Theorem 1; (Measures on nonmeasurable subsets)

Let $(X, \mathcal{A}, \mu)$ be a measure space and let Y be a subset of X such that if $B \subset Y'$ and $B \varepsilon \mathcal{A}$, then $\mu(B)=0$. Let $\mathcal{A}^+=\{Y \cap M \mid M \varepsilon \mathcal{A}\}$, μ^+: $\mathcal{A}^+ \to R^{\#}$ defined by $\mu^+(M^+)=\mu(M)$ if $M^+=Y \cap M$, $M \varepsilon \mathcal{A}$, then $(Y, \mathcal{A}^+, \mu^+)$ is a measure space.

Proof: (1) μ^+ is well defined.

If $M^+=Y \cap M_1=Y \cap M_2$ for M_1, $M_2 \varepsilon \mathcal{A}$

Since $Y \cap (M_1 \cap M'_2)=(Y \cap M_1) \cap M'_2=(Y \cap M_2) \cap M'_2=\phi$

then $M_1 \cap M'_2 \subset Y'$ and $M_1 \cap M'_2 \varepsilon \mathcal{A}$, this

implies $\mu(M_1)=\mu(M_1 \cap M_2)+\mu(M_1 \cap M'_2)=\mu(M_1 \cap M_2)$

Similarly $\mu(M_2)=\mu(M_2 \cap M_1)+\mu(M_2 \cap M'_1)=\mu(M_2 \cap M_1)=\mu(M_1 \cap M_2)=\mu(M_1)$

Hence for all $M^+ \varepsilon \mathcal{A}^+$, $\mu^+(M^+)$ is uniquely defined.

(2) $\mathcal{A}^+$ is a σ-algebra of subsets of Y.

For M^+, $N^+ \varepsilon \mathcal{A}^+$, there exist M, $N \varepsilon \mathcal{A}$ such that

$M^+=Y \cap M$ and $N^+=Y \cap N$. Since

$M^+ \cup N^+=(Y \cap M) \cup (Y \cap N)=Y \cap (M \cup N) \varepsilon \mathcal{A}^+$

and

$M_Y^{+\prime}=(Y \cap M)' \cap Y=(Y' \cup M') \cap Y=(Y' \cap Y) \cup (M' \cap Y)=M' \cap Y=Y \cap M' \varepsilon \mathcal{A}^+$

$M^+ \cap N^+=(Y \cap M) \cap (Y \cap N)=Y \cap (M \cap N) \varepsilon \mathcal{A}^+$.

if $M^+, \ldots, M_n^+, \ldots \varepsilon \mathcal{A}^+$

$$\bigcup_{k=1}^{\infty} M^{+}{}_{k}=\bigcup_{k=1}^{\infty}(Y\cap M_{k})=Y\cap(\bigcup_{k=1}^{\infty} M_{k})\varepsilon A^{+}$$

$$\phi=Y\cap\phi,\ Y\cap X\varepsilon A^{+}.$$

Hence A^{+} is a σ-algebra.

(3) (Y, A^{+}, μ^{+}) is a measure space.

(i) $0\leqq\mu^{+}(M^{+})=\mu(M)\leqq\infty$ for all $M^{+}\ \varepsilon A^{+}$;

(ii) $\mu^{+}(\phi)=\mu(\phi)=0$;

(iii) Let $(M^{+}{}_{k})_{k=1}^{\infty}$ be a sequence of pairwise disjoint set in A^{+}, $M^{+}{}_{k}=Y\cap M_{k}$ for all $k\varepsilon N$. Let $T=\bigcup_{k=1}^{\infty} M^{+}{}_{k}\varepsilon A^{+}$, then

$$\mu^{+}(\bigcup_{k=1}^{\infty} M^{+}{}_{k})=\mu^{+}(T)=\sum_{n=1}^{\infty}\mu^{+}(T\cap M^{+}{}_{n})+\mu^{+}(T\cap(\bigcup_{n=1}^{\infty}(M^{+}{}_{n})'_{Y})$$

$$=\sum_{n=1}^{\infty}\mu^{+}(\bigcup_{k=1}^{\infty} M^{+}{}_{n})+\mu^{+}(T\cap T'_{Y})=\sum_{n=1}^{\infty}\mu^{+}(M^{+}{}_{n})+\mu^{+}(\phi)$$

$$=\sum_{n=1}^{\infty}\mu^{+}(M^{+}{}_{n})+0=\sum_{n=1}^{\infty}\mu^{+}(M^{+}{}_{n})$$

([1] 10.9)

Thus (Y, A^{+}, μ^{+}) is a measure space.

Theorem 2: Every uncountable closed subset F of R has cardinal number c.

Proof: Since R is a topological space with a countable base, by the Cantor-Bendixson's theorem ([1]) 6.66) there is a perfect subset P and a countable subset C of R such that $F=P\cup C$. For F is uncountable, $P\neq\phi$, And since R is a complete metric space, $\overline{\overline{P}}=c$ ([1] 6.65). this shows that $\overline{\overline{F}}=c$.

Theorem 3: (F. Bernstein). there is a subset B of R such that $B\cap F\neq\phi$ and $B'\cap F\neq\phi$ for every uncountable subset F of R.

Proof: Let $\mathfrak{G}$ be the set of all open subsets of R and let $\mathfrak{F}$ be the set of all open subsets of R and let $\mathfrak{F}$ be the set of all uncountable closed subsets of R. For $\overline{\overline{\mathfrak{G}}}=c$ ([1] (4.34), (10.21)) then $\overline{\overline{\mathfrak{F}}}=c$. Let ω_c be the smallest ordinal number corresponding to the cardinal number c, that is let ω_c be the smallest ordinal number such that $\overline{\overline{P}}_{\omega_c}=c$ ([1]. p. 29). Now for P_{ω_c} is a well ordered set and $P_{\pi_c}\sim\mathfrak{F}$, $\mathfrak{F}$ can be well ordered by $F_\eta<F_r$ if $\eta<r<\omega_c$. Define set B by transfinite recursion and the axiom of choice as follows:

Let x_o and y_o be two distinct points of F_o

x_1 and y_1 be two distinct points of $F_1-\{x_o, y_o\}$......

Suppose x_r and y_r have been defined for all ordinal numbers $r<\eta$ where $\eta<\omega_c$. Let $A_\eta=\{x_r|r<\eta\}\cup\{y_r|r<\eta\}$ then $\overline{\overline{A}}_\eta<c$ for $\overline{\overline{A}}_\eta=\overline{\overline{P}}_\eta<c$ because ω_c is the smallest number such that $\overline{\overline{P}}_{\omega_c}=c$. Since $\overline{\overline{F}}_\eta=c$ (F_η is an uncountable closed subset of R) so $F_\eta\cap A'_\eta$ has cardinal number c, that is $F_\eta\cap A'_\eta$ is uncountable. Choose (any) two distinct points x_η and y_η in $F_\eta\cap A'_\eta$.

Finally let $B=\{x_\eta|\eta<\omega_c\}$. For each $\eta<\omega_c$, there exists $x_\eta \varepsilon B\cap F_\eta$, $B\cap F_\eta\neq\phi$. For each $\eta<\omega_c$, there exists $y_\eta \varepsilon F_\eta$ but $y_\eta\notin B$, $B'\cap F_\eta\neq\phi$. that is for each uncountable closed subset F of R $B\cap F\neq\phi$ and $B'\cap F\neq\phi$.

Theorem 4: The subset B of R as defined in theorem 3 is not λ_α-measurable if α is a countinuous nondecreasing real function and $\lambda_\alpha\neq 0$ is a Lebesgue-Stieltjes measure on R induced by α.

Proof: Suppose there is an α such that B is λ_α-measurable, then for $B\subset\bigcup_{n=1}^{\infty}[-n, n]$ and $\lambda(_\alpha[-n,n])<\infty$, so

$$\lambda_\alpha(B)=\sup\{\lambda_\alpha(F)|F \text{ is compact, } F\subset B\}$$

([1] 10.30)

But if F is a compact subset of B then F is clcsed in R. hence if F is not countable then $F\cap B'\neq\phi$, $F\not\subset B$. Hence each compact subset of B is a countable subset of R. Next if F is a countable subset of R then $F=\{x_n|n=1,2,\ldots\}$ and $\lambda_\alpha(x_n)=0$, each $\{x_n\}$ is a λ_α-measurable set. $\lambda_\alpha(F)=\lambda_\alpha(\bigcup_{n=1}^{\infty}\{x_n\})=\sum_{n=4}^{\infty}\lambda_\alpha(x_n)=0$

Therefore $\lambda_\alpha(B)=\sup\{\lambda_\alpha(F)|F \text{ is compact, } F\subset B\}=\sup\{0\}=0$.

On other hand, for B is λ_σ-measurable hence B' is λ_α-measurable, but B' is also a σ-finite set (about λ_α), $\lambda_\alpha(B')=\sup\{\lambda_\alpha(F)|F \text{ is compact, } F\subset B'\}=\sup\{0\}=0$, for each compact subset of B' is countable (if not, $B\cap F\neq\phi$, $F\not\subset B'$ is a contradition). Now, since B and B' are λ_α-measurable, $B\cap B'=\phi$ and $\lambda_\alpha(B)=(B')=0$

$$\lambda_\alpha(R)=\lambda_\alpha(B\subset B')=\lambda_\alpha(B)+\lambda_\alpha(B')=0,\ \lambda_\alpha=0.$$

These shows that if $\lambda_\alpha\neq 0$, B is a non λ_α-measurable set.

Corollary. B is not λ-measurable (*i.e.*, $B\notin M_\lambda$)

Proof. The Lebsegue measure λ is a special case of a Lebesgue-Stieltjes measure λ_α, *i.e.*, when α=identity function, now $\lambda\neq 0$ (for $\lambda(R)=\infty$, $\lambda([0, 1])=1$) by the theorem B is not λ-measurable.

Theorem 5: Consider the measure space (R, M_λ, λ) and the non λ-measurable, subset B constructed in theorem 3. If $E\in M_\lambda$ (*i. e.*, E is λ-measurable) and $E\subset B'$, then $\lambda(E)=0$.

Proof: From the proof of Theorem 4, we know that B' is λ σ-finite, so E is λ σ-finite if $E\subset B'$. By Lemma 5, if $E\varepsilon M_\lambda$, $E\subset B'$ we have

$$\lambda(E)=\sup\{\lambda(F)|F \text{ is compact, } F\subset E\}=\sup\{0\}=0$$

since each compact subset $F\subset E$ is countable.

Corollary: By Theorem 1 and Theorem 5, $(B, M^+{}_\lambda, \lambda^+)$ is a measure space.

*R*emark: There are many other methods to construct a non Lebesgue measurable subset of R. for example, by using the concept of equivalence class ([2]).

REFERENCES

1. E. Hewitt & K. Stromberg: Real and Abstract Analysis (1965)
2. W. Rudin: Real and Complex analysis.
3. Paul R. Halmos: Measure Theory.